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Program

13/1	Elementary statistics – 1
20/1	Elementary statistics - 2
27/1	Exercises – Computer room
10/2	Fourier Analysis -1
17/2	Fourier Analysis -2, stochastic processes
24/2	Exercises – Computer room
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9/3	Principal component analysis -1
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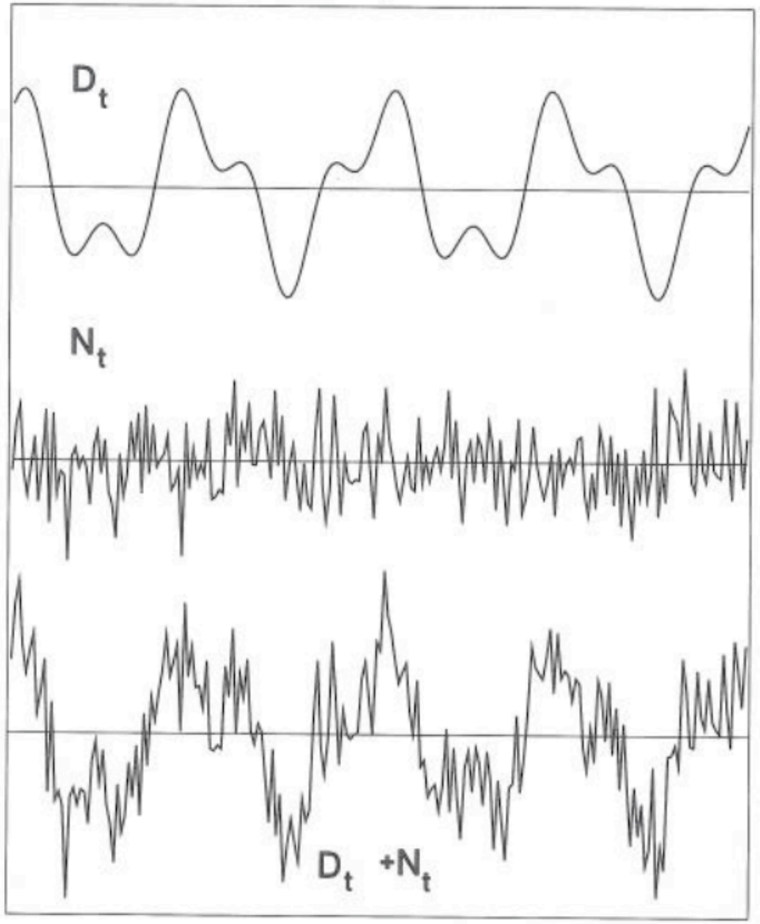
25/5	Principal component analysis: Complements
6/6	Exam.

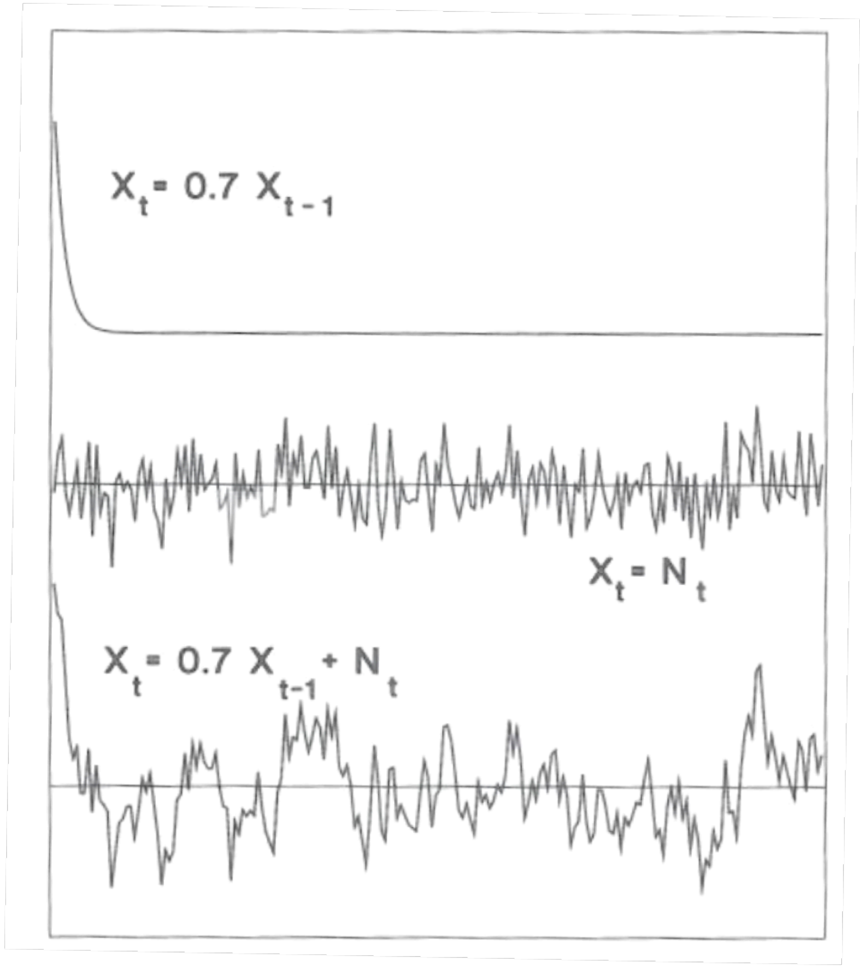
. Lesson 3.
Stochastic Processes, the role of Noise

Noise

A time series can be seen as the sum of a deterministic part and a noise part

$$x_t = D_t + N_t$$





A time series can be seen as a finite sample of a stochastic process.

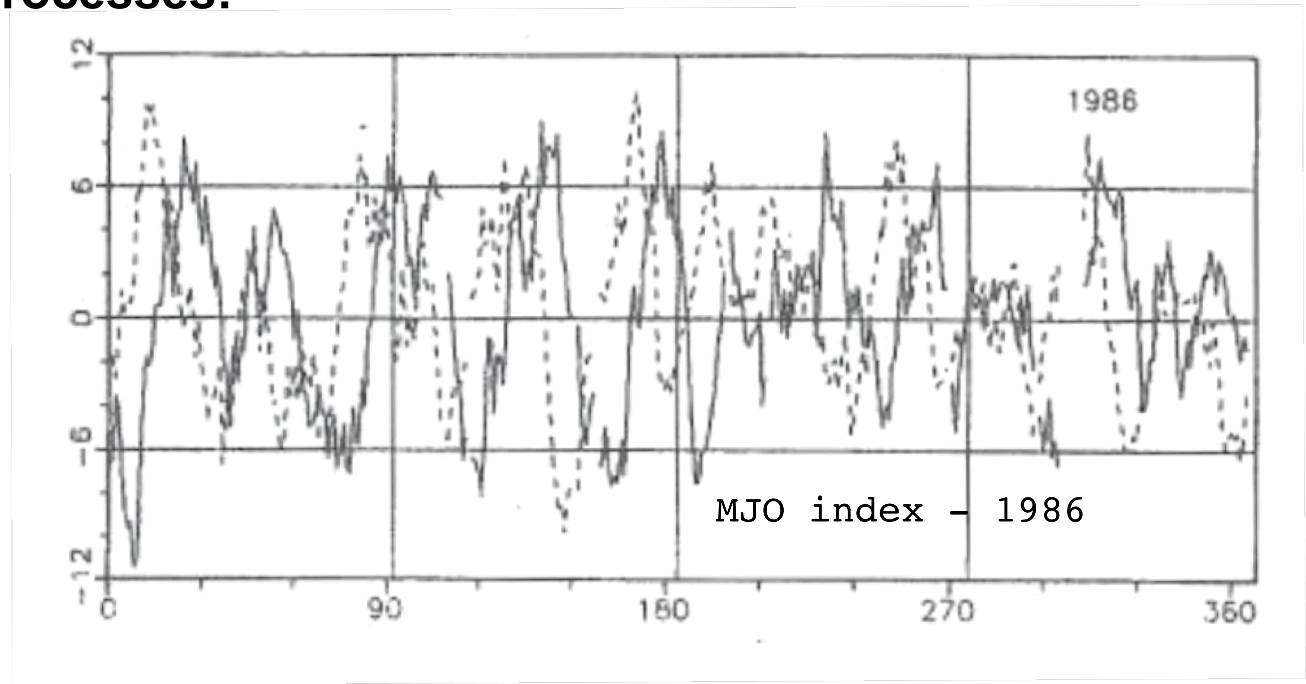
The state of a stochastic process x_t at time t depends on the state at all other times.

Properties of stochastic processes:

Characteristic time.
The series is
Autocorrelated:

$$P(x_{t+\tau} > 0 \mid x_t > 0) > 0.5.$$

The time τ for which the
Probability becomes 0.5
Is the characteristic time.



Stationarity
The statistical properties do not depend on time: x_t has the same PDF for all t, and the joint PDF of x_t and x_s depend only on $|t-s|$.

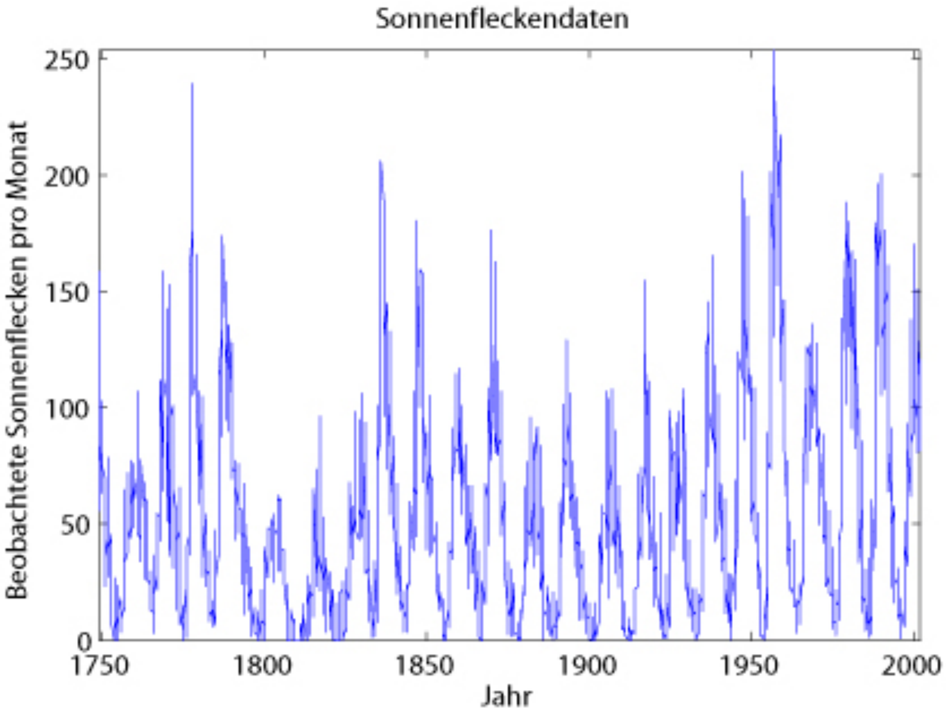
Weak Stationarity
Same but only for mean and variance.

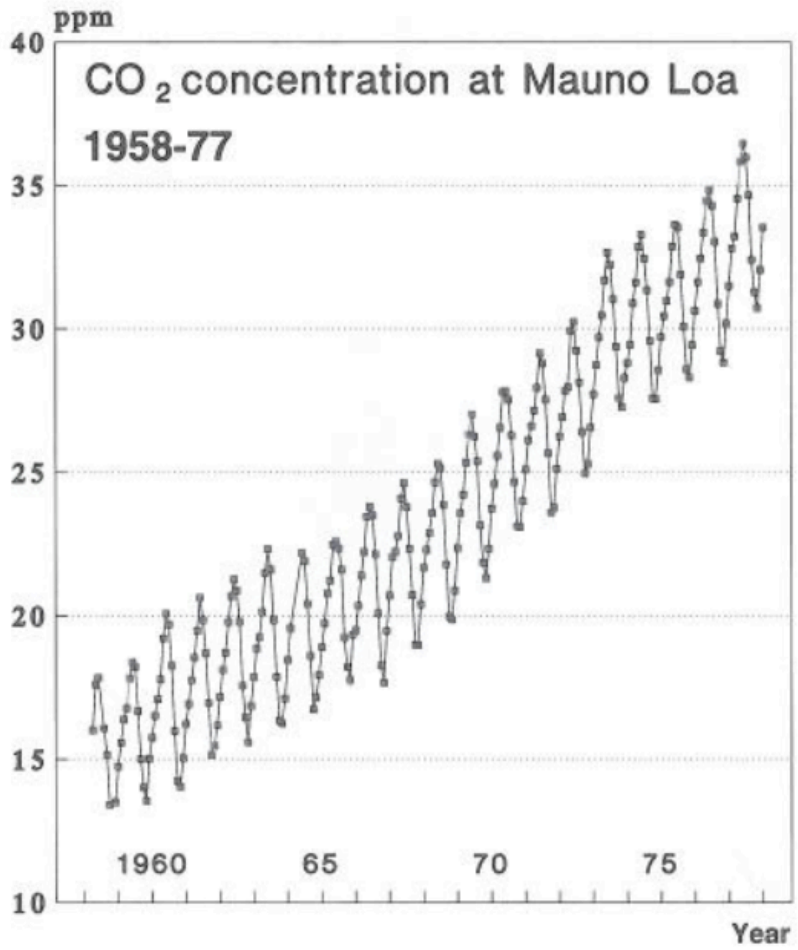
stationarity with trend

Cyclo-stationarity

Cyclo-stationarity with trend

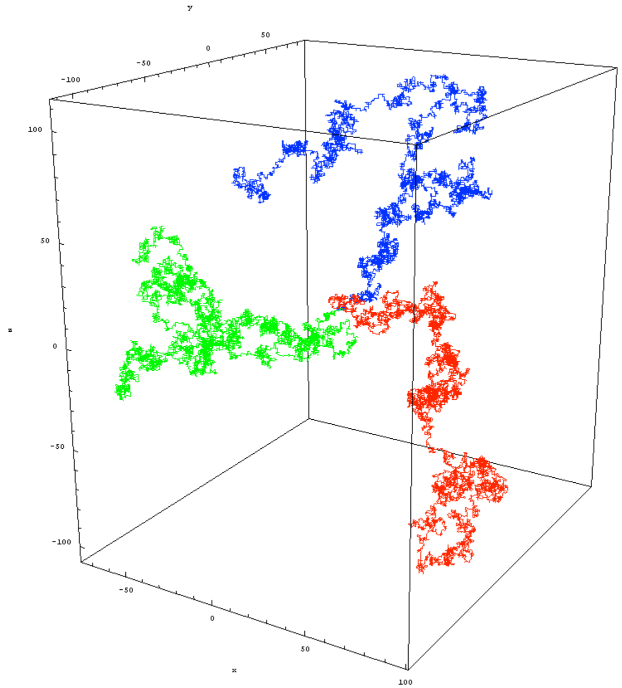
Ergodicity





Random Walk

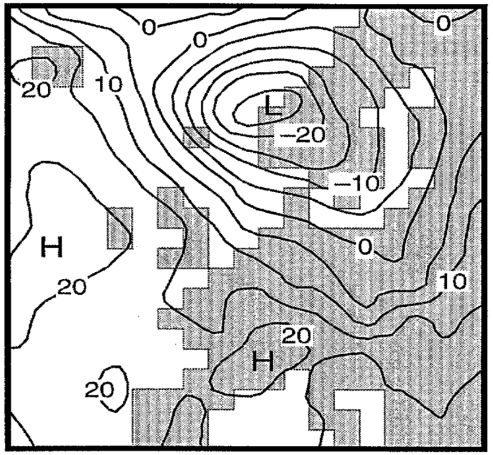
$$\mathbf{x}_t = \sum_{i=1}^{t} \mathbf{z}_i$$, where $\mathbf{z}_t$ is a white noise



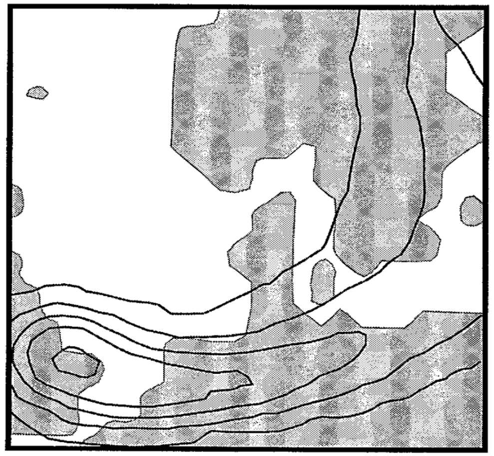
Example, pollutant transport:

$$\mathbf{x}_{t+1} = \mathbf{x}_t + U + \mathbf{z}_t$$

Example of a simulation of long-range transport of air pollutants.Left: Simulated 1000 hPa height field 24 hours after model initialization.Right: Distribution of pollutant continuously emitted in east England after 24 hours. From Lehmhaus et al.



1000 hPa height 24 hours after initialization



Concentration after 24 hours of emissions

Autoregressive processes

It's a sub-class of stochastic processes. But one can show that well behaved stochastic processes (stationary and ergodic) can be approximated by an autoregressive process.

$$x_t = a_0 + \sum_{k=1}^{p} a_k x_{t-k} + w_t$$

p is the *order.* Common writing is AR(p): AR(1) is an order 1, AR(2) etc etc.
If a process has mean zero it is $a_0 = 0$.

So, an order 1, zero mean process can be written (red noise):
$x_t = a_1 x_{t-1} + w_t$.

An order 0 (white noise):

$x_t = w_t$

Autoregressive processes are cool because:

1) Given any weakly stationary ergodic process it is possible to find an AR-process that approximates it arbitrarily closely

2) They can be seen as discretizations of ordinary differential equations. For example, order 1:

$$\frac{dx(t)}{dt} + \alpha x(t) = f(t)$$

$$\frac{x_t - x_{t-dt}}{dt} = \alpha x_t + f_t$$

$$x_t = a x_{t-1} + w_t \qquad a = \frac{1}{1-\alpha}, w_t = \frac{f_t}{1-\alpha}$$

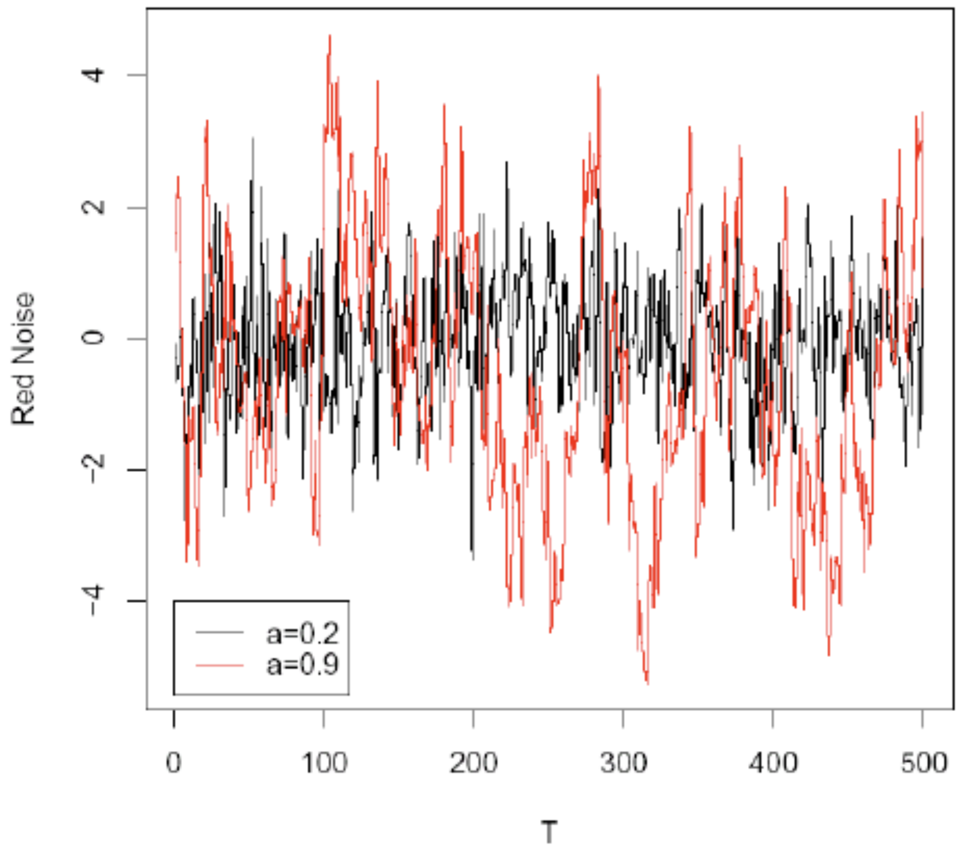
And order 2? Exercise!

AR(1) processes $x_{t+1} = a_1 x_t + w_t$

If we know the parameters a and σ_w we can compute the variance and the autocorrelation of the AR(1) process. The autocorrelation is simply given by a and

$$\sigma_x^2 = \frac{\sigma_w^2}{1-a^2}.$$

One can construct a “rednoise model” of a given timeseries. Equalling the rednoise variance and autocorrelation and the timeseries variance and autocorrelation. That gives two equations for a and σ_w^2.

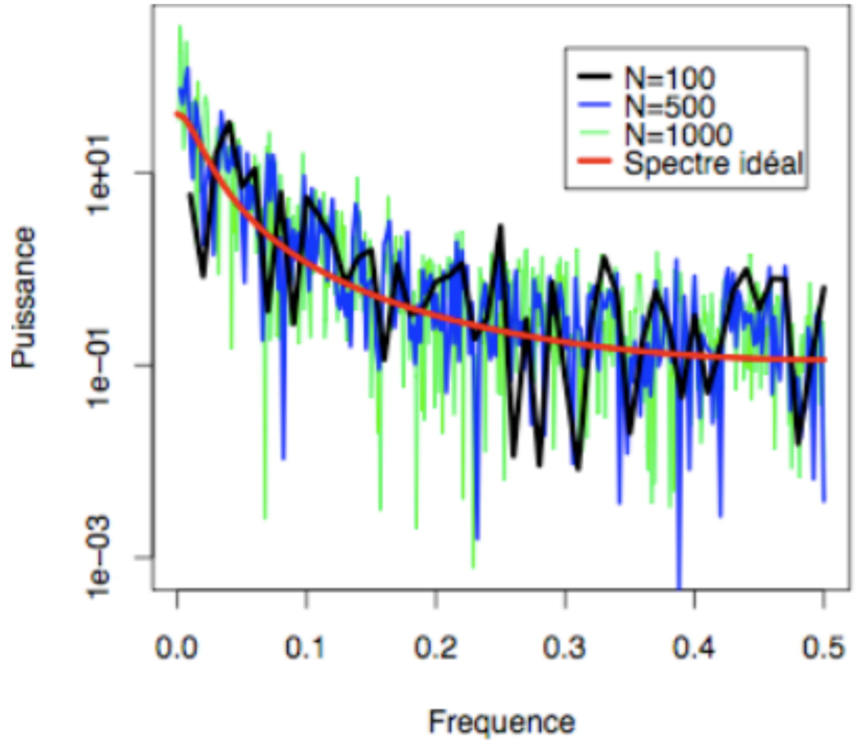


AR(p) spectrum. It can be shown that:

$$P(f) = \frac{\sigma_w^{\ 2}}{1 - \left|1 - \sum_{k=1}^{p} a_k e^{i2\pi fk}\right|^2}.$$

Which in the case of an AR(1) becomes:

$$P(f) = \sigma_x^2 \frac{1-a^2}{1-2a\cos(2\pi f)+a^2}.$$



Although there is no “memory” beyond lag 1, there is power at very low frequencies. Nonzero autocorrelation at short time lags can create high power at low frequency!

This is a feature of climatic – and geophysical – timeseries.

The rednoise process is used as a null hypothesis in spectral estimation.

Note that for $(2\pi f) << 1$

$$P(f) = \sigma_x^2 \frac{1-a^2}{(1+a)^2 + a(2\pi f)^2}$$

Hence:

$$P(f) = \frac{\sigma_w^2}{(1+a)^2} \quad for \quad (2\pi f)^2 << \frac{\left(1-a^2\right)^2}{a}$$

$$P(f) = \frac{\sigma_w^2}{a(2\pi f)^2} \quad for \quad (2\pi f)^2 >> \frac{\left(1-a^2\right)^2}{a}$$