The Geometry of the Universal Teichmüller Space and the Euler-Weil-Petersson Equation

François Gay-Balmaz *
Control and Dynamical Systems
California Institute of Technology 107-81
Pasadena, CA 91125 USA
Laboratoire de Météorologie Dyamique
Ecole Normale Supérieure/CNRS
Paris, France
fgbalmaz@cds.caltech.edu

Jerrold E. Marsden†
Control and Dynamical Systems
California Institute of Technology 107-81
Pasadena, CA 91125 USA
marsden@cds.caltech.edu

Tudor S. Ratiu‡
Section de Mathématiques
Ecole Polytechnique Fédérale de Lausanne
CH - 1015 Lausanne, Switzerland
Tudor.Ratiu@epfl.ch

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Abstract

On the identity component of the universal Teichmüller space endowed with the Takhtajan-Teo topology, the geodesics of the Weil-Petersson metric are shown to exist for all time. This component is naturally a subgroup of the quasisymmetric homeomorphisms of the circle. Viewed this way, the regularity of its elements is shown to be $H^{\frac{3}{2}-\varepsilon}$ for all $\varepsilon > 0$. The evolutionary PDE associated to the spatial representation of the geodesics of the Weil-Petersson metric is derived using multiplication and composition below the critical Sobolev index 3/2. Geodesic completeness is used to introduce special classes of solutions of this PDE analogous to peakons. Our setting is used to prove that there exists a unique geodesic between each two shapes in the plane in the context of the application of the Weil-Petersson metric in imaging.

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Contents

1 Introduction

This paper establishes a link between three distinct subjects: conservative evolutionary PDEs having a form similar to those appearing in fluid dynamics, the theory of the universal Teichmüller space, and the study of maps at critical Sobolev index.

It is well known since the work of Arnold [1966] that the solutions of the Euler equations are the spatial representation of the geodesics on the group of volume preserving diffeomorphisms. In material representation, the evolution is governed by a smooth vector field, the geodesic spray, on the tangent bundle of this diffeomorphism group (see Ebin and Marsden [1970], Bourguignon and Brezis [1974]). This point of view not only leads to an elegant proof of well posedness but to many other results regarding the Euler equations. This is a rare and remarkable property that, to our knowledge, appears in the conservative situation only for the incompressible non-homogeneous Euler equations (Marsden [1976]), the averaged Euler equations (Marsden, Ratiu, and Shkoller [2000], Shkoller [2000]), and the n-dimensional Camassa-Holm equations (Gay-Balmaz [2008]). Even equations that exhibit strong geometric properties, such as KdV, in general, do not have this property.

The universal Teichmüller space appears in many areas of mathematics and mathematical physics. For example, it is a special coadjoint orbit of the Bott-Virasoro group (Nag and Verjovsky [1990]) and plays an important role in the theory of Riemann surfaces, several complex variables, and quasiconformal maps (Gardiner and Lakic [2000], Lehto [1987], Nag [1988]).

The theory of the groups of diffeomorphisms of an n-dimenstional manifold endowed with a Sobolev manifold structure requires differentiability class strictly above $\frac{n}{2}+1$. It is not even clear how to define a group of diffeomorphisms at this critical index. This is reflected in the fact that a particle path flow associated to the

Euler equations in dimension at least 3, defined at every point in the reference configuration, is known only for differentiability class strictly bigger than this critical index.

In this paper we shall establish a connection between these three problems in the context of Weil-Petersson geometry on the universal Teichmüller space. The classical theory endows the universal Teichmüller space with a group structure and an infinite dimensional complex Banach manifold structure relative to which the inclusion of the Teichmüller spaces of Riemann surfaces is holomorphic. However, it is not a topological group and the formula for the Weil-Petersson metric proposed in Nag and Verjovsky [1990] is divergent and thus does not define a Riemannian metric. These basic problems were overcome in Takhtajan and Teo [2006] who endowed the universal Teichmüller space with a different complex Hilbert manifold structure in which the formula for the Weil-Petersson metric not only converges, but defines a strong metric, that is, a metric which induces the Hilbert space topology on each tangent spaces. They also show that the identity component of universal Teichmüller space is a topological group. With this manifold structure the tangent space at the identity is the space of functions on the circle of class $H^{3/2}$. Therefore, the identity component of universal Teichmüller space takes the place of diffeomorphisms of critical Sobolev class $H^{3/2}$.

We shall study this group from the point of view of manifolds of maps by identifying it with a subgroup of the quasisymmetric homeomorphisms of the circle. We shall prove that all elements of this group are of class $H^{\frac{3}{2}-\varepsilon}$ for all $\varepsilon > 0$. Then we shall use the fact that the metric is strong to show that all geodesics of the Weil-Petersson metric exist for all time, that is, we have geodesic completeness. We shall also prove that this space is Cauchy complete (something not generally implied by geodesic completeness in infinite dimensions) relative to the distance function defined by the Weil-Petersson metric.

The spatial formulation of the geodesic equations turns out to be considerably more involved than in the case of the Euler equations. We obtain a new equation, that we call the *Euler-Weil-Petersson equation*, and we show its solutions are C^0 in $H^{3/2}$ and C^1 in $H^{1/2}$. A comparison of the technical difficulties encountered in the study of the Euler equations and of the Euler-Weil-Petersson equation is in order. For the Euler equations, the main technical problem was the proof of the smoothness of the geodesic spray and the passage from material to spatial representation was easy. For the Euler-Weil-Petersson equation, the smoothness of the spray follows easily from the fact that the metric is strong, but the passage from material to spatial representation is difficult and requires composition and multiplication under the critical Sobolev exponent $3/2$. We close the paper with two applications. The first one is the proof of long time existence of special solutions called Teichons by analogy with the peakons for the Camassa-Holm equations. It turns out that these singular solutions are actually smoother than the generic geodesics. In the second application, we use again long time existence of Weil-Petersson geodesics to positively address a comment of Sharon and Mumford [2006], namely that there exists a unique geodesic between each two shapes in the plane.

Plan of the Paper. Section 2 reviews the basic facts concerning the universal Teichmüller space $T(1)$ endowed with its classical infinite dimensional complex Banach manifold structure. In particular, we recall that this manifold structure is not compatible with the natural group operation and that formula for the Weil-Petersson Riemannian metric on $T(1)$ is divergent. These difficulties are solved by endowing $T(1)$ with a new complex Hilbert manifold structure, the Takhtajan-Teo topology, that we review in Section 3. This approach allows us to define a new topological group and Hilbert manifold of homeomorphisms of the circle, that replaces the group of Sobolev diffeomorphisms in the case of the critical exponent $s = 3/2$. In particular, we show that the model space is given by Sobolev $H^{3/2}$ vector fields on the circle. In Section 4 we prove that this new Hilbert manifold is continuously embedded in the topological group of all homeomorphisms of the circle that are of Sobolev class $H^{3/2-\varepsilon}$ for all $\varepsilon > 0$. In Section 5 we exploit the strongness of the Weil-Petersson metric to show that the Hilbert manifold $T(1)$ is geodesically and Cauchy complete. The passage from the Lagrangian to the spatial formulation of geodesics is carried out in Section 6, using multiplication and composition under critical exponents in Sobolev spaces. Section 7 is concerned with particular solutions of the Euler-Weil-Petersson equation, called Teichons, by analogy with the peakons of the Camassa-Holm equations. These Teichons are shown to be particular Weil-Petersson geodesics. Finally, Section 8 considers application to imaging from the functional analytic point of view developed in the paper.

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2 The Universal Teichmüller Space

In this section we recall some basic classical facts we shall need about universal Teichmüller space, for the reader's convenience and to establish notation. A more complete exposition can be found in Ahlfors [1987], Gardiner [1987], Gardiner and Lakic [2000], Lehto [1987], and Nag [1988].

Notation and Some Important Facts. Let $\widehat{\mathbb{C}}$ be the Riemann sphere. Let the open unit disk in the complex plane be denoted by $\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$ and its exterior by $\mathbb{D}^* := \{z \in \widehat{\mathbb{C}} \mid |z| > 1\}$. We denote by d^2z the usual two dimensional Lebesgue measure on $\mathbb{C}$, that is, $d^2z := \frac{i}{2}(dz \wedge d\bar{z})$.

Consider the separable complex Banach space $L^1(\mathbb{D}^*)$ of integrable complex valued functions on $\mathbb{D}^*$. Its dual can be isometrically identified with the non separable complex Banach space $L^\infty(\mathbb{D}^*)$ of essentially bounded complex valued functions on $\mathbb{D}^*$. In the context of Teichmüller theory, the elements of $L^\infty(\mathbb{D}^*)$ are called the ***Beltrami differentials on*** $\mathbb{D}^*$.

Define the closed subspace

$$A_1(\mathbb{D}^*) = \left\{ \phi \in L^1(\mathbb{D}^*) \,\middle|\, \phi \text{ is holomorphic on } \mathbb{D}^* \right\}$$

of $L^1(\mathbb{D}^*)$. Its dual can be identified with the quotient Banach space $L^\infty(\mathbb{D}^*)/\mathcal{N}(\mathbb{D}^*)$, where

$$\mathcal{N}(\mathbb{D}^*) := \left\{ \mu \in L^\infty(\mathbb{D}^*) \;\middle|\; \int_{\mathbb{D}^*} \mu(z)\phi(z) d^2 z = 0, \, \forall \phi \in A_1(\mathbb{D}^*) \right\}$$

is the space of ***infinitesimally trivial Beltrami differentials***. There is a canonical splitting

$$L^\infty(\mathbb{D}^*) = \mathcal{N}(\mathbb{D}^*) \oplus \Omega^{-1,1}(\mathbb{D}^*), \tag{2.1}$$

where $\Omega^{-1,1}(\mathbb{D}^*)$ is the closed non separable Banach subspace of $L^\infty(\mathbb{D}^*)$ defined by

$$\Omega^{-1,1}(\mathbb{D}^*) := \left\{ \mu \in L^\infty(\mathbb{D}^*) \;\middle|\; \mu(z) = (1-|z|^2)^2\overline{\phi(z)}, \; \phi \text{ a holomorphic map on } \mathbb{D}^* \right\}.$$

This projection allows us to identify $L^\infty(\mathbb{D}^*)/\mathcal{N}(\mathbb{D}^*)$ with $\Omega^{-1,1}(\mathbb{D}^*)$, whose elements are called, by definition, ***harmonic Beltrami differentials on*** $\mathbb{D}^*$. We can write this space as

$$\Omega^{-1,1}(\mathbb{D}^*) := \left\{ \mu(z) = (1-|z|^2)^2\overline{\phi(z)} \;\middle|\; \phi \in A_\infty(\mathbb{D}^*) \right\},$$

where $A_\infty(\mathbb{D}^*)$ is the non separable complex Banach space

$$A_\infty(\mathbb{D}^*) = \left\{ \phi \text{ holomorphic in } \mathbb{D}^* \;\middle|\; \sup_{z\in\mathbb{D}^*} |\phi(z)(1-|z|^2)^2| < \infty \right\}.$$

All the results of this section remain valid when $\mathbb{D}^*$ is replaced by $\mathbb{D}$.

The Real Lie Group $\mathrm{PSU}(1,1)$**.** Recall that the biholomorphic maps of the Riemann sphere $\widehat{\mathbb{C}}$ are of the form

$$z \mapsto \frac{az+b}{cz+d} \quad \text{where} \quad a,b,c,d \in \mathbb{C}, \quad ad-bc=1.$$

The set of such maps form a group under composition that is readily checked to be isomorphic to the complex matrix Lie group

$$\mathrm{PSL}(2,\mathbb{C}) := \mathrm{SL}(2,\mathbb{C})/\{\pm\mathbb{I}\}$$

of complex dimension 3, called the ***group of Möbius transformations***. The subgroup of all biholomorphic maps of the Riemann sphere which preserve the unit disc $\mathbb{D}$ are of the form

$$z \mapsto \frac{az+b}{\bar{b}z+\bar{a}}, \quad \text{where} \quad a,b \in \mathbb{C}, \quad \text{and} \quad |a|^2-|b|^2=1.$$

This group is isomorphic to the real three dimensional Lie group

$$\mathrm{PSU}(1,1) := \mathrm{SU}(1,1)/\{\pm\mathbb{I}\}.$$

The elements of $\mathrm{PSU}(1,1)$ preserve the unit circle S^1 and are uniquely determined by their restriction to the circle S^1. Being conformal, these maps are orientation preserving on the disk. Since the circle inherits the natural boundary orientation, it follows that $\mathrm{PSU}(1,1)$ can be regarded as a subgroup of $\mathrm{Diff}_+(S^1)$, the orientation preserving C^∞ diffeomorphisms of the circle S^1.

View this way, and using the standard chart $\theta \mapsto e^{i\theta}$ of the circle, the Lie algebra of $\mathrm{PSU}(1,1)$ consists of periodic functions of the form

$$\mathfrak{psu}(1,1) = \{f_{a,b,c}(\theta) = a + b\sin\theta + c\cos\theta \mid a,b,c \in \mathbb{R}\}.$$

The Lie algebra bracket on this space of functions is minus the usual Jacobi-Lie bracket on vector fields. This Lie algebra bracket is given as follows: for Lie algebra elements $f(\theta)\partial/\partial\theta$ and $g(\theta)\partial/\partial\theta$, their bracket is

$$[f,g](\theta) = g(\theta)f'(\theta) - g'(\theta)f(\theta).$$

Quasiconformal Maps. Let $\phi : A \to \phi(A)$ be an orientation preserving homeomorphism defined on an open subset A of the complex plane. The map ϕ is said to be ***quasiconformal*** if it has all directional derivatives (in the sense of distributions) in $L^1_{\mathrm{loc}}(A)$ and if there is $\mu \in L^\infty(A)$ with $\|\mu\|_\infty < 1$ such that

$$\partial_{\bar{z}}\phi = \mu\partial_z\phi. \tag{2.2}$$

This is called the ***Beltrami equation*** with coefficient μ. If A and $\phi(A)$ have boundaries which are Jordan curves (that is, curves homeomorphic to a circle), then any quasiconformal map on A extends to an orientation preserving homeomorphism from $\mathrm{cl}(A)$ to $\mathrm{cl}(\phi(A))$ (see Theorem I.8.2 in Lehto and Virtanen [1973]).

In a similar way, an orientation preserving homeomorphism between Riemann surfaces is said to be quasiconformal if its local expressions are quasiconformal maps between open subsets of the complex plane. The only Riemann surface we will consider is the Riemann sphere $\widehat{\mathbb{C}}$.

The Universal Teichmüller Space. We recall below two equivalent models for the universal Teichmüller space, by following the presentation given in Takhtajan and Teo [2006]. See also Ahlfors [1987], Lehto [1987], and Nag [1988]. We denote by B_1^* the unit open ball in $L^\infty(\mathbb{D}^*)$.

- **Model A**. Extend every $\mu \in B_1^*$ to $\mathbb{D}$ by the reflection

$$\mu(z) = \overline{\mu\left(\frac{1}{\bar{z}}\right)}\frac{z^2}{\bar{z}^2},\ z \in \mathbb{D}.$$

Thus we get a new map, also denoted by $\mu \in L^\infty(\mathbb{C})$. We denote by $\omega_\mu : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ the unique solution of the Beltrami equation

$$\partial_{\bar{z}}\omega_\mu = \mu\partial_z\omega_\mu$$

which fixes $\pm 1, -i$. This ω_μ is obtained by applying the existence and uniqueness theorem of Ahlfors-Bers (see Ahlfors and Bers [1960]); ω_μ is a homeomorphism of $\widehat{\mathbb{C}}$ and it satisfies

$$\omega_\mu(z) = \overline{\omega_\mu\left(\frac{1}{\bar{z}}\right)}$$

due to the reflection symmetry of μ. As a result, $S^1, \mathbb{D}$ and $\mathbb{D}^*$ are invariant under ω_μ.

• **Model B**. Extend every $\mu \in B_1^*$ to be zero outside $\mathbb{D}^*$. We denote by $\omega^\mu : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ the unique solution of the Beltrami equation

$$\partial_{\bar{z}}\omega^\mu = \mu\partial_z\omega^\mu,$$

satisfying the conditions $f(0) = 0, \partial_z f(0) = 1$, and $\partial_z^2 f(0) = 0$, where f is the holomorphic mapping $f := \omega^\mu|_{\mathbb{D}}$. This ω^μ is a homeomorphism of $\widehat{\mathbb{C}}$ and is also obtained by applying the existence and uniqueness theorem of Ahlfors-Bers.

The relation between these two models is given by the following standard result.

Theorem 2.1. *For $\mu, \nu \in B_1^*$, we have the equivalence*

$$\omega_\mu|_{S^1} = \omega_\nu|_{S^1} \iff \omega^\mu|_{\mathbb{D}} = \omega^\nu|_{\mathbb{D}}.$$

See Lehto [1987], Chapter III, Theorem 1.2 for a proof. We now recall the definition of the universal Teichmüller space.

Definition 2.2. *The **universal Teichmüller** space is the quotient space:*

$$T(1) := B_1^*/\sim,$$

relative to following equivalence relation on B_1^*:

$$\mu \sim \nu \iff \omega_\mu|_{S^1} = \omega_\nu|_{S^1} \iff \omega^\mu|_{\mathbb{D}} = \omega^\nu|_{\mathbb{D}}.$$

In view of this definition $T(1)$, endowed with the quotient topology, is clearly connected. It turns out that $T(1)$ *is contractible* (Lehto [1987], Chapter III, Theorem 3.2).

The Bers Embedding and the complex Banach manifold structure. The embedding of $T(1)$ into $A_\infty(\mathbb{D})$ plays a crucial in the theory of Teichmüller spaces. We recall below the classical Bers theorem about this embedding.

Theorem 2.3. *The **Bers embedding***

$$\beta : T(1) \to A_\infty(\mathbb{D}), \ \ \beta([\mu]) := S\left(\omega^\mu|_{\mathbb{D}}\right),$$

is an injective mapping from $T(1)$ onto an open subset of $A_\infty(\mathbb{D})$. Its image contains the ball of radius 2 and is contained in the ball of radius 6. Here S denotes the Schwarzian derivative of a conformal map f, that is,

$$S(f) := \frac{\partial_z^3 f}{\partial_z f} - \frac{3}{2}\left(\frac{\partial_z^2 f}{\partial_z f}\right)^2.$$

Theorem 2.4. *There is a unique Banach manifold structure on $T(1)$ relative to which the projection map*
$$\pi : B_1^* \to T(1)^B$$
is a holomorphic submersion. Relative to this Banach manifold structure, the Bers embedding
$$\beta : T(1)^B \to A_\infty(\mathbb{D})$$
is a biholomorphic map onto its image.

It can be shown that the kernel of the tangent map $T_0\pi : L^\infty(\mathbb{D}^*) \to T_0T(1)^B$ is given by $\mathcal{N}(\mathbb{D}^*)$, so the tangent space $T_0T(1)^B$ can be identified with the Banach space $L^\infty(\mathbb{D}^*)/\mathcal{N}(\mathbb{D}^*) \simeq \Omega^{-1,1}(\mathbb{D}^*)$.

It is known that the topology of $T(1)$ is that of a metric space relative to the Teichmüller distance τ (see Lehto [1987], Chapter III, §2). Since this distance function will not be used in the sequel, we will not recall the definition. Nevertheless, it is interesting to recall that the metric space $(T(1), \tau)$ is complete. By Theorem 2.3, $T(1)$ is homeomorphic to an open subset of the Banach space $A_\infty(\mathbb{D})$, which is clearly incomplete.

Quasisymmetric Homeomorphisms of the Circle. An orientation preserving homeomorphism η of the circle S^1 is ***quasisymmetric*** if there is a constant M such that for every x and every $|t| \leq \pi/2$
$$\frac{1}{M} \leq \frac{\eta(x+t) - \eta(x)}{\eta(x) - \eta(x-t)} \leq M.$$
Note that the definition implies that $M \geq 1$. Here we identify the homeomorphisms of the circle with the strictly increasing homeomorphisms of the real line satisfying the condition $\eta(x+2\pi) = \eta(x)+2\pi$. The set of all quasisymmetric homeomorphisms of the circle is denoted by $\mathrm{QS}(S^1)$, it is a group under the composition of maps. The link with the quasiconformal mappings on the disc is given by the Beurling-Ahlfors extension theorem (see Beurling and Ahlfors [1956]).

Theorem 2.5. *An orientation preserving homeomorphism of the circle admits a quasiconformal extension to the disc if and only if it is quasisymmetric.*

Note that this extension is far from being unique. From this result, it follows that the restriction to the circle of a solution ω_μ of the Beltrami equation with coefficient $\mu \in B_1^*$ is a quasisymmetric homeomorphism of the circle. We therefore obtain that the map
$$\Phi : T(1) \longrightarrow \mathrm{QS}(S^1)_{\mathrm{fix}}, \quad [\mu] \longmapsto \omega_\mu|_{S^1} \tag{2.3}$$
is a bijection, where $\mathrm{QS}(S^1)_{\mathrm{fix}}$ denotes the subgroup of $\mathrm{QS}(S^1)$ consisting of quasisymmetric homeomorphisms fixing the points ± 1 and $-i$.

This bijection endows the group $\mathrm{QS}(S^1)_{\mathrm{fix}}$ with the structure of a complex Banach manifold by pushing forward this structure from $T(1)^B$. The resulting Banach manifold is denoted by $\mathrm{QS}(S^1)^B_{\mathrm{fix}}$. This bijection also endows the set $T(1)$ with a group structure by pulling back the group structure of $\mathrm{QS}(S^1)_{\mathrm{fix}}$. A straightforward computation shows that this group structure reads

$$[\nu]\cdot[\mu] = \left[\frac{\mu + (\nu\circ\omega_\mu)r_\mu}{1+\bar{\mu}(\nu\circ\omega_\mu)r_\mu}\right], \quad r_\mu = \frac{\overline{\partial_z\omega_\mu}}{\partial_z\omega_\mu}. \tag{2.4}$$

Relative to the complex Banach manifold structure, the right translations $R_{[\mu]}$ are biholomorphic mappings for all $[\mu]\in T(1)$. The left translations are not continuous in general, therefore $T(1)^B$ is not a topological group (see Theorem 3.3 in Lehto [1987]).

Note that $\mathrm{QS}(S^1)_{\mathrm{fix}}$ can be identified with the quotient space $\mathrm{QS}(S^1)/\,\mathrm{PSU}(1,1)$ (or $\mathrm{PSU}(1,1)\backslash\,\mathrm{QS}(S^1)$). Indeed, given $\eta\in\mathrm{QS}(S^1)$, there exists only one $\gamma\in\mathrm{PSU}(1,1)$ such that $\eta\circ\gamma$ (or $\gamma\circ\eta$) fixes the points ±1 and $-i$. Note that the projections $\mathrm{QS}(S^1)\to\mathrm{QS}(S^1)/\,\mathrm{PSU}(1,1)$ and $\mathrm{QS}(S^1)\to\mathrm{PSU}(1,1)\backslash\,\mathrm{QS}(S^1)$ are not group homomorphisms, when the quotient space is endowed with the group structure of $\mathrm{QS}(S^1)_{\mathrm{fix}}$.

The Tangent Space of $\mathrm{QS}(S^1)^B_{\mathrm{fix}}$. Recall that the tangent space to a point m of a Banach manifold M is defined as a space of equivalence of smooth curves. Two curves are said to be equivalent at m if they are tangent at this point in a chart. In general there is not a canonical realization of the tangent space. Nevertheless, in the case of manifolds of maps such a canonical realization exists.

We recall below how tangent spaces to manifolds of maps are concretely constructed (see Palais [1968] and Ebin and Marsden [1970]). If $s>\dim M/2$ then it is well-known that the set $H^s(M,N)$ of H^s-Sobolev class maps between two boundaryless compact manifolds M and N admits a smooth Hilbert manifold structure. Let us recall the basic ideas of this construction. To get a feeling of what a tangent vector at $f\in H^s(M,N)$ might be, let us take a path $t\in\,]-\varepsilon,\varepsilon[\,\mapsto f_t\in H^s(M,N)$ such that the map $f_t(m)$ is jointly smooth in $(t,m)\in\,]-\varepsilon,\varepsilon[\,\times M$. Then $t\in\,]-\varepsilon,\varepsilon[\,\mapsto f_t(m)\in N$ is a smooth path in N and hence $\partial f_t(m)/\partial t|_{t=0}$ is a tangent vector to N at the point $f_0(m)$. This suggests that a tangent vector at f is a H^s-map $U_f: M\to TN$ satisfying $U_f(m)\in T_{f(m)}N$ for every $m\in M$, that is, a vector field covering f. Hence the candidate tangent space is

$$T_fH^s(M,N) = \{U_f\in H^s(M,TN)\mid U_f(m)\in T_{f(m)}N\}.$$

Now one proceeds constructing charts for $H^s(M,N)$ with these Hilbert spaces as models, using the exponential map of some Riemannian metric on N. Once the manifold structure on $H^s(M,N)$ has been obtained, one proves the identity

$$\left(\frac{d}{dt}f_t\right)(m) = \frac{\partial}{\partial t}(f_t(m)).$$

for a smooth path $t\in\,]-\varepsilon,\varepsilon[\,\mapsto f_t\in H^s(M,N)$.

In our case, $\mathrm{QS}(S^1)^B_{\mathrm{fix}}$ is also a space of maps, as opposed to $T(1)^B$. Hence one would like to study $\mathrm{QS}(S^1)^B_{\mathrm{fix}}$ in the spirit of manifolds of maps. However, the topologies are different and to implement manifold of maps constructions one needs to use theorems in complex analysis as opposed to the standard facts in Sobolev space theory. Our goal is to obtain a concrete realization of the tangent space at $\eta := \Phi([\mu]) \in \mathrm{QS}(S^1)^B_{\mathrm{fix}}$ to the complex Banach manifold $\mathrm{QS}(S^1)^B_{\mathrm{fix}}$. Note that we already have an abstract description of this tangent space, namely, it is $T_{[\mu]}\Phi\left(T_{[\mu]}T(1)^B\right)$. However, so far we do not have any concrete realization of this complex Banach space. We will show below that it is equal to the right translate of a very concrete function space on S^1, the Zygmund space. In the process we will explicitly calculate TR_η.

Recall that the Banach manifold structure on $\mathrm{QS}(S^1)^B_{\mathrm{fix}}$ is defined by the condition that the bijection $\Phi : T(1)^B \to \mathrm{QS}(S^1)_{\mathrm{fix}}, \quad \Phi([\mu]) := \omega_\mu|_{S^1}$ is a diffeomorphism. This simply says that the manifold charts of $\mathrm{QS}(S^1)^B_{\mathrm{fix}}$ are of the form $(\varphi \circ \Phi^{-1}, \Phi^{-1}(U))$ where (φ, U) are the manifold charts of $T(1)^B$. A curve $\eta_t \in \mathrm{QS}(S^1)^B_{\mathrm{fix}}$ is smooth if it is of the form $\eta_t = \omega_{\mu(t)}|_{S^1}$, where $\mu(t)$ is a smooth curve in the open ball B_1^*. The problem of finding a concrete expression of the vector $\frac{d}{dt}\eta_t$ is equivalent to that of finding a concrete realization of the tangent spaces $T_\eta\,\mathrm{QS}(S^1)_{\mathrm{fix}}$ or a concrete expression for $T_\eta\Phi$. A first step in this direction is the following theorem. The first part is a direct consequence of Theorem 11 in Ahlfors and Bers [1960]. The expression (2.5) is the reformulation for the disk of equation (2.34) in Nag [1988]; see §1.2.11 - 1.2.12 of this book for additional information and the proof of this formula.

Theorem 2.6. *Let $\mu(t) \in B_1^*$ be a smooth curve such that $\mu(0) = 0$. Then for all $z \in S^1$, the curve $t \mapsto \omega_{\mu(t)}(z) \in S^1$ is smooth in a neighborhood of $t = 0$. The derivative at $t = 0$ is given by*

$$\left.\frac{\partial}{\partial t}\right|_{t=0} \omega_{\mu(t)}(z) = V_\nu(z),$$

where $\nu \in L^\infty(\mathbb{C})$ is $\dot{\mu}(0)$ extended to $\mathbb{C}$ by reflection, and

$$V_\nu(z) = -\frac{(z+1)(z+i)(z-1)}{\pi} \iint_{\mathbb{C}} \frac{\nu(w)}{(w+1)(w+i)(w-1)(w-z)} d^2w. \tag{2.5}$$

This theorem generalizes to the case where $\mu(0)$ is not 0. Indeed, since right translation on B_1^* is smooth, the curve $t \mapsto \mu(t) \cdot \mu(0)^{-1}$ is smooth. Here the dot denotes the group multiplication on B_1^* given in (2.4). We have

$$\left.\frac{\partial}{\partial t}\right|_{t=0} \omega_{\mu(t)}(z) = \left.\frac{\partial}{\partial t}\right|_{t=0} \omega_{\mu(t)\cdot\mu(0)^{-1}}(\omega_{\mu(0)}(z)) = V_\nu(\omega_{\mu(0)}(z)),$$

where ν is the extension of $\left.\frac{\partial}{\partial t}\right|_{t=0} \left(\mu(t) \cdot \mu(0)^{-1}\right)$ by reflection.

Here is a reformulation of these results. Let η_t be a smooth curve in $\mathrm{QS}(S^1)^B_{\mathrm{fix}}$. Then for all $z \in S^1$, the curve $\eta_t(z)$ is differentiable as a curve on S^1 and the time derivative is of the form

$$\left.\frac{\partial}{\partial s}\right|_{t=0} \eta_t(z) = V_\nu(\eta_0(z)), \tag{2.6}$$

where V_ν is a vector field on S^1 of the form (2.5).

The next theorem states that the vector field V_ν belongs to the ***Zygmund space*** on S^1 defined by

$$Z(S^1) := \big\{ u \in C^0(S^1) \,\big|\, \text{there is a } C \text{ such that} \\ |u(x+t) + u(x-t) - 2u(x)| \le C|t| \text{ for all } x, t \in S^1 \big\}.$$

Here, the continuous vector fields u on the circle are identified with continuous 2π-periodic functions on the real line. We also define the subspace

$$Z(S^1)_0 := \big\{ u \in Z(S^1) \mid u(\pm 1) = u(-i) = 0 \big\}.$$

Relative to the Zygmund norm

$$\|u\|_Z := \|u\|_\infty + \sup_{x,t\in S^1} \frac{|u(x+t) + u(x-t) - 2u(x)|}{|t|}, \tag{2.7}$$

$Z(S^1)$ is a nonseparable Banach space and $Z(S^1)_0$ a closed subspace (see Earle, Gardiner, and Lakic [2000], Gardiner and Lakic [2000]).

It is known that for all $0 < \alpha < 1$ and $s < 1$ we have the strict continuous inclusions

$$\Lambda^1(S^1) \subset Z(S^1) \subset \Lambda^\alpha(S^1), \quad \text{and} \quad Z(S^1) \subset H^s(S^1),$$

where $\Lambda^1(S^1)$ denotes the space of Lipschitz functions on the circle, $\Lambda^\alpha(S^1)$ denotes the space of α-Hölder functions on the circle, and $H^s(S^1)$ denotes the space of Sobolev class H^s functions on the circle. In terms of the Fourier series representation we have

$$H^s(S^1) = \left\{ u(x) = \sum_{n\in\mathbb{Z}} u_n e^{inx} \,\middle|\, u_{-n} = \overline{u}_n \text{ and } \sum_{n\in\mathbb{Z}} |n|^{2s} |u_n|^2 < \infty \right\}.$$

These continuous inclusions are particular cases of embedding theorems for spaces of Besov-Triebel-Lizorkin type, see Triebel [1983] and Runst and Sickel [1996] for example.

Theorem 2.7. *For all $\nu \in L^\infty(\mathbb{D}^*)$, we have $V_\nu \in Z(S^1)_0$. Moreover the linear map*

$$[\nu] \in L^\infty(\mathbb{D}^*)/N(\mathbb{D}^*) \simeq \Omega^{-1,1}(\mathbb{D}^*) \mapsto V_\nu \in Z(S^1)_0 \tag{2.8}$$

is an isomorphism of Banach spaces, where $L^\infty(\mathbb{D}^*)/N(\mathbb{D}^*)$ is endowed with the quotient norm and $Z(S^1)_0$ is endowed with the cross-ratio norm, a norm equivalent to the Zygmund norm defined in (2.7).

See Chapter 3 in Gardiner and Lakic [2000], Gardiner and Harvey [2002], and Gardiner and Sullivan [1992] for the definition of the cross-ratio norm and proofs of this theorem. On $Z(S^1)$ the cross-ratio norm is actually a seminorm whose kernel is given by $\mathfrak{psu}(1,1)$.

Using the preceding discussion, it follows that for two smooth curves η_t^1 and η_t^2 in $\mathrm{QS}(S^1)^B_{\text{fix}}$ such that $\eta_0^2 = \eta_0^1 = \eta$, they are tangent at the point η with respect to the Banach manifold structure if and only if

$$\frac{\partial}{\partial t}(\eta_t^1(z)) = \frac{\partial}{\partial t}(\eta_t^2(z)),\ \forall\, z \in S^1.$$

By formula (2.6) and Theorem 2.7 it follows that a realization of the tangent space to $\mathrm{QS}(S^1)^B_{\text{fix}}$ at η is

$$T_\eta \,\mathrm{QS}(S^1)^B_{\text{fix}} = Z(S^1)_0 \circ \eta.$$

Remark. It is worth noting here the difference between the usual theory of H^s-diffeomorphism groups and $\mathrm{QS}(S^1)^B_{\text{fix}}$. The formula above completely determines $T_\eta \,\mathrm{QS}(S^1)^B_{\text{fix}}$. The same formula is also valid for the group of H^s-diffeomorphisms $\mathrm{Diff}^s(M)$, namely, $T_\eta \,\mathrm{Diff}^s(M) = \mathfrak{X}^s(M) \circ \eta$, where $\mathfrak{X}^s(M)$ is the Hilbert space of H^s-vector fields on M. However, for the diffeomorphism group one can go further and say that $T_\eta \,\mathrm{Diff}^s(M) = \mathfrak{X}^s(M) \circ \eta = \{U_\eta \in H^s(M, TM) \mid U_\eta(m) \in T_{\eta(m)}M\}$, a realization that is not available for $T_\eta \,\mathrm{QS}(S^1)^B_{\text{fix}}$. ♦

With respect to this realization, the tangent map to Φ at $[0]$ is

$$T_{[0]}\Phi : L^\infty(\mathbb{D}^*)/\mathcal{N}(\mathbb{D}^*) \to T_{id}\,\mathrm{QS}_{\text{fix}}(S^1) = Z(S^1)_0, \quad T_{[0]}\Phi([\nu]) = V_\nu,$$

and the derivative of a smooth curve η_t is the vector in $T_{\eta_t}\,\mathrm{QS}(S^1)^B_{\text{fix}}$ given by

$$\left(\frac{d}{dt}\eta_t\right)(z) = \frac{\partial}{\partial t}(\eta_t(z)).$$

Of course, the previous equality holds only if the curve η_t is known to be smooth with respect to the Banach manifold structure. It is not sufficient that for all z, the curve $t \mapsto \eta_t(z)$ is smooth.

Right translation by γ is given by $R_\gamma(\xi) = \xi \circ \gamma$ and it is known to be a smooth map. Using the preceding results we have, for $u_\eta \in T_\eta\,\mathrm{QS}(S^1)^B_{\text{fix}}$,

$$\begin{aligned}(TR_\gamma(u_\eta))(z) &= \left(\left.\frac{d}{dt}\right|_{t=0} R_\gamma(\eta_t)\right)(z) = \left.\frac{d}{dt}\right|_{t=0} \frac{\partial}{\partial t}(\eta_t(\gamma(z))) \\ &= \left(\left.\frac{d}{dt}\right|_{t=0} \eta_t\right)(\gamma(z)) = u_\eta(\gamma(z)).\end{aligned}$$

Thus, we have $TR_\gamma(u_\eta) = u_\eta \circ \gamma$.

The isomorphism between the tangent space $T_{id}\,\mathrm{QS}(S^1)^B_{\text{fix}}$ and the Banach model space $A_\infty(\mathbb{D})$ of $T(1)^B$ is given by taking the tangent map at the identity to the map $\beta \circ \Phi^{-1} : \mathrm{QS}(S^1)^B_{\text{fix}} \to T(1)^B \to A_\infty(\mathbb{D})$, where β denotes the Bers embedding and Φ is the diffeomorphism defined in (2.3). It was proved Teo [2004], Theorem 2.11, that we have

$$T_{id}\left(\beta \circ \Phi^{-1}\right) : \sum_{n\in\mathbb{Z}} u_n e^{inx} \in T_{id}\,\mathrm{QS}(S^1)^B_{\text{fix}} \longmapsto i\sum_{n\geq 2}(n^3 - n)u_n z^{n-2} \in A_\infty(\mathbb{D}).$$

Using the complex Banach manifold structure of $\mathrm{QS}(S^1)^B_{\text{fix}}$ thought of as a real manifold, it is possible to endow the whole group $\mathrm{QS}(S^1)$ with a real Banach manifold structure, by declaring that the bijection

$$\Psi : \mathrm{QS}(S^1) \longrightarrow \mathrm{PSU}(1,1) \times \mathrm{QS}(S^1)^B_{\text{fix}}, \tag{2.9}$$

defined by the condition

$$\Psi(\eta) = (\hat{\eta}, \eta_0) \Longleftrightarrow \eta = \hat{\eta} \circ \eta_0, \tag{2.10}$$

is a diffeomorphism. The group $\mathrm{QS}(S^1)$ endowed with this Banach manifold structure is denoted by $\mathrm{QS}(S^1)^B$. Its properties are given in the theorem below. We will use the following lemma which shows that the choice of an other subgroup fixing three points does not change the Banach manifold structure on $\mathrm{QS}(S^1)$.

Lemma 2.8. *Let $\mathrm{QS}(S^1)_1$ be a subgroup of $\mathrm{QS}(S^1)$ consisting of quasisymmetric homeomorphisms fixing three points. Then $\mathrm{QS}(S^1)_1$ can be endowed with a Banach manifold structure in the same way as $\mathrm{QS}(S^1)_{\text{fix}}$. The bijection*

$$\mathrm{PSU}(1,1) \times \mathrm{QS}(S^1)^B_{\text{fix}} \to \mathrm{PSU}(1,1) \times \mathrm{QS}(S^1)^B_1$$

defined by

$$(\gamma_0, \eta_0) \mapsto (\gamma_1, \eta_1), \quad \textit{such that} \quad \gamma_0 \circ \eta_0 = \gamma_1 \circ \eta_1,$$

is a smooth diffeomorphism.

Proof. By definition of the Banach manifold structures on $\mathrm{QS}(S^1)_{\text{fix}}$ and $\mathrm{QS}(S^1)_1$, we obtain that the map

$$\mathrm{QS}(S^1)_{\text{fix}} \to \mathrm{QS}(S^1)_1, \quad \eta \mapsto \gamma \circ \eta, \tag{2.11}$$

where γ is the unique Möbius transformation such that $\gamma \circ \eta \in \mathrm{QS}(S^1)_1$, is a diffeomorphism.

We now show that the map $(\gamma_0, \eta_0) \mapsto (\gamma_1, \eta_1)$ is smooth. Since $\gamma_0 \circ \eta_0 = \gamma_1 \circ \eta_1$, we have $\eta_1 = (\gamma_1^{-1} \circ \gamma_2) \circ \eta_0$. Thus, using (2.11), we obtain that the map $(\gamma_0, \eta_0) \mapsto \eta_1$ is smooth. In order to show that $(\gamma_0, \eta_0) \mapsto \gamma_1$ is smooth, we consider the map

$$F : \mathrm{PSU}(1,1) \times \mathrm{QS}(S^1)_{\text{fix}} \times \mathrm{PSU}(1,1) \to \mathbb{R}^3, \quad F(\gamma, \eta, \xi) = (\gamma(\eta(x_i)) - \xi(x_i)),$$

where $(x_i), i = 1,2,3$, denote the fixed points associated to the group $\mathrm{QS}(S^1)_1$. Note that F is smooth and that $F(\gamma_0, \eta_0, \xi) = 0$ if and only if $\xi = \gamma_1$. The partial derivative of F with respect to the variable ξ and in the direction $V \in T_\xi \mathrm{PSU}(1,1)$ is computed to be

$$\frac{\partial F}{\partial \xi}(\gamma_0, \eta_0, \xi)(V) = -(V(x_i)),$$

therefore the linear map $\frac{\partial F}{\partial \xi}(\gamma_0, \eta_0, \xi) : T_\xi \mathrm{PSU}(1,1) \to \mathbb{R}^3$ is an isomorphism, and by the implicit function theorem, the correspondence $(\gamma_0, \eta_0) \mapsto \eta_1$ is smooth. ■

As a consequence of this lemma, we obtain that the identification of $\mathrm{QS}(S^1)$ with $\mathrm{PSU}(1,1)\times\mathrm{QS}(S^1)^B_{\text{fix}}$ or $\mathrm{PSU}(1,1)\times\mathrm{QS}(S^1)^B_1$ gives the same Banach manifold structure.

Theorem 2.9. *The tangent space at the identity to the real Banach manifold $\mathrm{QS}(S^1)^B$ is the Zygmund space $Z(S^1)$. The group $\mathrm{QS}(S^1)^B$ is not a topological group but the right translations are smooth; $\mathrm{QS}(S^1)^B$ contains the subgroup $\mathrm{QS}(S^1)^B_{\text{fix}}$ as a closed submanifold of codimension 3.*

Proof. From the definition of the Banach manifold structure we have

$$T_{id}\,\mathrm{QS}(S^1)^B = \mathfrak{psu}(1,1)\oplus T_{id}\,\mathrm{QS}(S^1)^B_{\text{fix}}.$$

Recall that $T_{id}\,\mathrm{QS}(S^1)^B_{\text{fix}}$ consists of vector fields in $Z(S^1)$ such that $u(\pm1)=u(-i)=0$. Therefore by a adding any element of $\mathfrak{psu}(1,1)$ we recover the whole space $Z(S^1)$. The set $\mathrm{QS}(S^1)^B_{\text{fix}}$ is clearly a subgroup of $\mathrm{QS}(S^1)^B$. It is also a closed submanifold since it is identified with the closed submanifold $\{e\}\times\mathrm{QS}(S^1)^B_{\text{fix}}$ in $\mathrm{PSU}(1,1)\times\mathrm{QS}(S^1)^B_{\text{fix}}$.

We now show that the right translations $R_\xi:\mathrm{QS}(S^1)^B\to\mathrm{QS}(S^1)^B, \eta\mapsto\eta\circ\xi$ are smooth for each fixed $\xi\in\mathrm{QS}(S^1)$. We first prove this for $\xi\in\mathrm{QS}(S^1)_{\text{fix}}$. Using the diffeomorphism (2.9), the correspondence $\eta\mapsto\eta\circ\xi$ reads $(\hat\eta,\eta_0)\mapsto(\hat\eta,\eta_0\circ\xi)$. This is a smooth map since right translations are known to be smooth on $\mathrm{QS}(S^1)^B_{\text{fix}}$. We now consider the case $\xi\in\mathrm{PSU}(1,1)$. Note that for any ξ we can define the subgroup $\mathrm{QS}(S^1)_\xi$ consisting of quasisymmetric homeomorphisms of the circle fixing the three points $\xi^{-1}(-1),\xi^{-1}(1)$ and $\xi^{-1}(-i)$. As a map from $\mathrm{PSU}(1,1)\times\mathrm{QS}(S^1)^B_{\text{fix}}$ to $\mathrm{PSU}(1,1)\times\mathrm{QS}(S^1)^B_\xi$, the correspondence $\eta\mapsto\eta\circ\xi$ reads $(\hat\eta,\eta_0)\mapsto(\hat\eta\circ\xi,\xi^{-1}\circ\eta_0\circ\xi)$. By the preceding lemma it suffices to show that this last correspondence is smooth. For the first factor this is trivial since right translation on $\mathrm{PSU}(1,1)$ is smooth. Hence it suffices to show that the map

$$\eta_0\in\mathrm{QS}(S^1)^B_{\text{fix}}\mapsto\xi^{-1}\circ\eta_0\circ\xi\in\mathrm{QS}(S^1)^B_\xi$$

is smooth. This follows from the fact this map is induced by the smooth map

$$\mu\in L^\infty(\mathbb{D}^*)\mapsto(\mu\circ\xi)\frac{\overline{\partial_z\xi}}{\partial_z\xi}\in L^\infty(\mathbb{D}^*).$$

To show that R_ξ is a smooth map for all $\xi\in\mathrm{QS}(S^1)$ it suffices to write $\xi=\hat\xi\circ\xi_0$ with $(\hat\xi,\xi_0)\in\mathrm{PSU}(1,1)\times\mathrm{QS}(S^1)_{\text{fix}}$. We then have $R_\xi=R_{\xi_0}\circ R_{\hat\xi}$, which is a composition of smooth maps by the preceding arguments. ∎

As in the case of $\mathrm{QS}(S^1)^B_{\text{fix}}$, we can show that $T_\eta\,\mathrm{QS}(S^1)^B=Z(S^1)\circ\eta$. Let η_t be a smooth curve in $\mathrm{QS}(S^1)^B$. By definition, see (2.10), we can write $\eta_t=\hat\eta_t\circ(\eta_t)_0$, where $\hat\eta_t$ is a smooth curve in $\mathrm{PSU}(1,1)$ and $(\eta_t)_0$ is a smooth curve in $\mathrm{QS}(S^1)^B_{\text{fix}}$. Therefore, we obtain that for all $z\in S^1$ the curve $\eta_t(z)$ is smooth. A direct computation shows for a smooth curve η_t we have

$$\left.\frac{\partial}{\partial t}\right|_{t=0}\eta_t(z)=V(\eta_0(z)),$$

where $V \in Z(S^1)$. This shows that a canonical realization of the tangent space $T_\eta\,\mathrm{QS}(S^1)^B$ is given by $Z(S^1) \circ \eta$, and that the tangent map to right translation is $TR_\gamma(u_\eta) = u_\eta \circ \gamma$.

A system of neighborhoods of the identity in $\mathrm{QS}(S^1)^B$ is given by $\{U(\varepsilon) \mid \varepsilon > 0\}$, where $U(\varepsilon)$ consists of all quasisymmetric homeomorphisms $\eta \in \mathrm{QS}(S^1)$ such that

$$\frac{1}{1+\varepsilon} \leq \frac{\eta(x+t)-\eta(x)}{\eta(x)-\eta(x-t)} \leq 1+\varepsilon \quad \text{and} \quad \sup_{x\in S^1} \left\{|\eta(x)-x|, |\eta^{-1}(x)-x|\right\} < \varepsilon.$$

At other points, the neighborhoods are obtained by right translation.

Relation with Diffeomorphism Groups. We have the following chain of subgroup inclusions

$$\mathrm{Diff}_+(S^1) \subset \mathrm{Diff}^s_+(S^1) \subset \mathrm{Diff}^{C^1}_+(S^1) \subset \mathrm{QS}(S^1), \tag{2.12}$$

for all $s > 3/2$. The differential properties are the following. The group $\mathrm{Diff}_+(S^1)$ is endowed with the C^∞ Fréchet manifold structure. The group $\mathrm{Diff}^s_+(S^1)$ denotes the group of all orientation preserving Sobolev class H^s diffeomorphisms of the circle. It is endowed with the Sobolev H^s Hilbert manifold structure; this is possible for all $s > 3/2$. The group $\mathrm{Diff}^{C^1}_+(S^1)$ is endowed with the C^1 Banach manifold structure. All these manifold structures are real and not complex. Recall that $\mathrm{Diff}_+(S^1)$ is a Fréchet Lie group (see Kriegl and Michor [1997]), $\mathrm{Diff}^s_+(S^1)$ and $\mathrm{Diff}^{C^1}_+(S^1)$ are topological groups with smooth right translations (Ebin and Marsden [1970] and Omori [1997]), $\mathrm{QS}(S^1)$ has smooth right translations but is not a topological group (Theorem 2.9). Note also that all the inclusions are smooth. The two first inclusions on the left have dense ranges and the last inclusion on the right is neither dense nor closed. The closure of $\mathrm{Diff}^{C^1}_+(S^1)$ in $\mathrm{QS}(S^1)$ determines the topological group $\mathrm{S}(S^1)$ of symmetric homeomorphisms of the circle. We refer to Gardiner and Sullivan [1992], Gardiner and Lakic [2000] for the definiton and the properties of $\mathrm{S}(S^1)$.

The same differential properties hold for the corresponding subgroups fixing the points ± 1 and $-i$. We get the inclusions

$$\mathrm{Diff}_+(S^1)_{\mathrm{fix}} \subset \mathrm{Diff}^s_+(S^1)_{\mathrm{fix}} \subset \mathrm{Diff}^{C^1}_+(S^1)_{\mathrm{fix}} \subset \mathrm{QS}(S^1)_{\mathrm{fix}}, \tag{2.13}$$

for all $s > 3/2$. These subgroups have the additional property to be also complex manifolds. The tangent spaces at the identity to these subgroups, denoted by

$$C^\infty(S^1)_0 \subset H^s(S^1)_0 \subset C^1(S^1)_0 \subset Z(S^1)_0, \tag{2.14}$$

are obtained by imposing the conditions $u(\pm 1) = u(-i) = 0$ on the elements of the tangent spaces at the identity to the corresponding large groups in (2.12).

Note that an other realization of the tangent spaces at the identity to these subgroups is given by imposing the conditions $u_{-1} = u_0 = u_1 = 0$ on the Fourier coefficients. This corresponds to thinking of these subgroups as quotient spaces of the corresponding groups by the Möbius group $\mathrm{PSU}(1,1)$; therefore the vector fields are taken modulo $\mathfrak{psu}(1,1)$. The tangent spaces at the identity in this interpretation are denoted by

$$\mathfrak{h}^\infty \subset \mathfrak{h}^s \subset \mathfrak{h}^{C^1} \subset \mathfrak{h}^{QS}.$$

More on the Complex Structure. Recall that the complex structure of $T(1)^B$ is the trivial one induced by the assumption that the projection $B_1^* \to T(1)^B$ is a holomorphic submersion. Therefore the complex structure operator is simply multiplication by i. We denote by J the complex structure operator induced on the Banach manifold $\mathrm{QS}(S^1)^B_{\text{fix}}$. The following Theorem due to Nag and Verjovsky [1990] shows that J takes a remarkably simple expression in terms of Fourier series.

Theorem 2.10. *The complex structure on the Banach manifold* $\mathrm{QS}(S^1)^B_{\text{fix}}$ *is the right-invariant structure given at the identity by the map* $\mathrm{J} : \mathfrak{h}^{QS} \to \mathfrak{h}^{QS}$ *defined by*

$$\mathrm{J}\left(\sum_{n\neq -1,0,1} u_n e^{inx}\right) = i \sum_{n\neq -1,0,1} \operatorname{sgn}(n) u_n e^{inx}.$$

The operator J is in fact the Hilbert transform on the circle

$$\mathrm{J}(u)(x) = \frac{1}{2\pi}\int_{S^1} u(s)\cot\left(\frac{s-x}{2}\right) ds.$$

The Weil-Petersson Metric. The ***Weil-Petersson Hermitian metric*** on $T(1)^B$ is the right-invariant metric whose value at the identity [0] is given by

$$\langle \mu, \nu\rangle := \int_{\mathbb{D}^*} \mu(z)\overline{\nu(z)}\frac{4}{(1-|z|^2)^2} d^2 z. \tag{2.15}$$

This metric was introduced by Nag and Verjovsky [1990] as a direct generalization of the Weil-Petersson metric on the finite dimensional Teichmüller spaces. As remarked by these authors, this Hermitian metric does not make sense for all $\mu, \nu \in \Omega^{-1,1}(\mathbb{D}^*)$. Indeed, it converges only for μ, ν in the Hilbert space $H^{-1,1}(\mathbb{D}^*) \subset \Omega^{-1,1}(\mathbb{D}^*)$ defined by

$$\begin{aligned} H^{-1,1}(\mathbb{D}^*) &= \left\{ \mu \in \Omega^{-1,1}(\mathbb{D}^*) \;\middle|\; \int_{\mathbb{D}^*} |\mu(z)|^2 \frac{1}{(1-|z|^2)^2} d^2 z < \infty \right\} \\ &= \left\{ \mu(z) = (1-|z|^2)^2\overline{\phi(z)} \;\middle|\; \phi \in A_2(\mathbb{D}^*)\right\}, \end{aligned}$$

where

$$A_2(\mathbb{D}^*) = \left\{ \phi \text{ holomorphic in } \mathbb{D}^* \;\middle|\; \int_{\mathbb{D}^*} |\phi(z)|^2 (1-|z|^2)^2 d^2 z < \infty \right\}.$$

Using the identifications

$$T_{id}\,\mathrm{QS}(S^1)^B_{\text{fix}} = \mathfrak{h}^{QS} \xleftarrow{T_{[0]}\Phi} T_{[0]}T(1)^B \xrightarrow{T_{[0]}\beta} A_\infty(\mathbb{D}),$$

the metric on $\mathfrak{h}^{QS}$ has the expression

$$h_{id}(u,v) = \frac{\pi}{2}\sum_{n=2}^{\infty} n(n^2-1) u_n \overline{v_n} \tag{2.16}$$

and one can see that it converges only for $u, v \in \mathfrak{h}^{\frac{3}{2}}$, the subspace of Sobolev class $H^{\frac{3}{2}}$ real vector fields on the circle with $u_0 = u_1 = 0$, a subspace strictly included in $\mathfrak{h}^{QS}$. Therefore we obtain that

$$T_{[0]}\Phi\left(H^{-1,1}(\mathbb{D})\right) = \mathfrak{h}^{\frac{3}{2}}. \tag{2.17}$$

On the other hand, the metric on $A_\infty(\mathbb{D})$ is given by the expression

$$h_{id}(\phi, \psi) = \frac{1}{4}\int_{\mathbb{D}} \phi(z)\overline{\psi(z)}(1-|z|^2)^2 d^2 z, \tag{2.18}$$

which converges only for ϕ, ψ in the strict subspace $A_2(\mathbb{D})$ of $A_\infty(\mathbb{D})$. We obtain hence that

$$T_{[0]}\beta\left(H^{-1,1}(\mathbb{D})\right) = A_2(\mathbb{D}). \tag{2.19}$$

The corresponding Weil-Petersson Riemannian metric on $\mathrm{QS}(S^1)^B_{\text{fix}}$, considered as a real manifold, is given by

$$g_{id}(u,v) = \frac{\pi}{2}\,\mathrm{Re}\left(\sum_{n=2}^{\infty} n(n^2-1)u_n\overline{v_n}\right) = \frac{\pi}{4}\sum_{n\neq -1,0,1} |n|(n^2-1)u_n\overline{v_n}. \tag{2.20}$$

The imaginary part of the Hermitian metric is the symplectic two-form

$$\omega_{id}(u,v) = \frac{\pi}{2}\,\mathrm{Im}\left(\sum_{n=2}^{\infty} n(n^2-1)u_n\overline{v_n}\right) = -\frac{i\pi}{4}\sum_{n\neq -1,0,1} n(n^2-1)u_n\overline{v_n}. \tag{2.21}$$

As it was the case for the Weil-Petersson Hermitian metric, g and ω are only defined on the subspace $\mathfrak{h}^{\frac{3}{2}}$ of $T_{id}\,\mathrm{QS}(S^1)^B_{\text{fix}} = \mathfrak{h}^{QS} \cong Z(S^1)_0$.

> *In order to solve the convergence problem in* (2.15) , Takhtajan and Teo [2006] *introduce a new complex Hilbert manifold structure on* $T(1)$*, such that the natural inner product is given by the Weil-Petersson Hermitian metric.*

3 Takhtajan–Teo Theory

The Complex Hilbert Manifold Structure on $T(1)$. The first step is to define an $A_2(\mathbb{D})$-Hilbert manifold structure on the set $A_\infty(\mathbb{D})$. This is done by defining a collection of charts on the set $A_\infty(\mathbb{D})$ as follows. Each point $\phi \in A_\infty(\mathbb{D})$ is declared to have a chart domain given by $\phi + A_2(\mathbb{D})$ and the chart map is given by mapping the point $\phi + \psi \in \phi + A_2(\mathbb{D})$ to $\psi \in A_2(\mathbb{D})$. This construction makes, in an elementary way, $A_\infty(\mathbb{D})$ into a $A_2(\mathbb{D})$-Hilbert manifold. Of course this sort of construction can be done for any vector space X that has a subspace Y that is, in its own right, a Hilbert manifold. Clearly, the resulting Hilbert manifold $A_\infty(\mathbb{D})$ modeled on $A_2(\mathbb{D})$ is not connected. Indeed, to each $[\phi] \in A_\infty(\mathbb{D})/A_2(\mathbb{D})$ we associate the connected component $\phi + A_2(\mathbb{D})$. This map is a bijection between the quotient space $A_\infty(\mathbb{D})/A_2(\mathbb{D})$ and the set of all uncountably many connected components of $A_\infty(\mathbb{D})$, the latter being viewed as a $A_2(\mathbb{D})$-Hilbert manifold.

In a similar way, Takhtajan and Teo [2006] define a $A_2(\mathbb{D})$-Hilbert manifold structure on the set $T(1)$, following the same approach as above for A_∞. The details of the construction are, however, considerably more technical. For details see Takhtajan and Teo [2006], Theorem 3.10. The resulting Hilbert manifold $T(1)$, modeled on $A_2(\mathbb{D})$, is also not connected, but rather has uncountably many components. The sets $T(1)$ and $A_\infty(\mathbb{D})$ endowed with the $A_2(\mathbb{D})$-Hilbert manifold structures are denoted by $T(1)^H$ and $A_\infty(\mathbb{D})^H$.

Normally one could view such a construction with some skepticism; however, as we shall see, Takhtajan and Teo [2006] prove some profound things about this type of construction. The main results of Takhtajan and Teo [2006] regarding $T(1)^H$ are as stated in the following theorem and in the properties below.

Theorem 3.1. *The Bers embedding*

$$\beta : T(1)^H \to A_\infty(\mathbb{D})^H, \;\; \beta([\mu]) := S\left(\omega^\mu|_{\mathbb{D}}\right),$$

is a biholomorphic mapping from $T(1)^H$ onto an open subset of $A_\infty(\mathbb{D})^H$. In particular, the tangent map $T_{[0]}\beta$ induces an isomorphism $H^{-1,1}(\mathbb{D}^*) \simeq A_2(\mathbb{D})$. The connected components of $T(1)^H$ are the inverse images of the connected components of $\beta\left(T(1)^H\right)$.

The connected component of $[0] \in T(1)^H$ is denoted by $T(1)^H_\circ$. The manifolds $T(1)^H$ and $T(1)^H_\circ$ have the following very attractive properties:

- The Weil-Petersson metric is strong on $T(1)^H$, since it is the natural Hermitian inner product on the tangent spaces. As a consequence, g and ω are also strong with respect to the Hilbert manifold structure. Moreover, $(T(1)^H, \mathrm{J}, \omega)$ is a strong Kähler-Einstein Hilbert manifold with negative constant Ricci curvature and negative sectional and holomorphic sectional curvatures.
- The right translations on $T(1)^H$ are biholomorphic mappings.
- The connected component $T(1)^H_\circ$ is a topological group.
- The connected component $T(1)^H_\circ$ is the closure in $T(1)^H$ of the subgroup $\Phi^{-1}\left(\mathrm{Diff}_+(S^1)_{\mathrm{fix}}\right)$; see (2.3) for the definition of Φ.

Recall that a Riemannian metric g on a Hilbert manifold Q is said to be strong if for all $q \in Q$, the associated flat map $T_qQ \to T_qQ^*, \;\; v_q \mapsto g_q(v_q, \cdot)$ is an isomorphism. Equivalently, g is strong if for all q, the inner product g_q induces the Hilbert space topology on T_qQ.

The Hilbert Manifold $\mathfrak{T}$. As in the Banach case, the bijection $\Phi : T(1) \to \mathrm{QS}(S^1)_{\mathrm{fix}}$ defined in (2.3) endows the group $\mathrm{QS}(S^1)_{\mathrm{fix}}$ with the structure of a complex Hilbert manifold denoted by $\mathrm{QS}(S^1)^H_{\mathrm{fix}}$. In addition, by (2.15), (2.16) and (2.17), the Banach space isomorphism

$$T_{[0]}\Phi : T_{[0]}\left(T(1)^B\right) = \Omega^{-1,1}(\mathbb{D}^*) \to T_{id}\,\mathrm{QS}(S^1)^B_{\mathrm{fix}} = Z(S^1)_0, \quad \nu \mapsto V_\nu$$

restricts to a Hilbert space isomorphism

$$T_{[0]}\Phi : T_{[0]}\left(T(1)^H\right) = H^{-1,1}(\mathbb{D}^*) \to T_{id}\,\mathrm{QS}(S^1)^H_{\mathrm{fix}} = H^{\frac{3}{2}}(S^1)_0.$$

Consider a smooth curve $t \mapsto \eta_t \in \mathrm{QS}(S^1)^H_{\mathrm{fix}}$, then η_t is smooth as a curve in $\mathrm{QS}(S^1)^B_{\mathrm{fix}}$. Therefore, by Theorem 2.6, the curve $t \mapsto \eta_t(z) \in S^1$ is smooth for all $z \in S^1$. As in the case of $\mathrm{QS}(S^1)^B_{\mathrm{fix}}$ we obtain that

$$\left(\frac{d}{dt}\eta_t\bigg|_{t=0}\right)(z) = \frac{\partial}{\partial t}\bigg|_{t=0}(\eta_t(z)) = V(\eta_0(z)), \tag{3.1}$$

where $V \in H^{\frac{3}{2}}(S^1)_0$. Therefore, a canonical realization of the tangent space at η is given by

$$T_\eta\,\mathrm{QS}(S^1)^H_{\mathrm{fix}} = H^{\frac{3}{2}}(S^1)_0 \circ \eta.$$

As before, we denote by $\mathrm{QS}(S^1)^H_{\mathrm{fix},\circ}$ the connected component of the identity of the Hilbert manifold $\mathrm{QS}(S^1)^H_{\mathrm{fix}}$. The key object of interest to us in this paper is the Hilbert manifold $\mathfrak{T}$ defined to be

$$\mathfrak{T} := \Phi(T(1)^H_\circ) = \mathrm{QS}(S^1)^H_{\mathrm{fix},\circ}.$$

Let us now summarize some of the key properties of $\mathfrak{T}$.

- $\mathfrak{T}$ is a complex Hilbert manifold and a connected topological group
- Right translations on $\mathfrak{T}$ are biholomorphic maps $R_\eta : \mathfrak{T} \to \mathfrak{T}$
- The group $\mathrm{Diff}_+(S^1)_{\mathrm{fix}}$ of smooth orientation preserving diffeomorphisms of S^1 that fix the three points $\pm 1, -i$ is dense in $\mathfrak{T}$
- The tangent space at the identity, $T_e\mathfrak{T}$ is equal to $H^{\frac{3}{2}}(S^1)_0 \simeq \mathfrak{h}^{\frac{3}{2}}$.
- The Weil-Petersson metric on $\mathfrak{T}$ is a right invariant strong and smooth metric that makes $\mathfrak{T}$ into a Kähler-Einstein manifold with negative constant Ricci curvature and negative sectional and holomorphic sectional curvatures.

As we have seen, the tangent space at the identity is the space of $H^{\frac{3}{2}}$ vector fields on S^1 vanishing at $\pm 1, -i$. This is a Hilbert space of real vector fields, identified with real valued functions that also has a complex structure given in Theorem 2.10. Since the derivative of right translation by $\eta \in \mathfrak{T}$ from $T_{id}\mathfrak{T} \to T_\eta\mathfrak{T}$ is a Hilbert space isomorphism, the topology on the other tangent space is equivalent to the $H^{\frac{3}{2}}$ topology as well. Thus, the topology induced on the tangent spaces of $\mathfrak{T}$ by the Weil-Petersson Riemannian metric (given by equation (2.20)) is also the $H^{\frac{3}{2}}$ topology, by strongness of the metric.

Moreover, again by strongness and smoothness of the Weil-Petersson Riemannian metric, its exponential maps form coordinate charts. Since these charts map into the space of $H^{\frac{3}{2}}$ functions, we can say that $\mathfrak{T}$ is an $H^{\frac{3}{2}}$ Hilbert manifold. Since, as explained, the topology on $\mathfrak{T}$ is the $H^{\frac{3}{2}}$ topology and since $\mathrm{Diff}_{+,\mathrm{fix}}(S^1)$ is dense in $\mathfrak{T}$, we can think of $\mathfrak{T}$ as the $H^{\frac{3}{2}}$-completion of $\mathrm{Diff}_{+,\mathrm{fix}}(S^1)$.

The Hilbert Manifold $\mathrm{QS}(S^1)^H_\circ$**.** Using the Hilbert manifold structure of $\mathrm{QS}(S^1)^H_{\mathrm{fix}}$, it is possible to endow the whole group $\mathrm{QS}(S^1)$ with a real Hilbert manifold structure, by declaring that the bijection

$$\Psi : \mathrm{QS}(S^1) \longrightarrow \mathrm{PSU}(1,1) \times \mathrm{QS}(S^1)^H_{\mathrm{fix}}, \tag{3.2}$$

defined by the condition $\Psi(\eta) = (\hat{\eta}, \eta_0) \iff \eta = \hat{\eta} \circ \eta_0$, is a diffeomorphism. The group $\mathrm{QS}(S^1)$ endowed with this Hilbert manifold structure is denoted by $\mathrm{QS}(S^1)^H$ and its connected component of the identity by $\mathrm{QS}(S^1)^H_\circ$.

In the same way as in Theorems 2.9 we prove the following result.

Theorem 3.2. *The tangent space at the identity to the real Hilbert manifold* $\mathrm{QS}(S^1)^H$ *is the space* $H^{\frac{3}{2}}(S^1)$ *of* $H^{\frac{3}{2}}$ *real valued functions on* S^1*. The manifold* $\mathrm{QS}(S^1)^H$ *has smooth right translations and contains the subgroup* $\mathrm{QS}(S^1)^H_{\mathrm{fix}}$ *as a closed submanifold of codimension* 3*. Moreover, the connected component of the identity* $\mathrm{QS}(S^1)^H_\circ$ *inherits from* $\mathrm{QS}(S^1)^H_{\mathrm{fix},\circ}$ *the property of being a topological group.*

Proof. It suffices to show that the composition and inversion are continuous maps on $\mathrm{QS}(S^1)^H_\circ$. Using the diffeomorphism

$$\eta \in \mathrm{QS}(S^1)^H_\circ \mapsto (\hat{\eta}, \eta_0) \in \mathrm{PSU}(1,1) \times \mathrm{QS}(S^1)^H_{\mathrm{fix},\circ},$$

the composition reads

$$(\hat{\eta}, \eta_0), (\hat{\xi}, \xi_0) \to (\hat{\eta} \circ a, b \circ \xi_0), \tag{3.3}$$

where $a \in \mathrm{PSU}(1,1)$ and $b \in \mathrm{QS}(S^1)^H_{\mathrm{fix},\circ}$ are uniquely determined by the condition $a \circ b = \eta_0 \circ \hat{\xi}$. Since a is determined by the condition $a(x_i) = \eta_0(\hat{\xi}(x_i)), x_i = -1, 1, -i$, the map $(\eta_0, \hat{\xi}) \mapsto a$ is clearly continuous. Now we have $b = a^{-1} \circ \eta_0 \circ \hat{\xi}$. The map $(a, \eta_0, \hat{\xi}) \mapsto b$ is continuous since, in terms of Beltrami coefficients it reads

$$(a, \mu, \hat{\xi}) \mapsto (\mu \circ \hat{\xi}) \frac{\overline{\partial_z \hat{\xi}}}{\partial_z \hat{\xi}}.$$

By combining these observations with the fact that composition on $\mathrm{QS}(S^1)^H_{\mathrm{fix},\circ}$ and $\mathrm{PSU}(1,1)$ is continuous, we obtain that the map (3.3) is continuous. The inversion reads

$$(\hat{\eta}, \eta_0) \mapsto \left(\widehat{\eta^{-1}}, (\eta^{-1})_0\right),$$

thus, we have $\eta_0^{-1} \circ \hat{\eta}^{-1} = \widehat{\eta^{-1}} \circ (\eta^{-1})_0$. As before, $\widehat{\eta^{-1}}$ and $(\eta^{-1})_0$ depends continuously on $(\eta_0^{-1}, \hat{\eta}^{-1})$. Using that inversion on $\mathrm{QS}(S^1)^H_{\mathrm{fix},\circ}$ and $\mathrm{PSU}(1,1)$ is continuous, we obtain the result. ■

As before, we can show that, for a smooth curve in $\mathrm{QS}(S^1)^H$, the curve $\eta_t(z)$ is smooth in S^1 for all $z \in S^1$. A canonical realization of the tangent space at η is $H^{\frac{3}{2}}(S^1) \circ \eta$.

From the preceding Theorem, the group $\mathrm{QS}(S^1)^H_\circ$ inherits from $\mathfrak{T}$ the following properties

- Elements $\eta \in \mathrm{QS}(S^1)^H_\circ$ are symmetric homeomorphisms $\eta : S^1 \to S^1$
- $\mathrm{QS}(S^1)^H_\circ$ is a real Hilbert manifold and a connected topological group
- Right translations on $\mathrm{QS}(S^1)^H_\circ$ are smooth diffeomorphisms $R_\eta : \mathrm{QS}(S^1)^H_\circ \to \mathrm{QS}(S^1)^H_\circ$
- The group $\mathrm{Diff}_+(S^1)$ of smooth orientation preserving diffeomorphisms of S^1 is dense in $\mathrm{QS}(S^1)^H_\circ$
- The tangent space at the identity, $T_e\, \mathrm{QS}(S^1)^H_\circ$ is equal to $H^{\frac{3}{2}}(S^1)$.

We summarize in the diagram below the various spaces that appear at the tangent space level. The maps $T_{[0]}\beta$ and $T_{[0]}\Phi$ are complex Banach space isomorphisms. All other arrows are continuous inclusions. The two horizontal arrows on the right are continuous inclusions whose images are codimension three subspaces. The vertical arrows have ranges that are neither closed nor dense.

$$\begin{tikzcd}
A_\infty(\mathbb{D}) & \Omega^{-1,1}(\mathbb{D}^*) \arrow[l, "T_{[0]}\beta"'] \arrow[r, "T_{[0]}\Phi"] & Z(S^1)_0 \arrow[r] & Z(S^1) \\
A_2(\mathbb{D}) \arrow[u] & H^{-1,1}(\mathbb{D}^*) \arrow[u] \arrow[l, "T_{[0]}\beta"'] \arrow[r, "T_{[0]}\Phi"] & H^{\frac{3}{2}}(S^1)_0 \arrow[u] \arrow[r] & H^{\frac{3}{2}}(S^1) \arrow[u]
\end{tikzcd}$$

Below is the corresponding diagram at manifold level. The maps β and Φ are diffeomorphisms relative to the indicated complex Banach and complex Hilbert manifold structures. The two images of β are open in the indicated Banach spaces. The two horizontal arrows on the right are codimension three embeddings with closed range. All spaces in this diagram are connected. The four vertical arrows are smooth inclusions whose inverses from their ranges are discontinuous; the first three vertical arrows are also holomorphic maps.

$$\begin{tikzcd}
A_\infty(\mathbb{D}) \supset \beta(T(1)^B) & T(1)^B \arrow[l, "\beta"'] \arrow[r, "\Phi"] & \mathrm{QS}(S^1)^B_{\mathrm{fix}} \arrow[r] & \mathrm{QS}(S^1)^B \\
A_2(\mathbb{D}) \supset \beta(T(1)^H_\circ) \arrow[u] & T(1)^H_\circ \arrow[u] \arrow[l, "\beta"'] \arrow[r, "\Phi"] & \mathrm{QS}(S^1)^H_{\mathrm{fix},\circ} \arrow[u] \arrow[r] & \mathrm{QS}(S^1)^H_\circ \arrow[u]
\end{tikzcd}$$

4 Regularity in $\mathfrak{T}$ and $\mathbf{QS}(S^1)^H_\circ$

As we have seen before, $\mathrm{QS}(S^1)^H_\circ$ is a Hilbert manifold and a topological group with smooth right translation, modeled on the Hilbert space $H^{\frac{3}{2}}(S^1)$. It contains the Fréchet Lie group $\mathrm{Diff}_+(S^1)$ as a dense subgroup.

Recall that for $s > 3/2$, the group $\mathrm{Diff}^s_+(S^1)$ has exactly the same properties, namely, it is a smooth Hilbert manifold and a topological group with smooth right translation, modeled on the Hilbert space $H^s(S^1)$. It contains the Fréchet Lie group $\mathrm{Diff}_+(S^1)$ as a dense subgroup. Also, it is known that if η is a C^1 diffeomorphism of S^1 and if η is of class $H^s, s > 3/2$, then its inverse η^{-1} is automatically of class H^s. Therefore, the set defined by

$$\mathrm{Diff}^s_+(S^1) := \mathrm{Diff}^{C^1}_+(S^1) \cap H^s(S^1, S^1)$$

consists of Sobolev class H^s diffeomorphisms of S^1. Moreover it is a group and an open subset of the Hilbert manifold $H^s(S^1, S^1)$.

The Flow of a $H^{3/2}$ Vector Field. A similar definition is not possible in the critical case $s = 3/2$, since $H^{\frac{3}{2}}$ functions on S^1 are not C^1 in general. It is therefore natural to ask if there exists a group, denoted by $\mathrm{Diff}^{\frac{3}{2}}_+(S^1)$, which replaces the group $\mathrm{Diff}^s_+(S^1)$ in the limiting case $s = 3/2$. The discussion above suggests that one can define $\mathrm{Diff}^{\frac{3}{2}}_+(S^1) := \mathrm{QS}(S^1)^H_\circ$. Note that in the case of $\mathrm{Diff}^s_+(S^1)$ with $s > 3/2$, the regularity of the group elements (the diffeomorphisms) and that of the model space (the vector fields) is the same, namely, the Sobolev H^s regularity. Since $\mathrm{QS}(S^1)^H_\circ$ is modeled on $H^{\frac{3}{2}}(S^1)$ it is natural to ask whether or not the group elements are in $H^{\frac{3}{2}}$. The answer depends on examining the flows of $H^{\frac{3}{2}}(S^1)$-vector fields. Since $H^{\frac{3}{2}}(S^1) \subset Z(S^1)$, by Reimann [1976], $H^{\frac{3}{2}}(S^1)$-vector fields have quasisymmetric flows; in particular, there is existence and uniqueness of integral curves. Unfortunately the flow need not be in $H^{\frac{3}{2}}(S^1)$, as shown by the following Theorem of Figalli [2009].

Theorem 4.1. *Let $u \in C^0\left([0,T], H^{\frac{3}{2}}(S^1)\right)$ and let $\eta(t,x)$ be the solution of the ODE*

$$\begin{cases} \dfrac{d}{dt}\eta(t,x) = u(t,\eta(t,x)) \\ \eta(0,x) = x. \end{cases} \tag{4.1}$$

Then $t \mapsto \eta(t,\cdot)$ is in $L^\infty\left([0,T], W^{1+r,p}(S^1)\right)$ for all $0 < r < 1/2$ and $1 \le p < 1/r$.

Consider the vector field v on S^1 given by

$$v(x) := \left(\int_0^x \int_y^{1/2} \frac{1}{s\sqrt{|\log(s)|}\log|\log(s)|} ds dy\right)\varphi(x),$$

where φ is a smooth function such that

$$0 \le \varphi \le 1, \quad \varphi(x) = 1 \text{ for } x \in [0,1/4], \quad \varphi(x) = 0 \text{ for } x \in [1/2,1].$$

Then, $v \in H^{\frac{3}{2}}(S^1)$ but its flow is neither Lipschitz nor in $W^{1+r,1/r}(S^1,S^1)$ for all $0 < r < 1$. In particular the flow of v is not in $H^{\frac{3}{2}}(S^1,S^1)$.

We now interpret the result of this theorem in terms of the Sobolev embedding. Recall that for all $s, r \geq 0$ and $p, q \geq 1$ such that $s \geq r$ and $s - 1/p \geq r - 1/q$ we have the continuous inclusion

$$W^{s,p}(S^1) \subset W^{r,q}(S^1).$$

Given a pair $(p, s) \in [1, \infty[\times [0, \infty[$, we consider the domain $D_{(p,s)}$ in the plane defined by

$$D_{(p,s)} := \left\{ (q, r) \in [1, \infty[\times [0, \infty[\,\middle|\, s \geq r \text{ and } s - \frac{1}{p} \geq r - \frac{1}{q} \right\},$$

and the curve

$$L_{(p,s)} := \big\{ (q, t(q)) \in [1, \infty[\times [0, \infty[\,\big|\, t(q) = \max\{ r \mid (q, r) \in D_{(p,s)} \} \big\} \subset D_{(p,s)},$$

as in Figure 4.1.

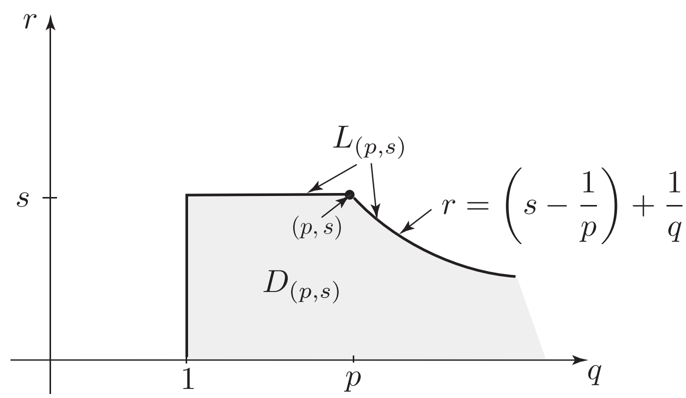

Figure 4.1: The graph in (q, r)-space of the Sobolev embedding $W^{s,p}(S^1) \subset W^{r,q}(S^1)$. If $u \in W^{s,p}(S^1)$, then $u \in W^{r,q}(S^1)$ for all $(q, r) \in D_{(p,s)}$ including the boundary $L_{(p,s)}$. The hyperbola part of $L_{(p,s)}$ has horizontal asymptote $r = s - \frac{1}{p}$.

The interpretation of this graph is the following. Given $u \in W^{s,p}(S^1)$, we have $u \in W^{r,q}(S^1)$ for all $(q, r) \in D_{(p,s)}$. In particular, this also holds if $(q, r) \in L_{(p,s)}$.

Now consider a given time dependent vector field $t \mapsto u(t, \cdot) \in C^0\left([0, T], H^{\frac{3}{2}}(S^1)\right)$ with flow $t \mapsto \eta(t, \cdot)$. Then from the above theorem and discussion, for all $t \in [0, T]$ we have

$$t \mapsto \eta(t, \cdot) \in L^\infty\left([0, T], W^{r,q}(S^1, S^1)\right), \text{ for all } (q, r) \in D_{(2,3/2)} \setminus L_{(2,3/2)}.$$

This figure illustrates that the result in Theorem 4.1 is sharp.

In particular, the flow map generated by the time dependent vector field $u \in C^0\left([0, T], H^{\frac{3}{2}}(S^1)\right)$ is in $L^\infty\left([0, T], H^{\frac{3}{2}-\varepsilon}(S^1, S^1)\right)$ for all $\varepsilon > 0$. This corresponds to the vertical interval $\{q = 2, 0 \leq r < 3/2\}$; see Figure 4.2. The counterexample in the theorem shows that the flow is not in the hyperbola part of $L_{(2,3/2)}$ and hence, in general, $\eta(t, \cdot)$ is not in $H^{\frac{3}{2}}(S^1, S^1)$.

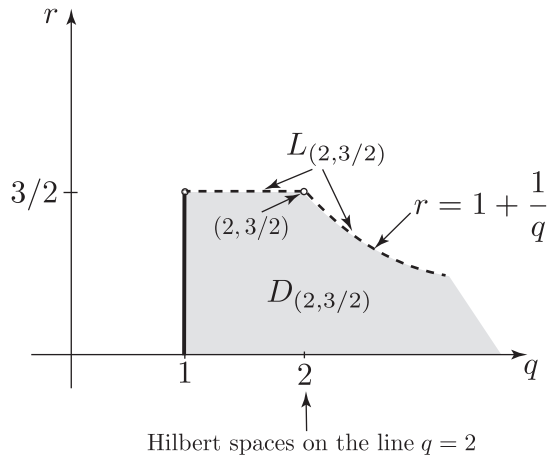

Figure 4.2: The graph in (q, r)-space of the regularity of the flow of an $H^{\frac{3}{2}}$ vector field on S^1. The flow $t \mapsto \eta(t, \cdot)$ of $u \in C^0\left([0,T], H^{\frac{3}{2}}(S^1)\right)$ is in $L^\infty\left([0,T], W^{r,q}(S^1, S^1)\right)$, for all $(q, r) \in D_{(2,3/2)} \setminus L_{(2,3/2)}$. The hyperbola part of $L_{(2,3/2)}$ has horizontal asymptote $r = 1$. The flow is not in the hyperbola part of $L_{(2,3/2)}$.

Regularity. We now apply this result to the study of the regularity of the elements in $\mathrm{QS}(S^1)^H_\circ$. Consider an element $\eta \in \mathrm{QS}(S^1)^H_\circ$. By connectedness, we can consider a smooth curve $\eta(t)$ such that $\eta(0) = id$ and $\eta(1) = \eta$. This defines a continuous curve $u(t) := TR_{\eta(t)^{-1}}(\dot{\eta}(t)) \in T_{id}\,\mathrm{QS}(S^1)^H_\circ = H^{\frac{3}{2}}(S^1)$ whose flow is given by η. Using the first equality in (3.1) and the previous theorem, we obtain that η and η^{-1} are in the Sobolev class $H^{\frac{3}{2}-\varepsilon}$ for all $\varepsilon > 0$. Thus, we have

$$\mathrm{QS}(S^1)^H_\circ \subset H^{\frac{3}{2}-\varepsilon}(S^1, S^1), \text{ for all } \varepsilon > 0.$$

The next theorem, shows that this inclusion is continuous.

Theorem 4.2. *Let $(\eta_n)_{n\in\mathbb{N}}$ be a sequence in $\mathrm{QS}(S^1)^H_\circ$ such that $\eta_n \to id$ relative to the Hilbert manifold topology of $\mathrm{QS}(S^1)^H_\circ$. Then*

$$\eta_n \to id \quad \textit{and} \quad \eta_n^{-1} \to id \quad \textit{in } H^{\frac{3}{2}-\varepsilon}, \textit{ for all } \varepsilon > 0.$$

Proof. By Theorem 3.2, it suffices to prove this in $\mathfrak{T} = \mathrm{QS}(S^1)^H_{\mathrm{fix},\circ}$. Let $\eta_n(t)$ be a smooth curve in $\mathfrak{T}$ such that $\eta_n(0) = id$ and $\eta_n(1) = \eta_n$. For n sufficiently large, η_n lies in a local chart U around id, and we can choose $\eta_n(t)$ to be locally a straight line. Thus, $\eta_n(t)$ can be seen as a sequence in $C^k([0,1], \mathfrak{T}), k \geq 1$, converging to the constant curve id in $C^k([0,1], \mathfrak{T}), k \geq 1$. Define $u_n(t) := \dot{\eta}_n(t) \circ \eta_n(t)^{-1} \in H^{\frac{3}{2}}(S^1)$. Note that $\dot{\eta}_n(t)$ converges to 0 in $C^{k-1}([0,1], \mathfrak{T})$. By Lemma 4.3 below, the curve $t \mapsto u_n(t)$ is continuous and the sequence $u_n(t)$ converge to the constant curve 0 in $C^0([0,1], \mathfrak{T})$. Since the Banach manifold structure of $\mathrm{QS}(S^1)^B$ is weaker than the Hilbert manifold structure, $\eta_n(t)$ also converges to id as a curve in $C^k\left([0,1], \mathrm{QS}(S^1)^B\right), k \geq 1$. This implies that for $\varepsilon > 0$, we have

$$\sup_{t\in[0,1]} \sup_{x\in S^1} |\eta_n(t)(x) - id| < \varepsilon,$$

for n sufficiently large. Thus, for all t, $\eta_n(t)$ converges to id in $C^0(S^1)$ and in particular in $L^p(S^1)$, for all p.

By Figalli [2009], we know that the flow of the time dependent vector field $u \in C^0([0,1], H^{\frac{3}{2}}(S^1))$ is in $H^{\frac{3}{2}-\varepsilon}(S^1, S^1)$ for all $\varepsilon > 0$. Moreover, by putting together all the estimations done in Figalli [2009], we obtain, for all $\varepsilon > 0$, the inequality

$$\|\eta(t)'\|_{H^{\frac{1}{2}-\varepsilon}} \leq A\|u'\|^B_{L^\infty\left([0,T],H^{\frac{1}{2}}\right)} \left(1 + \|\eta\|^C_{L^\infty([0,T],L^p)} + \|u'\|^D_{L^\infty\left([0,T],H^{\frac{1}{2}}\right)}\right),$$

where all constants $0 < A, B, C, D < \infty$ and $1 < p < \infty$ depend on ε.

Thus, if $\eta_n(t)$ converges to id in $C^0(S^1)$ and $u_n(t)$ converges to 0 in $H^{\frac{3}{2}}$, then, for all t, $\eta_n(t)$ converges to id in $H^{\frac{3}{2}-\varepsilon}(S^1)$, for all $\varepsilon > 0$. For $t = 1$, this implies that the sequence $(\eta_n)_{n\in\mathbb{N}}$ converges to id in $H^{\frac{3}{2}-\varepsilon}$ for all $\varepsilon > 0$. ∎

Lemma 4.3. *Let G be a Hilbert manifold and a topological group whose right-translations R_g are smooth. Suppose that G carries a strong and G-invariant Riemannian metric γ. Then the map*

$$(g, \xi_h) \in G \times TG \longmapsto TR_g(\xi_h) \in TG \tag{4.2}$$

is continuous. In particular, the map

$$(g, \xi) \in G \times \mathfrak{g} \longmapsto TR_g(\xi) \in TG$$

is an homeomorphism.

This result applies to $G = \mathfrak{T}$, endowed with the Weil-Petersson metric.

Proof. Since γ is a strong Riemannian metric, we can consider the associated exponential map $\exp : \mathcal{O} \subset TG \to G \times G$, where $\mathcal{O}$ is a neighborhood of the zero section. By restricting $\mathcal{O}$, exp is a diffeomorphism onto its image. Using this diffeomorphism, and the fact that exp is G-invariant, the map (4.2) reads $(g, h, f) \mapsto (hg, fg)$, which is continuous, since G is a topological group. ∎

Define the space

$$H^{\frac{1}{2}-}(S^1) := \bigcap_{s<1/2} H^s(S^1)$$

endowed with the least fine topology such that each inclusion $H^{\frac{1}{2}-}(S^1) \to H^s(S^1)$, $s < 1/2$ is continuous. This makes $H^{\frac{1}{2}-}(S^1)$ into a Fréchet space for which a fundamental system of neighborhood of 0 is given by

$$\mathcal{V} = \{U(r,R) \mid r < 1/2, R > 0\}, \quad U(r,R) := \left\{u \in H^{1/2-}(S^1) \;\middle|\; \|u\|_{H^r} < R\right\}.$$

Note that we have the continuous and strict inclusion

$$H^{1/2}(S^1) \subsetneq H^{\frac{1}{2}-}(S^1).$$

For example, the function $u : S^1 \to \mathbb{R}$ such that

$$u(x) = \begin{cases} 1 & \text{on } [0, 1/2[\\ 0 & \text{on } [1/2, 1[\end{cases},$$

is in $H^{\frac{1}{2}-}(S^1)$ but is not in $H^{\frac{1}{2}}(S^1)$, see Lemma 3, §2.3.1 in Runst and Sickel [1996].

We now recall from Gay-Balmaz, Marsden, and Ratiu [2009] two results about multiplication and composition in Sobolev spaces below the critical exponents 1/2 and 3/2 respectively.

Lemma 4.4. *For all $\varepsilon_1, \varepsilon_2 > 0$ such that $\max\{2\varepsilon_1 + \varepsilon_2, 2\varepsilon_2 + \varepsilon_1\} \leq 1/2$, pointwise multiplication on $C^\infty(S^1)$ extends to a continuous bilinear map*

$$H^{\frac{1}{2}-\varepsilon_1}(S^1) \times H^{\frac{1}{2}-\varepsilon_2}(S^1) \to H^{\frac{1}{2}-\varepsilon}(S^1),$$

for all ε such that $\max\{2\varepsilon_1 + \varepsilon_2, 2\varepsilon_2 + \varepsilon_1\} \leq \varepsilon \leq 1/2$. In particular, pointwise multiplication in $C^\infty(S^1)$ extends to a continuous bilinear map on $H^{\frac{1}{2}-}(S^1)$.

It is known that this result is false for $H^{\frac{1}{2}}(S^1)$, see Theorem 1 of §4.3.2 and Theorem 1 of §2.2.4 in Runst and Sickel [1996].

Lemma 4.5. *Let $u_n \in H^{\frac{3}{2}-\varepsilon}(S^1)$ for all $\varepsilon > 0$ and let η, η_n be orientation preserving homeomorphisms of S^1 such that $\eta, \eta^{-1}, \eta_n, \eta_n^{-1} \in H^{\frac{3}{2}-\varepsilon}(S^1, S^1)$ for all $\varepsilon > 0$. If $u_n \to u$, $\eta_n \to \eta$, and $\eta_n^{-1} \to \eta^{-1}$ in $H^{\frac{3}{2}-\varepsilon}$ for all $\varepsilon > 0$, then*

$$u_n \circ \eta_n \to u \circ \eta, \quad \text{in } H^{\frac{3}{2}-\varepsilon}(S^1, S^1) \text{ for all } \varepsilon > 0.$$

Consider the set $\mathrm{Diff}_+^{\frac{3}{2}-}(S^1)$ of all orientation preserving homeomorphism η of S^1 such that $\eta, \eta^{-1} \in H^{\frac{3}{2}-\varepsilon}(S^1, S^1)$ for all $\varepsilon > 0$. As a consequence of the previous lemma we obtain the following result.

Corollary 4.6. $\mathrm{Diff}_+^{\frac{3}{2}-}(S^1)$ *is a group under composition.*

As for $H^{\frac{1}{2}-}(S^1)$, there are elements η in $\mathrm{Diff}_+^{\frac{3}{2}-}(S^1)$ which are not in $H^{\frac{3}{2}}(S^1, S^1)$. The composition of two element in $H^{\frac{3}{2}}(S^1, S^1)$ is not in $H^{\frac{3}{2}}(S^1, S^1)$ in general, but only in $H^{\frac{3}{2}-\varepsilon}(S^1, S^1)$ for all $\varepsilon > 0$.

We endow the group $\mathrm{Diff}_+^{\frac{3}{2}-}(S^1)$ with the topology given by the system of neighborhoods $\mathcal{V}(\xi) = \{U(\xi, r, R) \mid r < 3/2, R > 0\}$ of an element ξ, where

$$U(\xi, r, R) := \left\{ \eta \in \mathcal{D}^{3/2-}(S^1) \,\middle|\, \|\eta - \xi\|_{H^r} < R,\ \|\eta^{-1} - \xi^{-1}\|_{H^r} < R \right\}.$$

Using Lemma 4.5 we immediately obtain the following result.

Corollary 4.7. *With this topology,* $\mathrm{Diff}_+^{\frac{3}{2}-}(S^1)$ *is a topological group.*

Using Theorem 4.2, we obtain the following result.

Theorem 4.8. *The natural inclusion*

$$\mathrm{QS}(S^1)^H_\circ \subset \mathrm{Diff}_+^{\frac{3}{2}-}(S^1)$$

is continuous.

Lie Algebra Structure. It is known that for all $s > 3/2$, the group $\mathrm{Diff}_+^s(S^1)$ is a Hilbert manifold modeled on the Hilbert space $\mathfrak{X}^s(S^1)$ of all Sobolev class H^s vector fields on S^1. More precisely, $\mathrm{Diff}_+^s(S^1)$ is an open set in the Hilbert manifold $H^s(S^1, S^1)$. It is also known that, with respect to the Hilbert manifold structure, $\mathrm{Diff}_+^s(S^1)$ is a topological group with smooth right translations. For $s > 3/2$, the subgroup $\mathrm{Diff}_+^s(S^1)_{\mathrm{fix}}$ of $\mathrm{Diff}_+^s(S^1)$ consisting of diffeomorphisms fixing the points ± 1 and $-i$, is a closed codimension 3 Hilbert submanifold. The tangent space at the identity is

$$\mathfrak{g}^s := H^s(S^1)_0 = \{u \in \mathfrak{X}^s(S^1) \mid u(\pm 1) = u(-i) = 0\}.$$

We now describe the Lie algebra bracket on the formal Lie algebra $\mathfrak{g}^{\frac{3}{2}}$ of $\mathfrak{T}$. Of course since $\mathfrak{T}$ is not a literal Lie group, we must proceed formally, as one does with diffeomorphism groups. The bracket should be the same as that on the Lie algebra of smooth vector fields, which, as we saw earlier, is, on Lie algebra elements $f(\theta)\partial/\partial\theta$ and $g(\theta)\partial/\partial\theta$,

$$[f, g](\theta) = g(\theta)f'(\theta) - g'(\theta)f(\theta)$$

We assert that this makes sense for $f, g \in \mathfrak{g}^{\frac{3}{2}}$, producing a vector field in $H^{\frac{1}{2}}$. To see this, we use a standard result about pointwise multiplication in H^r, namely the following (see for instance, Theorem 9.13 in Palais [1968]):

Theorem 4.9. *If $t > \frac{n}{2}$ and $r \geq -t$, pointwise multiplication extends from*

$$C^\infty(M, \mathbb{R}) \times C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R})$$

to a continuous bilinear map

$$H^t(M, \mathbb{R}) \times H^r(M, \mathbb{R}) \to H^{\min\{r,t\}}(M, \mathbb{R}). \tag{4.3}$$

In particular, for $|r| \leq t$, $H^r(M, \mathbb{R})$ is an $H^t(M, \mathbb{R})$-module.

5 Completeness of the Universal Teichmüller Space

We denote by g the Weil-Petersson Riemannian metric on $\mathfrak{T}$. Recall that this makes $\mathfrak{T}$ into a strong Riemannian Hilbert manifold that is also a topological group with smooth right translations and that g is right invariant. The results below apply equally well to $T(1)^H$; recall that $\mathfrak{T}$ is diffeomorphic to the connected component of the identity of $T(1)^H$.

By invariance of the metric under right translations and the fact that the metric is strong, we obtain the following result.

Proposition 5.1. *The Riemannian manifold $(\mathfrak{T}, g)$ is geodesically complete. The geodesic spray of the metric g^H is smooth and there is an associated Levi-Civita connection. The curvature and Ricci tensors are bounded operators, the sectional curvature is negative, and thus, there are no conjugate points.*

Proof. Recall that the geodesic spray of the metric is defined by the condition

$$\mathbf{i}_{\mathcal{Z}}\Omega = \mathbf{d}E, \tag{5.1}$$

where $E : T\mathfrak{T} \to \mathbb{R}$ is the right invariant kinetic energy of the Weil-Petersson metric g^H and where Ω is the strong symplectic form on $T\mathfrak{T}$ that is obtained by the Hilbert bundle isomorphism of $T\mathfrak{T}$ to $T^*\mathfrak{T}$ associated to the (strong) Weil-Petersson metric. Since Ω is a strong symplectic form and E is a smooth function, equation (5.1) defines a smooth vector field $\mathcal{Z}$ on the tangent bundle $T\mathfrak{T}$ of the Hilbert manifold $\mathfrak{T}$. Therefore we obtain the local existence and uniqueness of a smooth geodesics with initial velocity $u \in \mathfrak{g}^{\frac{3}{2}}$.

To show that the integral curves of $\mathcal{Z}$ are globally defined, we use the following standard argument. Indeed, one has

Lemma 5.2. *Let G be a smooth strong Riemannian Hilbert manifold that is also a topological group with smooth right translation. Then geodesics on G exist for all time.*

Proof. We begin by showing that for any $v \in T_{id}G$ the geodesic γ_v satisfying $\dot{\gamma}_v(0) = v$ exists for all time. Since $\gamma_{\lambda v}(t/\lambda) = \gamma_v(t)$ for any $\lambda > 0$ and t in the domain of definition of γ_v, it suffices to prove the statement for $\|v\|$ small.

The local existence and uniqueness theorem for $\mathcal{Z}$ implies that there is a ball $B \subset T_{id}G$ of initial velocities, $\tau > 0$, and a smooth map $F : B\times]-\tau,\tau[\to TG$, such that the curve $v_u :]-\tau,\tau[\to TG$, defined by $v_u(t) := F(u,t)$ is an integral curve of $\mathcal{Z}$ with initial condition $v_u(0) = u$. In particular, all integral curves of $\mathcal{Z}$ starting in B exist for a time at least $|t| < \tau$. Now extend the integral curve $v(t)$ starting at $v_0 \in B$ to a maximal time interval $[0,T[$. By conservation of energy, we have $\|v(t)\| = \|v_0\|$ for all $t \in [0,T[$. Since the topology of the manifold coincides with the topology defined by the geodesic distance, $v(t)$ lies in the ball obtained by right translation of B to the point $x(t)$, where $x(t) \in G$ is the base point of $v(t)$. Since right translation is a diffeomorphism and the vector field $\mathcal{Z}$ is smooth and right invariant, the time of existence for initial conditions in this translated ball is at least τ. Thus $v(t)$ can be extended beyond T which proves that it exists for all $t \in \mathbb{R}$.

For geodesics starting at points whose base is not the identity, one uses right translation to reduce to this cases. ■

The existence of the associated Levi-Civita connection ∇ follows as in the finite dimensional case, since the metric is strong. Consider the curvature tensor

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

In local charts, for $u, v, w \in \mathfrak{g}^{\frac{3}{2}}$, the curvature tensor has the expression

$$\begin{aligned} R(x)(u,v,w) = \mathbf{D}\Gamma(x)(v)\cdot(u,z) - \mathbf{D}\Gamma(x)(u)\cdot(v,z) \\ - \Gamma(x)(u,\Gamma(x)(v,w)) - \Gamma(x)(v,\Gamma(x)(u,w)), \end{aligned}$$

where $\Gamma(x)$ denotes the local Christoffel map associated to the geodesic spray, that is, we have $\mathcal{Z}(x,u) = (x,u,u,\Gamma(x)(u,u))$ in local coordinates. By strongness of the metric, we know that $\mathbf{D}$ is the Fréchet derivative with respect to the strong topology $H^{\frac{3}{2}}$. Therefore, for all $\eta \in \mathfrak{T}$, $R(\eta)$ is a bounded trilinear operator on $T_\eta T(1)^H$ with respect to the Hilbert topology. Note that this argument does not work in the case of a weak Riemannian metric with smooth geodesic spray. Indeed, when the metric is weak, the Fréchet derivative $\mathbf{D}$ is taken relative to a weaker topology and an explicit proof is needed to show that the curvature operator is bounded (see Misiolek [1993] for an example where this situation occurs; this is the case for the L^2 weak metric appearing in the study of the Euler equations).

Theorem 7.11 in Takhtajan and Teo [2006], states that the Ricci tensor Ric is well-defined as the trace of the curvature tensor and that

$$\mathrm{Ric} = -\frac{13}{12\pi} g.$$

It follows that the Ricci tensor is a bounded bilinear operator.

Given a WP geodesic γ, we consider the Jacobi equation along γ

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma}X - R(\gamma)(X,\dot\gamma)\dot\gamma = 0. \tag{5.2}$$

Since the curvature operator is bounded, using the same arguments as in Proposition 3.10 in Misiolek [1993], we obtain that for $u, v \in \mathfrak{g}^{\frac{3}{2}}$ there exists a unique vector field $X(t)$ along γ that is a solution of (5.2) satisfying $X(0) = u$ and $\nabla_{\dot\gamma}X(0) = v$. Since the sectional curvature is negative (see Theorem 7.14 in Takhtajan and Teo [2006]), there are no conjugate points. ∎

Corollary 5.3. *The Riemannian exponential map* $\exp_\eta : T_\eta T(1)^H_\circ \to T(1)^H_\circ$ *at any point* $\eta \in T(1)^H_\circ$ *is a covering map. The Riemannian manifold* $(\mathfrak{T}, g)$ *is a complete metric space relative to the distance function induced by the strong Weil-Petersson Riemannian metric.*

Proof. The first statement consists of the extension of the Cartan-Hadamard Theorem to the infinite dimensional case (Lang [1999], Chapter IX, §3, Theorem 3.8).

Recall that any strong Riemannian metric on a Hilbert manifold induces a distance function. For general infinite dimensional strong Riemannian Hilbert manifolds, Cauchy completeness implies geodesic completeness but the converse is, in general, false (Lang [1999], Chapter VIII, §6). However, if the sectional curvature is ≤ 0 and the manifold is connected, Corollary 3.9 in Chapter IX, §3 of Lang [1999] proves that geodesic completeness and Cauchy completeness are equivalent. ∎

6 The Euler-Weil-Petersson Equation

In this section we shall study the geodesic equation in Eulerian (spatial) representation.

The Associated Euler-Poincaré Equation. Using the general theory of Euler-Poincaré reduction, we know that for γ_t a geodesic of the Weil-Petersson metric on $\mathfrak{T}$, the curve

$$u_t := \dot{\gamma}_t \circ \gamma_t^{-1} \in T_{id}\mathfrak{T} = \mathfrak{g}^{\frac{3}{2}}$$

should *formally* be a solution of the Euler-Poincaré equation

$$\frac{d}{dt}\frac{\delta \ell}{\delta u_t} = -\operatorname{ad}^*_{u_t} \frac{\delta \ell}{\delta u_t},$$

where $\ell : \mathfrak{g}^{\frac{3}{2}} \to \mathbb{R}$, $l(u) = \frac{1}{2} g_{id}(u,u)$ is the Weil-Petersson Lagrangian. We call this equation the ***Euler-Weil-Petersson equation*** (or the ***EWP equation*** for short). For the moment we shall proceed somewhat formally and then after this, will turn to the rigorous interpretation of the equation. Below we derive the EWP equation explicitly and examine several representations of it.

The solution of the Euler-Poincaré equation formally does not depend on the choice of the duality pairing. Consider the L^2 pairing on $L^2(S^1)$

$$\langle u, v\rangle = \int_{S^1} uv. \tag{6.1}$$

This pairing extends to a pairing between $H^s(S^1)$ and $H^{-s}(S^1)$ for any $s \in \mathbb{R}$. Therefore the dual space to the closed subspace $\mathfrak{g}^s = \{u \in H^s(S^1) \mid u(\pm 1) = u(-i) = 0\}$ of $H^s(S^1), s > 1/2$, with respect to the pairing (6.1), is $H^{-s}(S^1)/N$, where $N = \{v \in H^{-s}(S^1) \mid \langle v, u\rangle = 0, \text{ for all } u \in \mathfrak{g}^s\}$.

With respect the pairing (6.1), the Weil-Petersson Lagrangian reads

$$\ell(u) = \frac{1}{2}\langle Q_{\mathrm{op}}(u), u\rangle, \tag{6.2}$$

where $Q_{\mathrm{op}} : H^s(S^1) \to H^{s-3}(S^1)$, $s \in \mathbb{R}$, is the symmetric operator given by

$$Q_{\mathrm{op}}\left(\sum_{n\in\mathbb{Z}} u_n e^{inx}\right) = \frac{1}{8}\sum_{n\in\mathbb{Z}} |n|(n^2-1)u_n e^{inx} = \frac{1}{8}\sum_{n\neq -1,0,1} |n|(n^2-1)u_n e^{inx}.$$

Properties of the Operator Q_{op}. We have $Q_{\mathrm{op}} = \frac{1}{8}\,\mathrm{J}\circ(\partial^3 + \partial)$ where J is the Hilbert-transform of Theorem 2.10. Thus, while Q_{op} is not literally a third order elliptic differential operator, it has similar properties. Namely, since $\mathrm{J} : H^s(S^1) \to H^s(S^1)$ is an isomorphism for all s and since $\partial^3 + \partial$ is literally a third order elliptic differential operator, we have the properties
(1) $Q_{\mathrm{op}} : H^s(S^1) \to H^{s-3}(S^1)$ and
(2) $Q_{\mathrm{op}}(u) \in H^s(S^1) \;\Rightarrow\; u \in H^{s+3}(S^1)$.

We have

$$\ker(Q_{\mathrm{op}}) = \mathfrak{psu}(1,1) \quad\text{and}\quad \operatorname{Im}(Q_{\mathrm{op}}) = \left\{ \sum_{n\neq -1,0,1} v_n e^{inx} \;\middle|\; v_{-n} = \overline{v_n} \right\} \cap H^{s-3}(S^1).$$

We now study the kernel and image of $Q_{\rm op}|_{\mathfrak{g}^s}$, the restriction of $Q_{\rm op}$ to $\mathfrak{g}^s$ for $s > 1/2$. Since $\ker(Q_{\rm op}) = \mathfrak{psu}(1,1)$ as we have observed, we see that $\ker Q_{\rm op}|_{\mathfrak{g}^s} = \ker Q_{\rm op} \cap \mathfrak{g}^s = \{0\}$ because elements of $\mathfrak{psu}(1,1)$ that vanish at three points are identically zero. Thus, $Q_{\rm op}|_{\mathfrak{g}^s}$ is injective.

Next, we consider the image of $Q_{\rm op}|_{\mathfrak{g}^s}$. We claim that

$$\mathrm{Im}(Q_{\rm op}|_{\mathfrak{g}^s}) = \mathrm{Im}(Q_{\rm op}).$$

Indeed, for $m \in \mathrm{Im}(Q_{\rm op})$ there exists $u \in H^s(S^1)$ such that $m = Q_{\rm op}(u)$. Let $\bar{u} \in \mathfrak{psu}(1,1)$ be such that $\bar{u}$ and u have the same values at ± 1 and $-i$. We have $u - \bar{u} \in \mathfrak{g}^s$ and $Q_{\rm op}(u - \bar{u}) = Q_{\rm op}(u) = m$, therefore $m \in \mathrm{Im}(Q_{\rm op}|_{\mathfrak{g}^s})$.

It follows, in particular, that

$$Q_{\rm op} : \mathfrak{g}^{\frac{3}{2}} \to \mathrm{Im}(Q_{\rm op}) \subset H^{-\frac{3}{2}}(S^1)$$

is an isomorphism. As a consequence, note that the representation of l given by (6.2) is well-defined on $\mathfrak{g}^{\frac{3}{2}}$ since $u \in H^{\frac{3}{2}}(S^1)$ and $Q_{\rm op}(u) \in H^{-\frac{3}{2}}(S^1)$.

With respect to the pairing (6.1), the infinitesimal coadjoint action reads

$$\mathrm{ad}^*_u m = 2mu' + m'u; \tag{6.3}$$

more precisely we should write $\mathrm{ad}^*_u[m] = [2mu' + m'u]$, where $[\,]$ denotes the equivalence class modulo N. One can check that $[2mu' + m'u]$ does not depend on the choice of $m \in [m]$.

The functional derivative of l is

$$\frac{\delta l}{\delta u} = Q_{\rm op}(u).$$

Thus, *the Euler-Weil-Petersson equation reads*

$$\dot{m} + 2mu' + m'u = 0, \quad m = Q_{\rm op}(u) \in H^{-\frac{3}{2}}(S^1). \tag{6.4}$$

However, there is a *major difficulty with this formal argument.* Namely, (6.3) as well as (6.4) make no sense as written. We will repair this deficiency shortly, but to appreciate the problem, we make some further remarks about this form of the EWP equation.

We assert that the expression $\mathrm{ad}^*_u m = 2mu' + m'u$ is not well defined for $u \in \mathfrak{g}^{\frac{3}{2}}$. Indeed, Theorem 4.9 and Lemma 4.4 clearly do not apply to our situation, and this suggests that pointwise multiplication of $u' \in H^{\frac{1}{2}}$ and $m \in H^{-\frac{3}{2}}$ is not defined. Thus, it seems that one cannot write the Euler-Poincaré equation. As another way of expressing the same essential difficulty, one can try to write the Euler-Poincaré equation in weak form to see if it makes sense. As we shall see, it does not, again as written. Indeed, write

$$\left\langle \frac{d}{dt} m, \varphi \right\rangle = \langle m, [u, \varphi] \rangle, \ \forall \varphi \in C^\infty(S^1), \quad m = Q_{\rm op}(u). \tag{6.5}$$

This is also not well-defined since on the right hand side there is a L^2 pairing between $m \in H^{-\frac{3}{2}}$ and $[u, \varphi] \in H^{\frac{1}{2}}$.

Remark. This sort of difficulty does not occur for the Camassa-Holm (or Euler, or Euler-alpha) equation. For example, for the Camassa-Holm equation, we have $\dot{m} + 2mu' + m'u = 0$; that is $\dot{u} + Q_{\text{op}}^{-1}(2mu' + m'u) = 0$, where here, $m = Q_{\text{op}}(u) = (1 - \alpha^2\partial^2)u$. Since $u \in H^s, s > 3/2$, we have $m \in H^{s-2}$ and so, by Theorem 4.9, $2mu' + m'u \in H^{s-3}$. Therefore $Q_{\text{op}}^{-1}(2mu' + m'u) \in H^{s-1}$. We also know that $u \in C^0(I, H^s) \cap C^1(I, H^{s-1})$. Thus, it is meaningful to write the Camassa-Holm equation in Euler-Poincaré form.

The Geometric Form of the EWP Equation. We now claim that the preceding difficulties disappear if one writes the equation directly in terms of u *without* introducing the dual space. In doing so, we will heavily exploit the fact that the spray of the WP metric is smooth.

Let $\gamma(x, t)$ be a WP geodesic, for $x \in S^1$. Thus, as a function of t, and thought of as a curve in $\mathfrak{T}$, it is smooth because the spray of the WP metric is smooth. Thus, $\dot{\gamma}$ and $\ddot{\gamma}$ are well defined. According to the fact that there is a smooth WP spray, we can write

$$\ddot{\gamma} = \mathcal{Z}(\dot{\gamma}).$$

By definition of u, we have

$$u(\gamma(x, t), t) = \dot{\gamma}(x, t).$$

This makes sense and defines a continuous curve $u_t \in \mathfrak{g}^{\frac{3}{2}}$ because, by Takhtajan-Teo theory, $\mathfrak{T}$ is a topological group, right multiplication is smooth and the tangent space at the identity is $\mathfrak{g}^{\frac{3}{2}}$. Since multiplication on the left is not smooth, u_t need not be differentiable as a curve in $\mathfrak{g}^{\frac{3}{2}}$. A formal computation shows that u_t can, at most, be differentiable as a curve in $H^{\frac{1}{2}}$. This is made precise in the following theorem.

Theorem 6.1. *Let γ_t be a geodesic of the Weil-Petersson metric. Then the continuous curve $t \in \mathbb{R} \mapsto u_t := \dot{\gamma}_t \circ \gamma_t^{-1} \in H^{\frac{3}{2}}(S^1)$ is continuously differentiable as a curve in $H^{\frac{1}{2}}(S^1)$. Its derivative is given by*

$$\dot{u}_t = -u_t' u_t + \mathcal{Z}(u_t). \tag{6.6}$$

Proof. Using Lemma 4.3, the map

$$U = u \circ \eta \in T\mathfrak{T} \mapsto (u, \eta) \in \mathfrak{g}^{\frac{3}{2}} \times \mathfrak{T}$$

is continuous. Thus, with Lemma 4.5, the inclusion

$$T\mathfrak{T} \subset H^{\frac{3}{2}-}(S^1) = \bigcap_{s<3/2} H^s(S^1)$$

is continuous. Since the candidate for the time-derivative of u_t is $\mathcal{Z}(u_t) - u_t' u_t$, we proceed as follows.

We have

$$\left\|\frac{u_{t+h}-u_t}{h}-\mathcal{Z}(u_t)+u_t'u_t\right\|_{1/2} = \left\|\frac{\dot\gamma_{t+h}\circ\gamma_{t+h}^{-1}-\dot\gamma_t\circ\gamma_t^{-1}}{h}-\mathcal{Z}(u_t)+u_t'u_t\right\|_{1/2}$$

$$\leq \left\|\frac{\dot\gamma_{t+h}\circ\gamma_{t+h}^{-1}-\dot\gamma_t\circ\gamma_{t+h}^{-1}}{h}-\mathcal{Z}(u_t)\right\|_{1/2} \tag{6.7}$$

$$+\left\|\frac{\dot\gamma_t\circ\gamma_{t+h}^{-1}-\dot\gamma_t\circ\gamma_t^{-1}}{h}+u_t'u_t\right\|_{1/2}. \tag{6.8}$$

We now treat the term (6.7). We have

$$\left\|\frac{\dot\gamma_{t+h}\circ\gamma_{t+h}^{-1}-\dot\gamma_t\circ\gamma_{t+h}^{-1}}{h}-\mathcal{Z}(u_t)\right\|_{1/2}$$
$$\leq \left\|\left(\frac{\dot\gamma_{t+h}-\dot\gamma_t}{h}-\mathcal{Z}(\dot\gamma_t)\right)\circ\gamma_{t+h}^{-1}\right\|_{1/2}+\left\|\mathcal{Z}(\dot\gamma_t\circ\gamma_{t+h}^{-1})-\mathcal{Z}(\dot\gamma_t\circ\gamma_t^{-1})\right\|_{1/2}.$$

Since $\mathcal{Z}(\dot\gamma_t)=\ddot\gamma_t$, the expression $\frac{\dot\gamma_{t+h}-\dot\gamma_t}{h}-\mathcal{Z}(\dot\gamma_t)$ converges to 0, as $h\to 0$, with respect to the Weil-Petersson topology. Thus, by Lemma 4.5 the limit of the first term is 0 in. The limit of the second term is clearly 0. Note that the previous estimations can be done with the stronger norms $H^{\frac{3}{2}-\varepsilon}$, for all $\varepsilon>0$.

We now treat the term (6.8). We have

$$\left\|\frac{\dot\gamma_t\circ\gamma_{t+h}^{-1}-\dot\gamma_t\circ\gamma_t^{-1}}{h}+u_t'u_t\right\|_{1/2} = \left\|\frac{u_t\circ(\gamma_t\circ\gamma_{t+h}^{-1})-u_t}{h}+u_t'u_t\right\|_{1/2}. \tag{6.9}$$

Thus, we need to show that for each t, the continuous curve

$$h\mapsto u_t\circ\gamma_t\circ\gamma_{t+h}^{-1}\in T\mathfrak{T}\subset H^{\frac{3}{2}-\varepsilon}(S^1),\quad \text{for all } \varepsilon>0, \tag{6.10}$$

is differentiable at $h=0$, as a curve in $H^{\frac{1}{2}}(S^1)$. We first prove that for all $0<\varepsilon_1<1$ the curve

$$h\mapsto\gamma_t\circ\gamma_{t+h}^{-1}\in\mathfrak{T}\subset H^{\frac{3}{2}-\varepsilon_1}(S^1,S^1),$$

is differentiable at $h=0$, where $H^{\frac{3}{2}-\varepsilon_1}(S^1,S^1)$ is endowed with its natural Hilbert manifold structure (which exists because $\frac{3}{2}-\varepsilon_1>\frac{1}{2}$). Note that

$$\left\|\frac{\gamma_t\circ\gamma_{t+h}^{-1}-id}{h}+u_t\right\|_{\frac{3}{2}-\varepsilon_1}\to 0$$

is equivalent to

$$\left\|\frac{\gamma_t-\gamma_{t+h}}{h}+u_t\circ\gamma_{t+h}\right\|_{\frac{3}{2}-\varepsilon_1}\to 0,$$

by Lemma 4.5. Furthermore, we have

$$\left\|\frac{\gamma_{t+h}-\gamma_t}{h}-u_t\circ\gamma_{t+h}\right\|_{\frac{3}{2}-\varepsilon_1}\leq\left\|\frac{\gamma_{t+h}-\gamma_t}{h}-u_t\circ\gamma_t\right\|_{\frac{3}{2}-\varepsilon_1}+\|u_t\circ\gamma_t-u_t\circ\gamma_{t+h}\|_{\frac{3}{2}-\varepsilon_1}.$$

The limit of the first term is zero since $\dot{\gamma}_t=u_t\circ\gamma_t$ with respect to the Weil-Petersson topology. The second term converges to zero since $h\mapsto\gamma_{t+h}$ is continuous with respect to the Weil-Petersson topology and by Lemma 4.5.

In order to show that the curve (6.10) is differentiable at 0, as a curve in $H^{\frac{1}{2}}(S^1)$, we need to show that the map

$$\eta\in H^{\frac{3}{2}-\varepsilon_1}(S^1,S^1)\mapsto u\circ\eta\in H^{\frac{1}{2}}(S^1)$$

is differentiable at id. Note that this map is only well-defined on a subset of $H^{\frac{3}{2}-\varepsilon_1}(S^1,S^1)$, which contains $\mathfrak{T}$, and is not continuous with respect to the topology of $H^{\frac{3}{2}-\varepsilon_1}(S^1,S^1)$. Moreover, $\eta\in H^{\frac{3}{2}-\varepsilon_1}(S^1,S^1)$ and $u\in H^{\frac{3}{2}}$ does not imply that $u\circ\eta\in H^{\frac{1}{2}}(S^1)$ because the hypotheses of Lemma 4.5 require that $\eta^{-1}\in H^{\frac{3}{2}-\varepsilon_1}(S^1,S^1)$ for small $\varepsilon_1>0$. Theorem 4.2 shows that the domain of this map contains $\mathfrak{T}$ because any $\eta\in\mathfrak{T}$ satisfies $\eta,\eta^{-1}\in H^{\frac{3}{2}-\varepsilon_1}(S^1,S^1)$ for all $\varepsilon_1>0$.

In spite of these problems, we shall prove that this map, defined only on its domain, is differentiable at id.

Recall that the functions in H^1 are absolutely continuous and so are differentiable a.e.. For absolutely continuous functions, the fundamental theorem of calculus is still valid (see Hewitt and Stromberg [1975], §18). Since u is in $H^{\frac{3}{2}}(S^1)$, we can write, for all $x,v\in\mathbb{R}$,

$$u(x+v)-u(x)=\int_0^1 u'(x+tv)v\,dt.$$

Thus, at points x at which u is differentiable, we have, for all v,

$$u(x+v)-u(x)-u'(x)v=\int_0^1(u'(x+tv)-u'(x))v\,dt.$$

Let $w\in H^{\frac{3}{2}-\varepsilon_1}(S^1)=T_eH^{\frac{3}{2}-\varepsilon_1}(S^1,S^1)$. Since the previous formula is valid for all v, we can write

$$u(x+w(x))-u(x)-u'(x)w(x)=\int_0^1\left(u'(x+tw(x))-u'(x)\right)w(x)dt. \tag{6.11}$$

By Lemma 4.5 and Theorem 4.9 we know that $t\in[0,1]\mapsto(u'\circ(id+tw)-u')\,w\in H^{\frac{1}{2}}$ is a continuous function if $\varepsilon_1>0$ sufficiently small, and for admissible w, that is, such that $id+w$ and its inverse are in $H^{\frac{3}{2}-\varepsilon_1}(S^1)$. For the same reasons $u\circ(id+w)-u-u'w\in H^{\frac{1}{2}}(S^1)$. Thus, it makes sense to ask whether the identity

$$u\circ(id+w)-u-u'w=\int_0^1\left(u'\circ(id+tw)-u'\right)wdt \tag{6.12}$$

holds as an equality in $H^{\frac{1}{2}}(S^1)$. The two sides of the equality above can be evaluated for all x in a subset of S^1 whose complement has measure zero. For x in this set, the evaluation of the left hand side of (6.12) clearly coincides with the left hand side of (6.11). The evaluation at x of the right hand side of (6.12) equals the right hand side of (6.11) by the definition of the integrals in both formulas.

Given $M > 0$, there is $\delta > 0$ such that if $\|w\|_{\frac{3}{2}-\varepsilon_1} < \delta$ we have

$$\|u' \circ (id + tw) - u'\|_{\frac{1}{2}} < M,$$

by Lemma 4.5. Using the properties of the integral in Hilbert spaces and Theorem 4.9, we obtain

$$\left\| \int_0^1 \left(u' \circ (id + tw) - u'\right) w dt \right\|_{\frac{1}{2}} \leq C\|u' \circ (id + tw) - u'\|_{1/2}\|w\|_{\frac{3}{2}-\varepsilon_1} \leq CM\|w\|_{\frac{3}{2}-\varepsilon_1}.$$

This proves that

$$\|u \circ (id + w) - u - u'w\|_{\frac{1}{2}} \leq CM\|w\|_{\frac{3}{2}-\varepsilon_1}.$$

Therefore, the map $\eta \in H^{\frac{3}{2}-\varepsilon_1}(S^1, S^1) \mapsto u \circ \eta \in H^{\frac{1}{2}}(S^1)$ is differentiable at id with respect to admissible directions. Since this map is well-defined on the curve $h \mapsto \gamma_t \circ \gamma_{t+h}^{-1}$, and since this curve is differentiable at 0, we obtain that the curve

$$h \mapsto u_t \circ \gamma_t \circ \gamma_{t+h}^{-1} \in H^{\frac{1}{2}}(S^1)$$

is differentiable at $h = 0$ and its derivative is $-u_t' u_t$. Thus we obtain that the expression (6.9) converges to zero, for all $t \in \mathbb{R}$. This proves that the curve $t \mapsto u_t$ is differentiable and its derivative is

$$\dot{u}_t = -u_t' u_t + \mathcal{Z}(u_t).$$

Since we already know that $t \mapsto u_t$ is continuous in $H^{\frac{3}{2}}(S^1)$, we obtain, by Theorem 4.9 and by the smoothness of the geodesic spray, that $t \mapsto -u_t' u_t + \mathcal{Z}(u_t)$ is a continuous curve in $H^{\frac{1}{2}}(S^1)$. This show that $t \mapsto u_t$ is in $C^1\left(\mathbb{R}, H^{\frac{1}{2}}(S^1)\right)$. ∎

This shows the remarkable fact that the Eulerian representation of the geodesics of the Weil-Petersson metric on the universal Teichmüller space have the same property as the Eulerian representation of geodesics in fluid mechanics (incompressible Euler, Euler-α, higher dimensional Camassa-Holm). Namely, the regularity of the Eulerian velocity is always of the form

$$v_t \in C^0([0, T], H^s) \cap C^1([0, T], H^{s-1}), \quad \text{for all } s > \frac{n}{2} + 1.$$

However, note that for the Euler-Weil-Petersson equation we are obliged to use the critical exponent $s = 3/2$. Long time existence follows from the strongness of the metric, as we have seen in Proposition 5.1, that is, we have

$$u_t \in C^0\left(\mathbb{R}, H^{\frac{3}{2}}\right) \cap C^1\left(\mathbb{R}, H^{1/2}\right).$$

Summary. We conclude this section by giving a brief summary of the properties of the WP geodesics on the Riemannian manifold $(\mathfrak{T}, g)$. Recall that $\mathfrak{T}$ is by definition the Hilbert manifold $\Phi(T(1)_\circ^H)$, where $T(1)_\circ$ is the connected component of the identity of the universal Teichmüller space endowed the with the Takhtajan-Teo Hilbert manifold structure. Alternatively, $\mathfrak{T}$ can also be seen as the closure of the diffeomorphism group $\mathrm{Diff}_{+,\mathrm{fix}}(S^1)$ in the group $\mathrm{QS}(S^1)^H_{\mathrm{fix}}$ of quasisymmetric homeomorphisms relative to the Hilbert manifold structure. The homeomorphisms in $\mathfrak{T}$ as well as their inverse, are of Sobolev class H^s for all $s < \frac{3}{2}$.

Given an homeomorphism γ in $\mathfrak{T}$ and a direction $u_\gamma \in T_\gamma\mathfrak{T}$, there exists a unique geodesic $\gamma(t)$ starting at γ with initial velocity u_γ. Moreover, this geodesic can be extended for all time. The associated Eulerian velocity $u(t) := \dot{\gamma}(t)\circ\gamma(t)^{-1}$ solves the Euler-Weil-Petersson equations and defines a continuous curve in $H^{\frac{3}{2}}$, differentiable as a curve in $H^{\frac{1}{2}}$.

7 Teichons as Particular WP Geodesics

In this section we consider particular solutions of the Euler-Weil-Petersson equation, analogue to the peakons solutions of the Camassa-Holm equations. They are naturally called Teichons. Recall that in the case of Camassa-Holm equations, these peakons are singular solutions and are not obtained by solving the geodesic spray equations on the group of Sobolev diffeomorphism $\mathrm{Diff}^s_+(S^1)$, $s > 3/2$. As we will see below, the situation is different for the Teichons. For the Camassa-Holm equations, the peakons are given by expression of the form $u(x) = \sum_{j=1}^N p_j G(x - q_j)$, where G denotes the Green's function of the operator $(1 - \alpha^2\partial_x^2)$. The associated momentum $m = (1 - \alpha^2\partial_x^2)u$ is interpreted as a momentum mapping for the left action of the diffeomorphism group $\mathrm{Diff}_+(S^1)$ on the Cartesian product of N circles.

Teichons. In the case of the Euler-Weil-Petersson equation, the Green's function is

$$G(x) = \frac{4}{\pi} \sum_{n\neq -1,0,1} \frac{e^{inx}}{|n|(n^2-1)},$$

since for $Q_{\mathrm{op}} = \frac{1}{8}\,\mathrm{J}\circ(\partial_x^3 + \partial_x)$ we have

$$Q_{\mathrm{op}}(G) = \frac{1}{2\pi} \sum_{n\neq -1,0,1} e^{inx} = \mathbf{P}(\delta) = \frac{1}{2\pi} \sum_{n\neq -1,0,1} e^{inx},$$

where δ is the Dirac distribution and $\mathbf{P}$ is the projection onto the space $\mathfrak{h}^s = \{u \in H^s(S^1) \mid u_0 = u_1 = u_{-1} = 0\}, s \in \mathbb{R}$. From the expression for the Green's function, we see that

$$G \in \mathfrak{h}^{5/2-\varepsilon}, \quad \text{for all} \quad \varepsilon > 0.$$

Recall that $Q_{\mathrm{op}} : \mathfrak{h}^s \to \mathfrak{h}^{s-3}$ is an isomorphism. For $s > 1/2$, $Q_{\mathrm{op}} : \mathfrak{g}^s \to \mathfrak{h}^{s-3}$ is also an isomorphism.

We search for solutions of the weak Euler-Weil-Petersson equation (6.5) of the form

$$u(x,t)=\sum_{j=1}^{N} p_j(t)G(x-q_j(t)) \tag{7.1}$$

and want to determine $(q_j(t),p_j(t))$. We begin by noting that it makes sense to search for solutions of this type. Indeed, all the terms in the equation $\langle \dot{m},\varphi\rangle+\langle m,u'\varphi\rangle-\langle m,u\varphi'\rangle=0$ make sense. In the second term $m\in H^{-\frac{1}{2}-\varepsilon}(S^1)$ and $u'\in H^{\frac{3}{2}-\varepsilon}(S^1)$ so the L^2 integral of their product makes sense for this form of u. In the third term, $m\in H^{-\frac{1}{2}-\varepsilon}(S^1)$ and $u\in H^{\frac{5}{2}-\varepsilon}(S^1)$ so the L^2 integral of their product also makes sense.

Let

$$m(x,t)=(Q_{\mathrm{op}}u)(x,t)=\sum_{j=1}^{N} p_j(t)(\mathbf{P}\delta)(x-q_j(t))=\mathbf{P}\left(\sum_{j=1}^{N} p_j(t)\delta(x-q_j(t))\right).$$

Assume that

$$\sum_{j=1}^{N} p_j(t)=\sum_{j=1}^{N} p_j(t)e^{iq_j(t)}=\sum_{j=1}^{N} p_j(t)e^{-iq_j(t)}=0. \tag{7.2}$$

Now note these three expressions are 2π times the $n=0,\pm1$ Fourier coefficients of $\sum_{j=1}^{N} p_j(t)\delta(x-q_j(t))$. Therefore, under the conditions (7.2),

$$m(x,t)=\sum_{j=1}^{N} p_j(t)\delta(x-q_j(t)).$$

Note that $m(\cdot,t)\in\mathfrak{h}^{-1/2-\varepsilon}$. A direct computation shows that $\langle \dot{m},\varphi\rangle+\langle m,u'\varphi\rangle-\langle m,u\varphi'\rangle=0$ for every $\varphi\in C^\infty(S^1)$ if and only if

$$\partial_t q_j=\sum_{i=1}^{N} p_i G(q_i-q_j)=\frac{\partial H}{\partial p_j},\qquad \partial_t p_j=-\sum_{i=1}^{N} p_i p_j G'(q_i-q_j)=-\frac{\partial H}{\partial q_j}, \tag{7.3}$$

relative to the collective Hamiltonian $H:[T^*S^1]^N\to\mathbb{R}$ given by

$$H(\mathbf{q},\mathbf{p})=\frac{1}{2}\sum_{i,j=1}^{N} p_i p_j G(q_i-q_j).$$

The conditions (7.2) are conserved along the flow of (7.3) and thus we need to impose them only on the initial conditions, that is,

$$\sum_{j=1}^{N} p_j(0)=\sum_{j=1}^{N} p_j(0)e^{iq_j(0)}=\sum_{j=1}^{N} p_j(0)e^{-iq_j(0)}=0. \tag{7.4}$$

Note that since $G \in H^{\frac{5}{2}-\varepsilon}(S^1)$, the right hand side of (7.3) is in $H^{\frac{3}{2}-\varepsilon}(S^1) \subset C^0(S^1)$ and thus the solution is C^1 in time.

Recall that the Euler-Weil-Petersson equation in weak form makes no sense if u is only in $H^{3/2}(S^1)$, but here the ansatz $u(x,t) = \sum_{j=1}^N p_j(t)G(x - q_j(t))$ is in $H^{5/2-\varepsilon}(S^1)$, for all $\varepsilon > 0$, therefore it makes sense to say that $u(x,t) = \sum_{j=1}^N p_j(t)G(x - q_j(t))$ is a weak solution of the Euler-Weil-Petersson equation.

Note that the solutions of (6.5) are in $H^{\frac{5}{2}-\varepsilon}(S^1)$ for all t near zero and so they necessarily lie in $H^{\frac{3}{2}}(S^1)$. Thus they are solutions of the spatial representation (6.6) of the geodesic spray of the Weil-Petersson metric on $T(1)^H_\circ$. Consequently $t \mapsto \eta_t$, where η_t is the solution of the equation $\dot{\eta}_t(x) = u(t, \eta_t(x))$, is a solution of the geodesic spray of the Weil-Petersson metric. We summarize all these comments in the following statement.

Theorem 7.1. *The Hamiltonian system (7.3) with conditions (7.4) has infinite time solutions. The functions $u(x,t)$ given by (7.1) with $(q_j(t), p_j(t))$ solution of (7.3) are particular solutions of the geodesic spray of the Weil-Petersson metric on $T(1)^H_\circ$ in spatial representation.*

The long time existence result of solutions of the type (7.1) was independently obtained by Kushnarev [2009] by a direct analysis of the Hamiltonian system (7.3). Due to the similarity to soliton and peakon solutions for the KdV and Camassa-Holm equations, solutions of the type (7.1) are called *Teichons*.

Momentum Mapping Interpretation. As in the case of the peakons of the Camassa-Holm equations, see Holm and Marsden [2004], the momentum $m = Q_{\text{op}}u$ associated to the Teichons solutions is given by momentum mapping. To see this, it suffices to remark that

$$\mathbf{J} : [T^*S^1]^N \to \mathfrak{h}^{-3/2}, \quad \mathbb{J}(\mathbf{q}, \mathbf{p}) = \mathbf{P}\left(\sum_{j=1}^N p_j \delta(x - q_j)\right)$$

is the momentum mapping associated to the cotangent lift of left composition by $\mathfrak{T}$.

Teichons Versus Peakons. The situation is totally different for the peakon solution of the Camassa-Holm equations. In this case the Green's function is in $H^{\frac{3}{2}-\varepsilon}$, therefore, the peakons solutions for Camassa-Holm are really less regular than the solutions obtained by geometric means which require $s > 3/2$. This is exactly backwards for EWP, the Teichons are more regular than the geodesics.

We now mention another difference between Peakons and Teichons. For peakons for the Camassa-Holm equations, to make sense of weak solutions, one has to multiply by a test function of space-time, not just space, see Alber, Camassa, Fedorov, Holm, and Marsden [2001]

8 Application to Pattern Recognition

The goal of this section is to apply the properties of the Euler-Weil-Petersson spray to image recognition. Sharon and Mumford [2006] used the Weil-Petersson metric to compare images and introduced the fingerprint map to identify a space of smooth shapes with the group of diffeomorphisms by means of conformal welding. We shall prove that the fingerprint map extends to certain completions of these spaces relative to the Weil-Petersson metric. This enables us to make use of the Takhtajan-Teo theory and the results of the present paper. The fact that the universal Teichmüller space is geodesically complete with negative curvature enables us to positively address a comment of Sharon and Mumford [2006], namely that there exists a unique geodesic between each two shapes in the plane.

Conformal Welding. Given a closed Jordan curve Γ in the Riemann sphere $\widehat{\mathbb{C}}$, we denote by Ω and Ω^* the two components of the open subset $\widehat{\mathbb{C}}\backslash\Gamma$ of $\widehat{\mathbb{C}}$. By the Riemann mapping theorem we can find conformal maps $f : \mathbb{D} \to \Omega$ and $g : \mathbb{D}^* \to \Omega^*$. These maps extend to homeomorphisms between the closures of the domains and, by restriction of these extensions to S^1, we can form the orientation preserving homeomorphism

$$h := g^{-1} \circ f$$

of the circle. A homeomorphism of the circle arising this way is called a ***conformal welding***. If Γ is a smooth curve, then f and g extend to smooth maps of the corresponding boundaries. In this case h is a smooth diffeomorphism of S^1. It is well known that there exist homeomorphisms which are not conformal weldings. We have the following well-known existence and uniqueness result in the case of quasisymmetric homeomorphisms.

Theorem 8.1. *Let $\eta \in \mathrm{QS}(S^1)_{\mathrm{fix}}$. Then η is a conformal welding and the decomposition $\eta = g^{-1} \circ f$ is unique up to left composition of f and g by a Möbius transformation.*

This theorem follows easily form the fact that for each $\eta \in \mathrm{QS}(S^1)_{\mathrm{fix}}$ we can write

$$\eta = \omega_\mu|_{S^1} = (\omega^\mu \circ \omega_\mu^{-1})^{-1} \circ \omega^\mu|_{S^1}. \tag{8.1}$$

To the decomposition $\eta = g^{-1} \circ f$ we can associate the Jordan curve $\Gamma := f(S^1) = g(S^1)$, and from (8.1) we know that Γ is the image of S^1 by a quasiconformal mapping of $\widehat{\mathbb{C}}$. Such a curve is called a ***quasicircle***.

The Fingerprint Map. The set of all quasicircles in the complex plane is denoted by $\mathcal{QC}$. It is known (see Lemma p.123 in Kirillov [1986]) that if η is a smooth diffeomorphism, then the quasicircle Γ associated to the decomposition $\eta = g^{-1} \circ f$ is a smooth curve. We denote by $\mathcal{S}$ the subset of all smooth and simple closed curves in $\mathbb{C}$ and by $\overline{\mathcal{QC}}$ and $\overline{\mathcal{S}}$ the corresponding quotient spaces associated to the action of the group of transformations $\{az + b \mid a > 0, b \in \mathbb{C}\}$.

For a quasicircle $\Gamma \in \mathcal{QC}$ we denote by Γ_+ the open subset of $\widehat{\mathbb{C}}$ bounded by Γ and containing the point ∞ and by Γ_- the other open subset bounded by Γ. Using the fact that each quasicircle is the image of S^1 by a quasiconformal mapping of $\widehat{\mathbb{C}}$ which is conformal on $\mathbb{D}$ (see Lemma I.6.2 in Lehto [1987]) we can find a quasiconformal map ϕ_- of $\widehat{\mathbb{C}}$ verifying the following two conditions:

- $\phi_-(\mathbb{D}) = \Gamma_-$,
- $\phi_-|_{\mathbb{D}}$ is conformal.

Similarly, we can find a quasiconformal map ϕ_+ of $\widehat{\mathbb{C}}$ such that

- $\phi_+(\mathbb{D}^*) = \Gamma_+$,
- $\phi_+|_{\mathbb{D}}$ is conformal,
- $\phi_+(\infty) = \infty$ and $\partial_z \phi_+(\infty) > 0$.

Note that we have $\phi_-(S^1) = \phi_+(S^1) = \Gamma$. Note also that given $\Gamma \in \mathcal{QC}$, the conformal map $\phi_-|_{\mathbb{D}}$ is uniquely determined up to right composition with an element in $\mathrm{PSU}(1,1)$ and the conformal map $\phi_+|\mathbb{D}^*$ is uniquely determined by the three previous conditions. Therefore, we can consider the map

$$\mathcal{QC} \to \mathrm{PSU}(1,1)\backslash \mathrm{QS}(S^1) \simeq \mathrm{QS}(S^1)_{\mathrm{fix}}, \qquad \Gamma \mapsto [\phi_+^{-1} \circ \phi_-|_{S^1}].$$

This map is invariant under scalings and translations in $\mathcal{QC}$ and therefore induces a map

$$\mathcal{F} : \overline{\mathcal{QC}} \to \mathrm{QS}(S^1)_{\mathrm{fix}}.$$

By Theorem 8.1, each $\eta \in \mathrm{QS}(S^1)_{\mathrm{fix}}$ is a conformal welding, therefore we can write $\eta = g^{-1} \circ f$. Since f and g are unique up to left composition with a Möbius transformation, we can impose the normalization conditions $g(\infty) = \infty$ and $\partial_z g(\infty) > 0$. Setting $\Gamma := g(S^1)$, we obtain that $\mathcal{F}(\Gamma) = \eta$. This proves that this map is surjective. Suppose now that we have an other decomposition $\eta = \bar{g}^{-1} \circ \bar{f}$ where $\bar{g}$ verifies the same normalization conditions. Since $\bar{g} = \gamma \circ g$ for γ a Möbius transformation, we must have $\gamma(z) = az + b$ where $a > 0$ and $b \in \mathbb{C}$. This proves that $\mathcal{F}$ is a bijection. Note that the map $\mathcal{F}$ restricts to a map

$$\mathcal{F} : \overline{\mathcal{S}} \to \mathrm{Diff}_+(S^1)_{\mathrm{fix}}.$$

This restricted map is actually the *fingerprint map* considered in Sharon and Mumford [2006]. In this paper, the set $\overline{\mathcal{S}}$ is seen as the space of all smooth shapes in the plane up to scalings and translations. Using the map $\mathcal{F}$, each equivalence class of shapes is identified with a unique diffeomorphism $\eta \in \mathrm{Diff}_+(S^1)_{\mathrm{fix}}$. The Weil-Petersson metric on $\mathrm{Diff}_+(S^1)_{\mathrm{fix}}$ can be used as a metric on this space of shapes. This point of view is developed in Sharon and Mumford [2006] with applications to 2D-shape analysis and leads to interesting numerical computations.

Hilbert Manifold Structure. Recall from section 3 the following two important facts.

- The group $\mathrm{QS}(S^1)_{\mathrm{fix}}$ can be endowed with a smooth Hilbert manifold structure on which the Weil-Petersson metric is strong. This manifold is denoted by $\mathrm{QS}(S^1)^H_{\mathrm{fix}}$ and has uncountably many connected components.
- The completion in $\mathrm{QS}(S^1)^H_{\mathrm{fix}}$ of the subgroup $\mathrm{Diff}_+(S^1)_{\mathrm{fix}}$ of smooth diffeomorphisms is the connected component of the identity in $\mathrm{QS}(S^1)^H_{\mathrm{fix}}$.

Using the fingerprint map, we can pull back the Hilbert manifold structure of $\mathrm{QS}(S^1)^H_{\mathrm{fix}}$ to the set $\overline{\mathcal{QC}}$ of all quasicircles up to scalings and translations. The connected component $\overline{\mathcal{QC}}_\circ$ of S^1 in $\overline{\mathcal{QC}}$ is therefore the completion of the shape space $\overline{\mathcal{S}}$. We thus have the smooth inclusions

$$\overline{\mathcal{S}} \subset \overline{\mathcal{QC}}_\circ \subset \overline{\mathcal{QC}}.$$

Since in the completion $\overline{\mathcal{QC}}_\circ$ the Weil-Petersson metric is strong and has non-positive sectional curvature, we have the following theorem.

Theorem 8.2. *Let Γ_1, Γ_2 be two shapes in $\overline{\mathcal{QC}}_\circ$. Then in any homotopy class of curves from Γ_1 to Γ_2, there is precisely one Weil-Petersson geodesic from Γ_1 to Γ_2.*

This discussion gives a positive answer to a question posed by Sharon and Mumford [2006] in the introduction. Thus, integrating the Weil-Petersson metric along a geodesic gives the distance between two shapes.

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