E2C2-GIACS Advanced School

Bifurcations in the computer lab

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Illustration of non-linear systems and bifurcations

- 5 examples of non-linear systems;
- use Mini_Ker for time integration and parameter variation, compute eigenvalues with lapack, and do graphics with gnuplot;
- codes are already prepared;
- concentrate in trajectories analyses, and relation between local linear system and bifurcation.

The plan

1. Manage to run an example, and do graphs.
2. Make some modifications.
3. Relate bifurcations with local linear stability.

Moving around with the shell

- Accessories → terminal
- ‘ls’ lists current directory. Directories are in blue.
- ‘pwd’ gives your current position in the directories hierarchy.
- Most important command (for us!): ‘cd’, change directory. Directories are separated by /
  - .. means the parent directory,
  - . means current directory.

  The shell drops you in your home directory, and the examples are below Desktop/bifurcations.
- Go to the main directory, then to the directory that demonstrates the saddle node bifurcation, and from that directory, to the hopf bifurcation directory:

```
$ cd Desktop/bifurcations
$ ls
common              hopf    Makefile         pitchfork
generate_serie.sh   lorenz  Makefile.common  saddle_node
$ cd saddle_node
$ cd ../hopf
```

Running a model, and doing a graph

1. Go to the `results` subdirectory of an example directory, here the saddle node example:

   ```
   $ cd saddle_node/results
   ```

2. Run the `run.sh` script (program) that is in the directory, it may be verbose, no problem with that:

   ```
   $ ./run.sh
   ```

3. Start gnuplot. At the gnuplot prompt, plot the results for the fourth run, corresponding with a parameter value of 0.5, time (column 1) on the x axis and the variable trajectory (column 2) on the y axis:

   ```
   $ gnuplot
   gnuplot> plot 'res-4.data' u 1:2
   ```

Modifying bifurcation parameters

The example taken here is the saddle node:

$$\dot{x} = \lambda - x^2$$

- Open a new terminal and go to the `results` subdirectory of the example directory.
- In `parameters.dat` are the values used for the bifurcation parameter, here λ. The results are in the files `res-1.data` for the first parameter value, `res-2.data` for the second parameter value and so on and so forth.
- To add or remove a parameter, edit the `parameters.dat` file with your preferred editor. If you don't have one, use nano:

  ```
  $ nano parameters.dat
  ```
- add or remove parameters. Open another shell, rerun the models. Open another shell and plot the results.

Modifying starting points, length, time step

- To modify these elements, you have to edit the code. It is in the `zinit.mti` file, directly in the example directory.
- At the beginning of the file, you can change:

  dt the time step length,

  nstep the number of steps.
- the model code is within a `set_eta` block. In that block `var:` sets a variable name, while `fun:` sets the time derivative of the variable. Here the variable name for x is '`variable`' the name for λ is '`saddle_param`' and $\lambda - x^2$ corresponds with

  ```
  saddle_param - variable**2
  ```
- to change the starting point value, you should go to the end of the file, and change the value assigned to '`variable`', for example, set:

  ```
  ! initial value
  variable = 5.;
  ```

Computing and displaying linear model eigenvalues

- 2 sets of eigenvalues of the linear tangent model are computed: eigenvalues at an arbitrary point, and eigenvalues at the terminating point. It allows to compute these eigenvalues at an unstable point, for example.
- for each parameter value/model run, a line is output in `bifurcation.dat`, with:

```
<parameter> <last var. values> <last var. real EV>
 <last var. complex EV> <point var. real EV>
 <point var. complex EV> <last var. speed>
```

- to change the arbitrary point coordinates, you should change the variables assignements right after

```
! fixed point
```

- example of graphs for the saddle node bifurcation:

```
gnuplot> plot [0:] 'bifurcation.dat' u 1:2
```

Saddle node bifurcation

- A simple system showing a saddle node bifurcation is:

$$\dot{x} = \lambda - x^2$$

- In this case you can calculate the fixed point by hand, and also determine the fixed point linear stability.

Pitchfork bifurcation

- A simple system showing a pitchfork bifurcation is:

$$\dot{x} = x(\lambda - x^2)$$

- It is still possible to calculate the fixed points linear stability by hand.

Hopf bifurcation

- At least 2 dimensions for that bifurcation ($\alpha < 0$), λ is the bifurcation parameter:

$$\begin{aligned} \dot{x_1} &= \lambda x_1 + (x_1\alpha - x_2\beta)(x_1^2 + x_2^2) \\ \dot{x_2} &= \lambda x_2 + (x_1\beta + x_2\alpha)(x_1^2 + x_2^2) \end{aligned}$$

- In polar coordinate it gives:

$$\begin{aligned} \dot{\rho} &= \rho(\lambda + \alpha\rho^2) \\ \dot{\theta} &= \beta\rho^2 \end{aligned}$$

- Fixed points and stability of ρ may still be calculated, allowing to determine the limit circle radius.
- Still possible to calculate the Jacobian eigen values at the fixed point...
- but a graph showing that real eigen value part cross the zero line at the bifurcation can also be done.

Lorenz model

- This well-known model can give chaos:

$$\begin{aligned}\dot{x}_1 &= \sigma(x_2 - x_1)\\ \dot{x}_2 &= x_1(\rho - x_3) - x_2\\ \dot{x}_3 &= x_1 x_2 - \beta x_3\end{aligned}$$

- You can calculcate fixed points by hand. Stability is less easy.
- You can look at eigen values to find
  1. the change from stable node to stable spiral
  2. the bifurcation to chaos
  3. (haven't checked) the change from stable node to saddle point

Simmonet model

- from E. Simonnet, M. Ghil, and H. A. Dijkstra (2005), Homoclinic bifurcations in the quasi-geostrophic double-gyre circulation.

$$
\begin{aligned}
\dot{A}_1 &= c_1 A_1 A_2 + c_2 A_2 A_3 + c_3 A_3 A_4 - \mu A_1 \\
\dot{A}_2 &= c_4 A_2 A_4 + c_5 A_1 A_3 - c_1 A_1^2 - \mu A_2 + \sigma \\
\dot{A}_3 &= c_6 A_1 A_4 - (c_2 + c_5) A_1 A_2 - \mu A_3 \\
\dot{A}_4 &= -c_4 A_2^2 - (c_3 + c_6) A_1 A_3 - \mu A_4
\end{aligned}
$$

- In the article, the $A_1 + A_3$, A_4 plane is used for diagrams
- pitchfork bifurcation, Hopf bifurcation leading to double loop cycle, then 3 loops, and an homoclinic transition to chaos.