Ecole Polytechnique **Printemps 2019**

Modal de Géophysique

Elements of geophysical fluid dynamics

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2.1 The Boussinesq approximation

The seawater density is a complicated function of the temperature, salinity and pressure. At first order, one can linearize this equation: we get

$$\rho = \rho_0[1 - \alpha(T - T_0) + \beta(S - S_0)] = \rho_0 + \rho' \,, \tag{2.1}$$

with ρ the density T the temperature and S the salinity, ρ_0 the reference density at temperature T_0 and salinity S_0, $\alpha = 2 \times 10^{-4}\ \mathrm{K}^{-1}$, the thermal expansion coefficient and $\beta = 7 \times 10^{-4}\ \mathrm{g\,kg}^{-1}$ the haline expansion coefficient. We write ρ' the deviation with respect to the reference density

$$\rho' = \rho_0(-\alpha(T - T_0) + \beta(S - S_0))\,. \tag{2.2}$$

The reference density of a fresh water parcel at 4^o C is $\rho_0 = 1000\ \mathrm{kg\,m}^{-3}$. Densities for typical values of temperature and salinity found in the ocean are plotted in Fig. 2.1 with the full equation of state (at constant pressure). The density variations (ρ') between the surface and the bottom of the ocean are $\mathcal{O}(1\%)$.

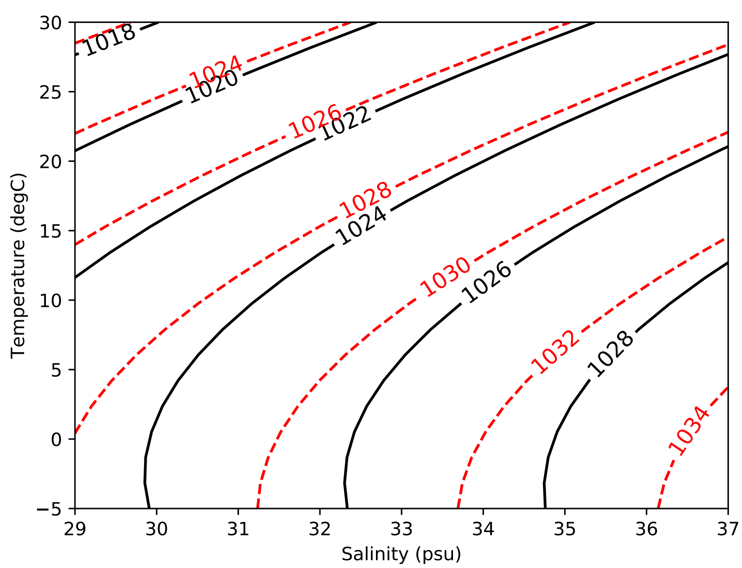

Figure 2.1: Density ($\mathrm{kg\,m}^{-3}$) as a function of temperature and salinity with a full equation of state

The hydrostatic balance is

$$\frac{\partial p}{\partial z} = -\rho g = -(\rho_0 + \rho')g\,. \tag{2.3}$$

We can then split the pressure in two components: the background pressure p_0 and the dynamic pressure p' such that $p = p_0 + p'$, and

$$\frac{\partial p'}{\partial z} = -\rho' g\,, \tag{2.4}$$

and p_0 is a function of z only. In the momentum equations, we have

$$(\rho_0 + \rho')\frac{Du}{Dt} = -\frac{\partial p}{\partial x} = -\frac{\partial p'}{\partial x} \tag{2.5}$$

$$(\rho_0 + \rho')\frac{Dv}{Dt} = -\frac{\partial p}{\partial y} = -\frac{\partial p'}{\partial y}\,, \tag{2.6}$$

because the horizontal derivatives along the x and y directions are at constant z. If we divide this equation by ρ_0 and use the fact that $\rho'/\rho_0 \ll 1$, we have at first order

$$\frac{Du}{Dt} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x} \tag{2.7}$$

$$\frac{Dv}{Dt} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial y}\,, \tag{2.8}$$

which correspond to the Boussinesq approximation. We still need to take into account the variations of density but just to compute the pressure field.

2.2 Static stability

We consider a stratified ocean where the density ρ is function of z only. At $t = 0$ we displace a water parcel from its initial height z_0 to the height $z' = z_0 + \delta z$. This water parcel has a density $\rho(z_0)$ and we wish to describe the dynamics of this parcel. The parcel is only subject to the action of gravity so its acceleration is given by the archimedes' principle

$$\rho(z_0)\frac{d^2 z'}{dt} = -g(\rho(z_0) - \rho(z_0 + \delta z))\,. \tag{2.9}$$

For small displacements δz, one can approximate $(\rho(z_0) - \rho(z_0 + \delta z))/\delta z$ as the vertical derivative of the density

$$\frac{d^2 \delta z}{dt} = -\frac{g}{\rho(z_0)}\frac{d\rho(z)}{dz}\delta z\,. \tag{2.10}$$

We write

$$N^2 = -\frac{g}{\rho_0}\frac{d\rho}{dz}\,, \tag{2.11}$$

the Brunt Vaisalla frequency (squared). If $N^2 > 0$ the ocean is stably stratified and the water parcel oscillate around its initial position z_0. If $N^2 < 0$, the water column is unstable and the water parcel convects. This occurs for exemple after a cold event when the water at the surface is cooled by a storm. In the ocean, typical values for N are $N = 10^{-3}\,\mathrm{s}^{-1}$ (a periodicity of ~ 2 hours).

2.3 Shallow water equations

Both the ocean and the atmosphere are thin layers of fluid: the mean depth of the ocean is $H_o = 4$ km and the thickness of the troposphere (main atmospheric layer) is $H_a = 10$ km, such that the aspect ratio

$$a = \frac{H}{R_{earth}} \ll 1\,. \tag{2.12}$$

With these considerations in mind, it is sometimes helpful to describe the ocean as a layer of homogeneous fluid of uniform density (cf. Fig. 2.2). We note H is the mean depth of the fluid, h the actual depth and η the surface height anomaly: $h = H + \eta$.

The pressure in this layer of fluid is given by the hydrostatic balance integrated between a height z and the free surface

$$p(z) = p_s + \rho g(\eta - z)\,, \tag{2.13}$$

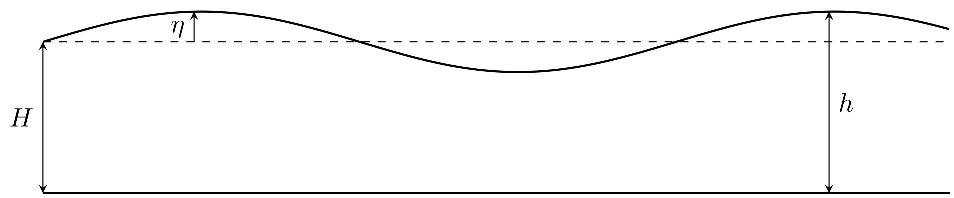

Figure 2.2: The Shallow water model. H is the mean depth of the fluid, h is its actual depth and η the surface height anomaly: $h = H + \eta$.

with p_s the pressure at the surface that we suppose uniform. If we use this expression of pressure in the horizontal momentum equation, we get

$$\frac{D\boldsymbol{u}}{Dt} = -\frac{1}{\rho_0}\nabla p = -g\nabla\eta \,. \tag{2.14}$$

The rhs is independent of z and if the initial condition for $\boldsymbol{u}$ is independent of z then it will remain independent of z during the entire evolution of the system. Hence the advection operator simplifies in

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial x} \,. \tag{2.15}$$

which looks like a 2d operator even though the vertical velocity is non zero. The horizontal momentum equations are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -g\frac{\partial \eta}{\partial x} \tag{2.16}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -g\frac{\partial \eta}{\partial y} \,. \tag{2.17}$$

To get the equation of evolution of h, we use the incompressibility condition

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \,, \tag{2.18}$$

and we know that u and v are independent of z. We can then integrate this equation over the entire layer depth

$$h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + w(\eta) - w(-H) = 0 \,, \tag{2.19}$$

and we use the kinematic boundary condition at the surface

$$w(\eta) = \left.\frac{D\eta}{Dt}\right|_\eta \,. \tag{2.20}$$

At the bottom, the impermeability condition is $w(-H) = 0$ (for a flat bottom). We thus get

$$\frac{D\eta}{Dt} + h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \,. \tag{2.21}$$

We have three equations and three unknowns (u, v and h); this set of partial differential equations is called the shallow water system. In a rotating framework, we simply add the Coriolis force in the horizontal momentum equations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -g\frac{\partial \eta}{\partial x} \tag{2.22}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -g\frac{\partial \eta}{\partial y} \tag{2.23}$$

$$\frac{D\eta}{Dt} + (H+\eta)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \tag{2.24}$$

We define the vorticity (which is the vertical component of the curl of $\boldsymbol{u}$ in this 2d system)

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\,, \tag{2.25}$$

and we can show that the quantity

$$q = \frac{\omega + f}{h}\,, \tag{2.26}$$

is conserved:

$$\frac{\partial q}{\partial t} + u\frac{\partial q}{\partial x} + v\frac{\partial q}{\partial y} = 0 \tag{2.27}$$

We call q the potential vorticity.

2.4 Linear adjustment

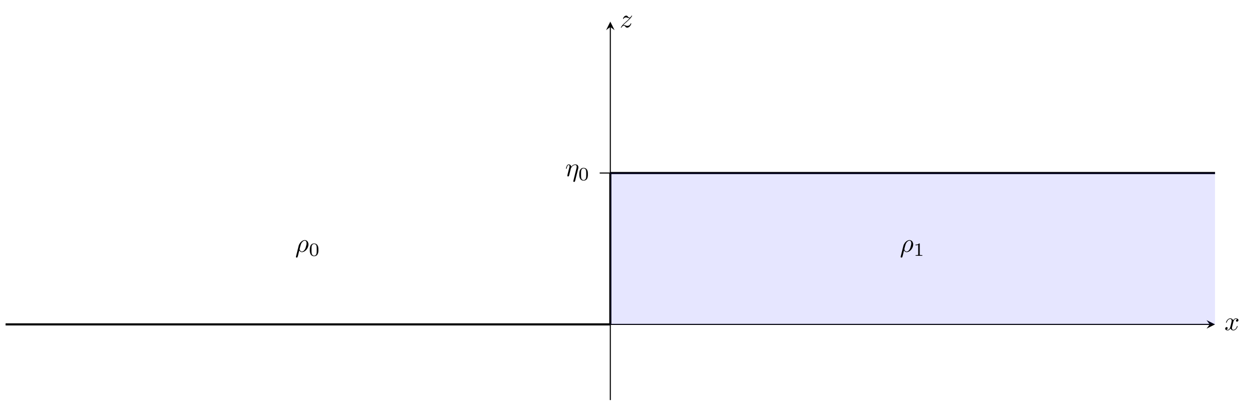

Figure 2.3: Initial configuration of the unstable configuration. In this exemple $\rho_1 > \rho_0$

We consider the problem shown in Fig. 2.3: we fill the right part of the domain ($x > 0$) with water of density ρ_1 and the rest of the domain with water of density ρ_0

$$\begin{aligned} \eta_i &= 0 \quad \text{for} \quad x < 0 \\ &= \eta_0 \quad \text{for} \quad x \geq 0\,, \end{aligned} \tag{2.28}$$

As soon as we release the gate at $x = 0$, we expect that the dense water will fill the left part of the domain. . We suppose that the dynamics is invariant along the y direction ($\partial/\partial y = 0$).

2.4.1 Non rotating linear solution

In the non rotating case, the linear shallow water equations are

$$\frac{\partial u}{\partial t} + g\frac{\partial \eta}{\partial x} = 0 \tag{2.29}$$

$$\frac{\partial \eta}{\partial t} + H\frac{\partial u}{\partial x} = 0 \tag{2.30}$$

which can also be written as

$$\frac{\partial^2 \eta}{\partial t^2} - gH\frac{\partial^2 \eta}{\partial x^2} = 0\,. \tag{2.31}$$

and which admits solutions of the form of non dispersive waves $F(x \pm ct)$, with $c = \sqrt{gH}$, the velocity at which the front propagates. The initial discontinuity propagates to the left and to the right (cf. Fig. 2.4). The final state for an infinite domain is $\eta = \eta_0/2$.

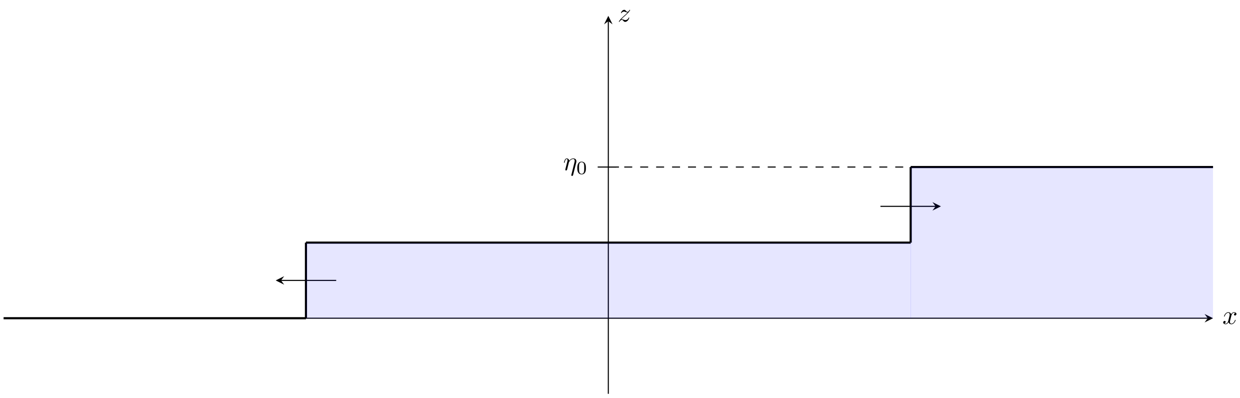

Figure 2.4: Adjustment in the linear non rotating case (transient state)

2.4.2 Linear solution in the rotating case

Rotation completely changes the outcome of the problem. The linear rotating shallow water equations are (with $\partial/\partial y = 0$)

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x} \tag{2.32}$$

$$\frac{\partial v}{\partial t} + fu = 0 \tag{2.33}$$

$$\frac{\partial \eta}{\partial t} + H\frac{\partial u}{\partial x} = 0 \tag{2.34}$$

$$\frac{\partial q}{\partial t} = 0 \tag{2.35}$$

with the linear potential vorticity

$$q = \frac{\omega + f}{H} - f\frac{\eta}{H^2}\,, \tag{2.36}$$

with

$$\omega = \frac{\partial v}{\partial x} \tag{2.37}$$

The initial linear potential vorticity distribution is

$$q_i = \frac{f}{H} - f\frac{\eta_i}{H^2}\,, \tag{2.38}$$

with η_i given in Eq. (2.28). The final potential vorticity distribution is

$$q_f = \frac{f}{H} + \frac{1}{H}\frac{\partial v_f}{\partial x} - f\frac{\eta_f}{H^2}\,, \tag{2.39}$$

and we know $q_i = q_f$ (because of Eq. 2.35), such that

$$\frac{\partial v_f}{\partial x} - f\frac{\eta_f}{H} = f\frac{\eta_i}{H}\,. \tag{2.40}$$

and with equation (2.32), we have

$$fv_f = g\frac{\partial \eta_f}{\partial x}\,. \tag{2.41}$$

We combine these last two equations and get

$$\frac{\partial^2 \eta_f}{\partial x^2} - \frac{f^2}{gH}\eta_f = \frac{f^2}{gH}\eta_i\,. \tag{2.42}$$

We can solve this equation for $x > 0$ and $x < 0$ and use the conservation of volume to adjust the constants of integration. We find that the surface deviation profile is an exponantial with characteristic length scale

$$R_d = \frac{\sqrt{gH}}{f} \tag{2.43}$$

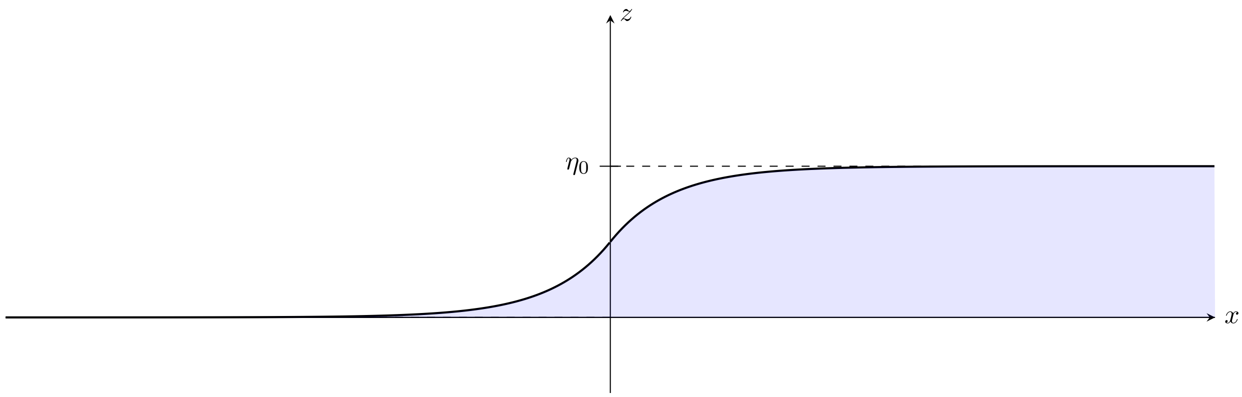

Figure 2.5: Linear adjustment in the rotating case (final state)

Asoociated to this elevation profile, there is a non zero velocity field in the page.