Clebsch optimal control formulation in mechanics

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Abstract

This paper introduces and studies a class of optimal control problems based on the Clebsch approach to Euler-Poincaré dynamics. This approach unifies and generalizes a wide range of examples appearing in the literature: the symmetric formulation of N-dimensional rigid body and its generalization to other matrix groups; optimal control for ideal flow using the back-to-label map; the double bracket equations associated to symmetric spaces. New examples are provided such as the optimal control formulation for the N-Camassa-Holm equation and a new geodesic interpretation of its singular solutions.

1 Introduction

This paper introduces a new class of optimal control problems that unifies many known results and generalizes their key properties to new models, such as the N-Camassa-Holm equation, optimization on reductive homogeneous spaces, and optimal control formulations for ideal fluid flow using Clebsch variables. The basic idea combines the classical Clebsch variable approach in hydrodynamics, interpreted in terms of symplectic and Poisson geometry, with the standard theory of optimal control. Historically, Clebsch variables appeared in fluid mechanics in order to recast the Euler equations in a standard canonical Hamiltonian form. It is well known that from the point of view of Poisson and symplectic geometry, the Clebsch variables form a momentum map of the action of the particle relabeling group on various vector spaces. The general framework is the following.

Let P be a Poisson manifold, thought of as the phase space of a physical system. By *Clebsch variables* (or *symplectic variables*) for this system, we mean a symplectic manifold R and a Poisson map

$$\Psi : R \to P, \tag{1.1}$$

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see Marsden and Weinstein [1983]. Classical examples of such Clebsch variables are the *Cayley-Klein* parameters for the free rigid body, the *Kustaanheimo-Stiefel* coordinates in quantum mechanics, and, of course, the *classical Clebsch variables* in various fluid dynamical models (homogeneous incompressible, isentropic, MHD, etc).

A fundamental example of Poisson manifold is provided by a dual Lie algebra $\mathfrak{g}^*$ endowed with the $\pm$ Lie-Poisson bracket defined by

$$\{f, g\}_\pm(\mu) = \pm \left\langle \mu, \left[\frac{\delta f}{\delta \mu}, \frac{\delta g}{\delta \mu}\right] \right\rangle, \quad \mu \in \mathfrak{g}^*,$$

for smooth maps $f, g \in \mathcal{F}(\mathfrak{g}^*)$. If in (1.1) the Poisson manifold is $P = \mathfrak{g}^*$ endowed with the Lie-Poisson structure, then Ψ is necessarily the momentum map of a Hamiltonian action of the Lie algebra $\mathfrak{g}$ on the symplectic manifold R. In many examples, as is the case in this paper, the symplectic manifold R is a cotangent bundle T^*Q on which the Lie group G, with Lie algebra $\mathfrak{g}$, acts by point transformations. In this case, a Poisson map Ψ is provided by the cotangent bundle momentum map $\mathbf{J} : T^*Q \to \mathfrak{g}^*$, defined by

$$\langle \mathbf{J}(\alpha_q), u \rangle := \langle \alpha_q, u_Q(q) \rangle ,$$

where u_Q is the infinitesimal vector field on Q associated to the G-action and to the Lie algebra element $u \in \mathfrak{g}$. Using the map $\mathbf{J}$, any Hamiltonian function h on $\mathfrak{g}^*$ induces a so called *collective Hamiltonian* $H := h \circ \mathbf{J}$ on T^*Q. Since $\mathbf{J}$ is a Poisson map, each solution of the canonical Hamilton equations on T^*Q projects through $\mathbf{J}$ to a solution of the *Lie-Poisson equations* on $\mathfrak{g}^*$ given by

$$\dot{\mu} = \mp \operatorname{ad}^*_{\frac{\delta h}{\delta \mu}} \mu. \tag{1.2}$$

When the Hamiltonian h is associated to a Lagrangian $\ell : \mathfrak{g} \to \mathbb{R}$ by Legendre transformation, these equations are equivalent to the *Euler-Poincaré equations*

$$\frac{d}{dt} \frac{\delta \ell}{\delta u} = \mp \operatorname{ad}^*_u \frac{\delta \ell}{\delta u}. \tag{1.3}$$

Besides the standard Euler-Poincaré variational principle (Marsden and Ratiu [1999]), there is also a variational formulation of the Euler-Poincaré equations (1.3) based on the Clebsch approach. It reads

$$\delta \int_0^T \left(\langle \alpha_q, u_Q(q) - \dot{q} \rangle - \ell(u) \right) dt = 0,$$

where, as before, u_Q denotes the infinitesimal vector field on Q determined by the G-action. The key observation here is that this variational principle can be associated to an optimal control problem that we naturally call a *Clebsch optimal control problem.* This observation will be widely expanded and studied in this paper. Remarkably, the Clebsch optimal control problem that we introduce here, recovers and unifies many examples such as various double bracket equations and systems on Lie algebras and symmetric spaces (Brockett [1994], Bloch and Crouch [1996], Bloch, Brockett, and Crouch [1997]), optimal control problems for fluids (Bloch, Crouch, Holm, and Marsden [2000], Holm [2009]), and the symmetric representation of the rigid body and its generalizations (Bloch and Crouch [1996], Bloch, Crouch, Marsden, and Ratiu [1998], Bloch, Crouch, and Sanyal [2006], Bloch, Crouch, Marsden, and Sanyal [2008]). In particular, our approach allows us to generalize some results that were only known for particular examples and proved by ad hoc methods. This includes, in particular, the geodesic interpretation of the double bracket equations on adjoint orbits relative to the normal metric. Besides recovering and unifying known results, we also present new optimal control formulations in hydrodynamics such as the optimal control formulation for the N-Camassa-Holm equation using its singular solutions, and the optimal control formulation for ideal flows using the classical Clebsch variables. This gives a new geodesic interpretation of the singular solutions relative to a normal metric.

The Clebsch optimal control problem introduced in this paper is associated to an arbitrary action of a Lie group G on a manifold Q and to a cost function $\ell : \mathfrak{g} \to \mathbb{R}$. By specializing to particular examples of actions and cost functions, the corresponding Hamilton's equations obtained by the Pontryagin Maximum Principle recover the well-known *coupled double bracket equations*

$$\dot{x} = [x, [p, x]], \quad \dot{p} = [p, [p, x]]$$

on a Lie algebra $\mathfrak{g}$, or the *symmetric representation of the N-dimensional rigid body*

$$\dot{Q} = PU, \quad \dot{P} = PU, \quad \text{for} \quad U = J^{-1}\mathbb{P}\left(Q^T P\right),$$

where $\mathbb{P}$ is the orthogonal projection onto the antisymmetric matrices. Our formalism allows us to obtain analogous equations for more complicated examples by simply applying our general results to different actions of Lie groups. For example, applying the Pontryagin Maximum Principle to the cost function given by the kinetic energy of ideal two dimensional fluid flow we get the *Euler equations in double bracket form*

$$\dot{\lambda} + \{\lambda, \Delta^{-1}\{\mu, \lambda\}\} = 0, \quad \dot{\mu} + \{\mu, \Delta^{-1}\{\mu, \lambda\}\} = 0$$

for the canonical variables (λ, μ) which are the classical Clebsch variables for the Eulerian velocity of the fluid. Using the back-to-label map for ideal flows, in any dimension, the Pontryagin Maximum principle yields the *symmetric representation of the Euler equations*

$$\dot{q} = -Tq \circ u, \quad \dot{\pi} = -T\pi \circ u, \quad \text{for} \quad u = -\mathbb{P}\left(Tq^{\dagger} \circ \pi\right).$$

The paper is organized as follows. In Section 2, we define the Clebsch optimal control problem associated to an arbitrary action of a Lie group on a manifold Q and study its general properties. In particular, we define the normal metric induced on G-orbits by an inner product on the Lie algebra of G. Then we show that the extremal solutions of the problem are geodesics on orbits relative to this normal metric, when the cost function is given by a kinetic energy. In Section 3 we specialize the results obtained so far to the simple case when Q is a Lie group acted on by a subgroup G of Q. This allows us to obtain the symmetric representation for the N-dimensional rigid body and its generalization to quadratic Lie groups, as well as the optimal control formulation for ideal fluids. The Clebsch optimal control formulations for the N-dimensional Camassa-Holm equation is presented in Section 4, where it is also shown that its singular solutions can be seen as geodesics of a normal metric. In Section 5 we study the Clebsch optimal control problem in the particular case when the manifold Q is a Lie algebra acted on by the adjoint representation. This allows us to recover the coupled double bracket equations and their geodesic properties on adjoint orbits. We also consider the closely related case of the coupled double bracket equation on symmetric spaces or, more generally, on reductive homogeneous spaces. The classical Clebsch variables for ideal fluids are used in Section 6 to write an alternative optimal control formulation for ideal fluid equations. When specialized to two-dimensional flows, the Pontryagin Maximum Principle yields the equations in coupled double bracket form. Finally, Section 7 presents the more general case when the cost function ℓ also depends on a variable in the manifold Q. The main properties of the associated Clebsch optimal control problem are investigated. As applications, we consider an optimization problem on Stiefel manifolds and the case of Euler-Poincaré equations for semidirect products.

2 The Clebsch optimal control problem and its properties

Let Q be a smooth manifold, U a vector space, $g : Q \times U \to \mathbb{R}$ a cost function, and $X = X(q, u)$ a vector field on Q depending smoothly on the variable $u \in U$. Given two elements $q_0, q_T \in Q$, we consider the following typical optimal control problem:

Find the curves $q = q(t) \in Q$ and $u = u(t) \in U$ that minimize the integral

$$\int_0^T g(q(t), u(t))dt \tag{2.1}$$

subject to the following conditions:

(A) $\dot{q}(t) = X(q(t), u(t))$;

(B) $q(0) = q_0$ *and* $q(T) = q_T$.

The first approach one considers when studying an optimal control problem is to apply the *Pontryagin maximum principle* which gives a necessary condition on the extremal of the problem. The first step is to define the *Pontryagin function* $\hat{H} : T^*Q \times U \to \mathbb{R}$ associated to the optimal control problem (2.1). It is given by

$$\hat{H}(\alpha_q, u) := \langle \alpha_q, X(q, u) \rangle - p_0 g(q, u),$$

where $p_0 \geq 0$ is a fixed positive constant and the bracket denotes the duality pairing between T^*Q and TQ.

The Pontryagin maximum principle (see Agrachev and Sachkov [2004], Theorem 12.10) asserts that if $(q(t), u(t))$ is a solution of the optimal control problem (2.1) then there is a curve $\alpha(t) \in T^*Q$ covering $q(t)$ such that

$$\frac{d}{dt}\alpha(t) = X_{\hat{H}_{u(t)}}(\alpha(t)), \qquad \hat{H}(\alpha(t), u(t)) = \max_{u \in U} \hat{H}(\alpha(t), u), \tag{2.2}$$

where $X_{\hat{H}_u}$ is the Hamiltonian vector field defined by $\hat{H}_u(\alpha) := \hat{H}(\alpha, u)$. Note that if $p_0 \neq 0$, replacing $\alpha(t)$ by $\alpha(t)/p_0$ shows that in $\hat{H}$ one can always assume that $p_0 = 1$. Solutions of the optimal control problem (2.1) with $p_0 \neq 0$ are called *normal extremals*. Solutions with $p_0 = 0$ are called *abnormal extremals*. We shall deal in this paper exclusively with normal extremals and set $p_0 = 1$.

Let us assume that $\hat{H}$ is of class C^1. Then the optimal control $u(t)$ is found by solving the equation

$$\frac{\partial \hat{H}}{\partial u}(\alpha(t), u(t)) = 0.$$

A sufficient condition that guarantees that maximum is achieved along the control $u(t)$ is that X is linear in u and g is strictly convex in u.

So, locally the Pontryagin maximum principle states that $\alpha(t) = (q(t), p(t))$ and $u(t)$ are determined by the system of equations

$$\frac{\partial \hat{H}}{\partial u} = 0, \quad \dot{q} = \frac{\partial \hat{H}}{\partial p} = X(q, u), \quad \dot{p} = -\frac{\partial \hat{H}}{\partial q}. \tag{2.3}$$

If $\partial \hat{H}/\partial u = 0$ can be solved for $u = u(\alpha)$, then we define $H(\alpha) := \hat{H}(\alpha, u(\alpha))$ and (2.2) become the usual Hamilton equations for H. This happens locally, for example, if $\hat{H}$ is of class C^2 and $\partial^2 \hat{H}/\partial u^2 : U \to U^*$, computed at every point of U, is an isomorphism. If X is linear in u and g is strictly convex in u, then this holds at every point. It is easily seen that equations (2.3) can be obtained by the variational principle

$$\delta \int_0^T \left(\hat{H}(\alpha_q, u) - \langle \alpha_q, \dot{q} \rangle \right) dt = 0.$$

We define below a class of optimal control problems associated to group actions that will be the main object of study for this paper. A slight generalization of this optimal control problem will be considered in §7.

Definition 2.1 *Let Φ be a (left or right) action of a Lie group G on a manifold Q. For a Lie algebra element $u \in \mathfrak{g}$ let*

$$u_Q(q) := \left. \frac{d}{dt} \right|_{t=0} \Phi_{\exp(tu)}(q)$$

denote the corresponding infinitesimal generator of the action. Given a cost function $\ell : \mathfrak{g} \to \mathbb{R}$, the ***Clebsch optimal control problem*** *is, by definition*

$$\min_{u(t)} \int_0^T \ell(u(t)) dt \tag{2.4}$$

subject to the following conditions:

(A) $\dot{q}(t) = u(t)_Q(q(t))$ *or* (A)′ $\dot{q}(t) = -u(t)_Q(q(t))$*;*

(B) $q(0) = q_0$ *and* $q(T) = q_T$*.*

The case when (A)$'$ is assumed instead of (A) is referred to as the *inverse representation*. Note that the Clebsch optimal control problem (2.4) is obtained from the general problem (2.1) by choosing the space of control variables $U = \mathfrak{g}$, the cost function $g(q,u) = \ell(u)$, and the vector field $X(q,u) = u_Q(q)$. Thus, X is linear in u. The Pontryagin function reads

$$\hat{H}(\alpha_q, u) = \pm\langle \alpha_q, u_Q(q)\rangle - \ell(u),$$

where the sign depends on whether one chooses (A) or (A)$'$. The function $\hat{H}$ can be rewritten in terms of the momentum map $\mathbf{J} : T^*Q \to \mathfrak{g}^*$ associated to the cotangent-lifted action on T^*Q as

$$\hat{H}(\alpha_q, u) = \pm\langle \mathbf{J}(\alpha_q), u\rangle - \ell(u).$$

Note that ℓ is strictly convex if and only if $\hat{H}$ is strictly concave in u. In this case, the unique critical point of $\hat{H}$ relative to u and keeping α fixed is necessarily a maximum of $\hat{H}(\alpha, \cdot)$. From now on we suppose that all the Lagrangians we consider are strictly convex in u.

We will denote by $\Phi_g^{T^*} := T^*\Phi_{g^{-1}}$ the cotangent lifted action of Φ_g on T^*Q. The equations giving the extremal solution of the Clebsch optimal control problem are determined in the following theorem.

Theorem 2.2 *Assume that $u \in \mathfrak{g} \mapsto \frac{\delta \ell}{\delta u} \in \mathfrak{g}^*$ is a diffeomorphism. Let G act on the left (resp. on the right) on Q. Then, an extremal solution of the Clebsch optimal control problem* (2.4) *with condition* (A) *is a solution of*

$$\frac{\delta \ell}{\delta u} = \mathbf{J}(\alpha), \quad \dot{\alpha} = u_{T^*Q}(\alpha).$$

The first equation can be equivalently written $u = \frac{\delta h}{\delta \mathbf{J}}$, where $h : \mathfrak{g}^* \to \mathbb{R}$ is the Hamiltonian associated to ℓ by Legendre transformation. Moreover, the solution reads $\alpha(t) = \Phi^{T^*}_{g(t)}(\alpha(0))$, where

$$\dot{g}(t)g(t)^{-1} = u(t), \quad \textit{resp.} \quad g(t)^{-1}\dot{g}(t) = u(t).$$

These equations imply the Euler-Poincaré equations (see Remark 2.5 below)

$$\frac{d}{dt}\frac{\delta \ell}{\delta u} = -\operatorname{ad}^*_u \frac{\delta \ell}{\delta u}, \quad \textit{resp.} \quad \frac{d}{dt}\frac{\delta \ell}{\delta u} = \operatorname{ad}^*_u \frac{\delta \ell}{\delta u}$$

for the control u.

Proof. We give the proof in the case of a left action. The first equation in (2.3) gives the condition

$$\frac{\delta \ell}{\delta u} = \mathbf{J}(\alpha).$$

on the optimal control u, which can be solved to give $u = f(\alpha)$. We now compute Hamilton's equations associated to the Hamiltonian $\hat{H}_u : T^*Q \to \mathbb{R}$. As is well known (see e.g., Marsden and Ratiu [1999], Proposition 12.1.3), the Hamiltonian vector field associated to the momentum function $\mathcal{P}(u)(\alpha_q) = \langle \alpha_q, u_Q(q)\rangle$ is

$$X_{\mathcal{P}(u)}(\alpha) = u_{T^*Q}(\alpha),$$

where u_{T^*Q} is the infinitesimal generator of the cotangent lifted action $\Phi_g^{T^*} := T^*\Phi_{g^{-1}}$ of G on T^*Q.

A solution $\alpha(t)$ of $\dot{\alpha} = u_{T^*Q}(\alpha)$ is necessarily of the form $\alpha(t) = \Phi^{T^*}_{g(t)}(\alpha(0))$, where $g(0) = e$ and $\dot{g}(t)g(t)^{-1} = u(t)$. Therefore, since u is a function of α, we get $\dot{g}(t)g(t)^{-1} = f(\alpha(t)) = f\left(\Phi^{T^*}_{g(t)}(\alpha(0))\right)$ which is an ordinary differential equation for $g(t)$. We take the unique solution of this equation with initial condition $g(0) = e$. We thus obtain

$$\frac{\delta \ell}{\delta u(t)} = \mathbf{J}(\alpha(t)) = \mathbf{J}\left(\Phi^{T^*}_{g(t)}(\alpha(0))\right) = \operatorname{Ad}^*_{g(t)^{-1}} \mathbf{J}(\alpha(0)),$$

and by differentiating with respect to time, we get the Euler-Poincaré equations.

$$\frac{d}{dt}\frac{\delta \ell}{\delta u(t)} = -\operatorname{ad}^*_{\dot{g}(t)g(t)^{-1}} \frac{\delta \ell}{\delta u(t)} = -\operatorname{ad}^*_{u(t)} \frac{\delta \ell}{\delta u(t)}. \qquad \blacksquare$$

We now give the same theorem but in the case of the inverse representation, that is, when (A)$'$ is assumed instead of (A).

Theorem 2.3 (Inverse representation) *Assume that* $u \in \mathfrak{g} \mapsto \frac{\delta \ell}{\delta u} \in \mathfrak{g}^*$ *is a diffeomorphism. Let* G *act on the left (resp. on the right) on* Q*. Then, an extremal solution of the Clebsch optimal control problem* (2.4) *with condition* (A)$'$ *is a solution of*

$$\frac{\delta \ell}{\delta u} = -\mathbf{J}(\alpha), \quad \dot{\alpha} = -u_{T^*Q}(\alpha).$$

The first equation can be equivalently written $u = \frac{\delta h}{\delta(-\mathbf{J})}$*, where* $h : \mathfrak{g}^* \to \mathbb{R}$ *is the Hamiltonian associated to* ℓ *by Legendre transformation. Moreover, the solution reads* $\alpha(t) = \Phi^{T^*}_{g(t)^{-1}}(\alpha(0))$*, where*

$$g(t)^{-1}\dot{g}(t) = u(t) \quad \textit{resp.} \quad \dot{g}(t)g(t)^{-1} = u(t).$$

These equations imply the Euler-Poincaré equations

$$\frac{d}{dt}\frac{\delta \ell}{\delta u} = \operatorname{ad}^*_u \frac{\delta \ell}{\delta u}, \quad \textit{resp.} \quad \frac{d}{dt}\frac{\delta \ell}{\delta u} = -\operatorname{ad}^*_u \frac{\delta \ell}{\delta u}.$$

Remark 2.4 Note that the solution $q(t)$ of the Clebsch optimal control problem (2.4) necessarily evolves in a G-orbit. Therefore, we shall assume that q_0 and q_T belong to the same G-orbit, in order to have a well posed problem.

Remark 2.5 (Euler-Poincaré and Lie-Poisson) Recall that the *Euler-Poincaré equations* are obtained by Lagrangian reduction of the Euler-Lagrange equations associated to a Lagrangian $L : TG \to \mathbb{R}$ defined on the tangent bundle of G and invariant under the tangent lift of left (resp. right) translations. A curve $g(t) \in G$ is a solution of the Euler-Lagrange equation for L if and only if the curve $\xi(t) = g(t)^{-1}\dot{g}(t)$ (resp. $\xi(t) = \dot{g}(t)g(t)^{-1}$) is a solution of the Euler-Poincaré equation for the reduced Lagrangian $\ell := L|_{\mathfrak{g}} : \mathfrak{g} \to \mathbb{R}$. The same procedure for the Hamiltonian side leads to the so called *Lie-Poisson equations*

$$\dot{\mu} = -\operatorname{ad}_{\frac{\delta h}{\delta \mu}} \mu, \quad \text{resp.} \quad \dot{\mu} = -\operatorname{ad}_{\frac{\delta h}{\delta \mu}} \mu,$$

associated to a right (resp. left) invariant Hamiltonian H on T^*G. If $u \mapsto \frac{\delta \ell}{\delta u}$ is a diffeomorphism (which we assume throughout the paper), then Euler-Poincaré and Lie-Poisson equations are equivalent, since one can pass from one to the other by the Legendre transformation

$$h(\mu) = \langle \mu, \xi \rangle - l(\xi), \qquad \mu = \frac{\delta \ell}{\delta \xi}.$$

The collective Hamiltonian. Recall that the Hamiltonian H is defined by $H(\alpha) = \hat{H}(\alpha, u(\alpha))$ where the optimal control $u(\alpha)$ is uniquely determined by the condition $\delta\ell/\delta u = \mathbf{J}(\alpha)$. We thus obtain

$$H(\alpha) = \left\langle \frac{\delta \ell}{\delta u}, u \right\rangle - \ell(u) = h\left(\frac{\delta \ell}{\delta u}\right) = h(\mathbf{J}(\alpha)), \tag{2.5}$$

where $h : \mathfrak{g}^* \to \mathbb{R}$ is the Hamiltonian associated to ℓ via the Legendre transformation $u \mapsto \delta\ell/\delta u$. The Hamiltonian H is thus the *collective Hamiltonian* associated to h via the cotangent lifted action of G on T^*Q. When the constraint (A)$'$ is chosen instead of (A), the Hamiltonian is $H(\alpha) = h(-\mathbf{J}(\alpha))$. An important example is $\ell(u) = \frac{1}{2}\|u\|^2$, where the norm is associated to a inner product on $\mathfrak{g}$. In this case, identifying the dual Lie algebra $\mathfrak{g}^*$ with $\mathfrak{g}$ via the inner product, we have

$$H(\alpha) = \frac{1}{2}\|\mathbf{J}(\alpha)\|^2.$$

The case of an Ad-invariant Lagrangian. Let $\ell : \mathfrak{g} \to \mathbb{R}$ be a Lagrangian such that $u \in \mathfrak{g} \mapsto \delta\ell/\delta u \in \mathfrak{g}^*$ is a diffeomorphism. If ℓ is invariant under the adjoint representation then, by equivariance of the momentum map $\mathbf{J}$, the Hamiltonian H is G-invariant. This happens, for example, when ℓ is given by the kinetic energy associated to an Ad-invariant positive definite inner product on $\mathfrak{g}$. Examples of such inner products are given by (minus) the Killing form on compact semisimple Lie algebras.

Restriction to G-orbits, geodesics, and the normal metric. We now make some simple observations that will turn out to be crucial in examples. We first note that the canonical Hamilton equations $\dot{\alpha} = u_{T^*Q}(\alpha)$ on T^*Q induce canonical equations on $T^*\mathcal{O}$, where $\mathcal{O} := \{\Phi_g(q) \mid g \in G\}$ is an orbit of the G-action on Q. Recall that, given a Lie group G and a Lagrangian $\ell : \mathfrak{g} \to \mathbb{R}$, Theorems 2.2 and 2.3 is valid for any manifold Q acted on by G. Therefore, in the Clebsch optimal control problem (2.4), one can always choose the manifold Q to be a G-orbit $\mathcal{O}$ in Q and obtain the same conclusions. In particular, the solution of the Euler-Poincaré equations for ℓ are obtained by solving the canonical Hamilton's equations

$$\dot{\alpha} = u_{T^*\mathcal{O}}(\alpha)$$

on $T^*\mathcal{O}$. Such a result was clearly expected, since the solution α of Hamilton's equations $\dot{\alpha} = u_{T^*Q}(\alpha)$ are of the form $\alpha(t) = \Phi^{T^*}_{g(t)}(\alpha_0)$. Recall that the infinitesimal generator $u_{T^*\mathcal{O}} \in \mathfrak{X}(T^*\mathcal{O})$ is the Hamiltonian vector field associated to the canonical symplectic form $\Omega_{\mathcal{O}}$ on $T^*\mathcal{O}$ and to the momentum function $\mathcal{P}(u) \in \mathcal{F}(T^*\mathcal{O})$ defined by $\mathcal{P}(u)(\alpha_q) := \langle \alpha_q, \xi_{\mathcal{O}}(q)\rangle$.

We now consider the particular case when ℓ is the kinetic energy associated to a positive definite inner product γ on $\mathfrak{g}$. Given $q \in Q$, let $\mathfrak{g}_q := \{\xi \in \mathfrak{g} \mid \xi_Q(q) = 0\}$ denote the isotropy Lie algebra of q. Using the inner product γ on $\mathfrak{g}$ we have the orthogonal decomposition $\mathfrak{g} = \mathfrak{g}_q \oplus \mathfrak{g}_q^\perp$, and we denote by $\xi = \xi_q + \xi^q$ the associated splitting of $\xi \in \mathfrak{g}$ in this direct sum. With these notation we can state the following definition.

Definition 2.6 *Let $\mathcal{O}$ be an orbit of an action $\Phi : G \times Q \to Q$ of Lie group G on a manifold Q and let γ be a positive definite inner product on the Lie algebra $\mathfrak{g}$. The associated* ***normal metric*** *$\gamma_{\mathcal{O}}$ is defined by*

$$\gamma_{\mathcal{O}}(\xi_{\mathcal{O}}(q), \eta_{\mathcal{O}}(q)) := \gamma(\xi^q, \eta^q), \tag{2.6}$$

for all $q \in \mathcal{O}$.

Note that if $\xi_Q(q) = \zeta_Q(q)$ for two Lie algebra elements $\xi, \zeta \in \mathfrak{g}$, then $\xi^q = \zeta^q$, therefore, $\gamma_{\mathcal{O}}$ is well-defined by the above formula. One easily checks that $\gamma_{\mathcal{O}}$ is indeed a Riemannian metric on $\mathcal{O}$. It is called a normal metric since it recovers the usual normal metric on adjoint orbits on compact Lie algebras, as we will see below. Note that if the action is locally free, then we simply have $\gamma_{\mathcal{O}}(\xi_{\mathcal{O}}(q), \eta_{\mathcal{O}}(q)) = \gamma(\xi, \eta)$, since $\mathfrak{g}_q = \{0\}$.

We now compute the associated Hamiltonian H on $T^*\mathcal{O}$. Denoting by $\sharp$ the usual index raising operator associated to γ and $\gamma_{\mathcal{O}}$, we have

$$\begin{aligned} 2H(\alpha_q) &= \|\mathbf{J}(\alpha_q)\|^2 = \gamma\left(\mathbf{J}(\alpha_q)^\sharp, \mathbf{J}(\alpha_q)^\sharp\right) = \left\langle \mathbf{J}(\alpha_q), \mathbf{J}(\alpha_q)^\sharp \right\rangle \\ &= \left\langle \alpha_q, (\mathbf{J}(\alpha_q)^\sharp)_Q(q) \right\rangle = \left\langle \alpha_q, \alpha_q^\sharp \right\rangle = \gamma_{\mathcal{O}}(\alpha_q^\sharp, \alpha_q^\sharp), \end{aligned}$$

where we used the equality $(\mathbf{J}(\alpha_q)^\sharp)_Q(q) = \alpha_q^\sharp$ that can be checked as follows. For all $\eta \in \mathfrak{g}$ we have

$$\begin{aligned} \gamma_{\mathcal{O}}\left((\mathbf{J}(\alpha_q)^\sharp)_Q(q), \eta_Q(q)\right) &= \gamma((\mathbf{J}(\alpha_q)^\sharp)^q, \eta^q) = \gamma(\mathbf{J}(\alpha_q)^\sharp, \eta^q) = \langle \mathbf{J}(\alpha_q), \eta^q \rangle \\ &= \langle \alpha_q, \eta_Q(q) \rangle = \gamma_{\mathcal{O}}(\alpha_q^\sharp, \eta_Q(q)). \end{aligned}$$

Of course, H can be seen as the Hamiltonian associated to the Lagrangian $L : T\mathcal{O} \to \mathbb{R}$ associated to the Riemannian metric $\gamma_{\mathcal{O}}$. Note that we have the equality $H(\alpha_q) = \|\mathbf{J}(\alpha_q)\|^2/2$ on the whole cotangent bundle T^*Q, but it is only on $T^*\mathcal{O}$ that H corresponds to the kinetic energy of a nondegenerate metric. Indeed, the kernel of $\mathbf{J}|_{T_q^*Q}$ is given by the annihilator $(T_q\mathcal{O})^\circ$ in T_q^*Q of the tangent space to the orbit.

Note that if the inner product γ is Ad-invariant, then we have

$$\begin{aligned} \gamma_{\mathcal{O}}(T\Phi_g(\xi_Q(q)), T\Phi_g(\xi_Q(q))) &= \gamma_{\mathcal{O}}\left((\mathrm{Ad}_g\,\xi)_Q(\Phi_g(q)), (\mathrm{Ad}_g\,\xi)_Q(\Phi_g(q))\right) \\ &= \gamma\left((\mathrm{Ad}_g\,\xi)^{\Phi_g(q)}, (\mathrm{Ad}_g\,\xi)^{\Phi_g(q)}\right) \\ &= \gamma\left(\mathrm{Ad}_g(\xi^q), \mathrm{Ad}_g(\xi^q)\right) \\ &= \gamma(\xi^q, \eta^q) = \gamma_{\mathcal{O}}(\xi_Q(q), \eta_Q(q)), \end{aligned}$$

therefore, the normal metric $\gamma_{\mathcal{O}}$ is G-invariant. This is just a particular case of the observation we made earlier, that an Ad-invariant Lagrangian ℓ induces G-invariant Hamiltonian H.

The above discussion is summarized as follows.

Theorem 2.7 *Let G act on the left (resp. on the right) on Q and let $\ell : \mathfrak{g} \to \mathbb{R}$ a Lagrangian. Then, if the points q_0 and q_T belong to the same G-orbit $\mathcal{O}$, then problem* (2.4) *restricts to a Clebsch optimal control problem on the orbit $\mathcal{O}$, with the same extremal solution.*

For the particular case where γ is a positive definite inner product on $\mathfrak{g}$ and ℓ is the Lagrangian given by the kinetic energy, an extremal solution of the Clebsch optimal control problem (2.4) *with condition* (A) *is given by $u(t) = \mathbf{J}(\alpha(t))^\sharp$, where $\alpha(t)^\sharp = \dot{q}(t)$ and $q(t)$ is a geodesic on a G-orbit $\mathcal{O} \subset Q$, relative to the normal Riemannian metric $\gamma_{\mathcal{O}}$ on $\mathcal{O}$ defined by*

$$\gamma_{\mathcal{O}}(\xi_Q(q), \eta_Q(q) := \gamma\left(\xi^q, \eta^q\right).$$

Moreover, since $\alpha(t) = \Phi^{T^*}_{g(t)}(\alpha(0))$, the geodesic is $q(t) = \Phi_{g(t)}(q(0))$, where

$$\dot{g}(t)g(t)^{-1} = u(t), \quad \textit{resp.} \quad g(t)^{-1}\dot{g}(t) = u(t).$$

Remarkably, by Euler-Poincaré theory, the curve $g(t)$ itself is known to be a geodesic on G relative to the right (resp. left) invariant metric induced on G by γ.

In the case of condition (A)$'$, *the optimal control is given by $u(t) = -\mathbf{J}(\alpha(t))^\sharp$, where $\alpha(t)^\sharp = \dot{q}(t)$ and $q(t)$ is a geodesic on a G-orbit $\mathcal{O} \subset Q$, relative to the normal Riemannian metric $\gamma_{\mathcal{O}}$ on $\mathcal{O}$ defined as before by*

$$\gamma_{\mathcal{O}}(\xi_Q(q), \eta_Q(q) := \gamma\left(\xi^q, \eta^q\right).$$

Moreover, since $\alpha(t) = \Phi^{T^*}_{g(t)^{-1}}(\alpha(0))$, the geodesic is $q(t) = \Phi_{g(t)^{-1}}(q(0))$, where

$$g(t)^{-1}\dot{g}(t) = u(t), \quad \textit{resp.} \quad \dot{g}(t)g(t)^{-1} = u(t).$$

Remarkably, by Euler-Poincaré theory, the curve $g(t)$ itself is known to be a geodesic on G relative to the left (resp. right) invariant metric induced on G by γ.

3 The case of subgroup actions

We now specialize Theorems 2.2 and 2.3 to the case where Q is a Lie group H, containing G as a subgroup and the action is given by multiplication $G \times H \to H$. Of course, one can choose $H = G$. Given $u \in \mathfrak{g}$, the infinitesimal generator associated to left multiplication by G on H reads $u_H(q) = T_e R_q(u) =: uq$, and for right multiplication we have $u_H(q) = T_e L_q(u) =: qu$. This trivial setting will be important to understand geometrically the symmetric representation for rigid bodies and fluids.

Given $q_0, q_T \in H$, the Clebsch optimal control problem is

$$\min_{u(t)} \int_0^T \ell(u(t))dt$$

subject to the following conditions:

(A) $\dot{q}(t) = u(t)q(t)$, resp. $\dot{q}(t) = q(t)u(t)$;

(B) $q(0) = q_0$ and $q(T) = q_T$.

The associated variational principles are

$$\delta \int_0^T \left(\langle \alpha_q, uq - \dot{q} \rangle - \ell(u) \right) dt = 0, \quad \text{resp.} \quad \delta \int_0^T \left(\langle \alpha_q, qu - \dot{q} \rangle - \ell(u) \right) dt = 0, \tag{3.1}$$

the optimal control u is given by

$$\frac{\delta \ell}{\delta u} = \mathbf{J}(\alpha_q) = i^* \left(\alpha_q q^{-1} \right), \quad \text{resp.} \quad \frac{\delta \ell}{\delta u} = \mathbf{J}(\alpha_q) = i^* \left(q^{-1} \alpha_q \right), \tag{3.2}$$

where $i^* : \mathfrak{h}^* \to \mathfrak{g}^*$ is the dual map to the inclusion, and the canonical Hamilton's equations on T^*H read

$$\dot{\alpha} = u\alpha, \quad \text{resp.} \quad \dot{\alpha} = \alpha u. \tag{3.3}$$

The solution can be written $\alpha(t) = g(t)\alpha_0$, resp. $\alpha(t) = \alpha_0 g(t)$, where the concatenation means the cotangent-lift of translation by G on H. In particular, for the base curves, we have $q(t) = g(t)q_0$, resp. $q(t) = q_0 g(t)$. If the initial condition α_0 lies at the identity, then we have $q(t) = g(t)$. When u verifies the optimal control condition $\frac{\delta \ell}{\delta u} = \mathbf{J}(\alpha)$, then equations (3.3) are the canonical Hamilton's equation of the collective Hamiltonian $H(\alpha) = h(\mathbf{J}(\alpha))$, where h is the Hamiltonian associated to ℓ.

We now examine the statements of Theorem 2.7 in this particular case. For left (resp. right) translation by G, the orbits are $\mathcal{O}_q = \{gq \mid g \in G\}$ (resp. $\mathcal{O}_q = \{qg \mid g \in G\}$), where $q \in H$ is fixed. Hamilton's equations (3.3) restrict to canonical Hamilton's equations on any G-orbit $\mathcal{O}_q$. For the particular case when the orbit is the subgroup G, then these Hamilton's equations coincide with the usual canonical Hamilton's equations on T^*G associated to h by right (resp. left) Lie-Poisson reconstruction. Indeed, the momentum map $\mathbf{J} : T^*G \to \mathfrak{g}^*$ associated to left (resp. right) translations is also the projection associated to right (resp. left) Lie-Poisson reduction.

We now consider the particular case when ℓ is the kinetic energy of the inner product γ. Since the action is free, the normal metric defined in (2.6) reads

$$\gamma_{\mathcal{O}_q}(uf, vf) = \gamma(u, v) \quad \text{resp.} \quad \gamma_{\mathcal{O}_q}(fu, fv) = \gamma(u, v), \tag{3.4}$$

where $f \in \mathcal{O}_q$. By Theorem 2.7, if the Lagrangian is given by the kinetic energy of γ, then the base curves $q(t) = g(t)q_0$, resp. $q(t) = q_0 g(t)$ are geodesics on $\mathcal{O}_{q_0}$ with respect to $\gamma_{\mathcal{O}_q}$. This is coherent with the Euler-Poincaré interpretation saying that $g(t)$ is a geodesic on G with respect to the G-invariant metric induced by γ. In the particular case when the orbit $\mathcal{O}_q$ is the subgroup G, the normal metric is the right (resp. left) invariant extension of γ to G. In this case, the two interpretations of geodesics coincide, as remarked before for an arbitrary Lagrangian.

We now quickly give the formulas in the case of the inverse representation, that is, if condition (A) is replaced by the condition (A)′. In the case of left (resp. right) multiplication by the subgroup G of H, condition (A)′ reads

$$\dot{q} = -uq, \quad \text{resp.} \quad \dot{q} = -qu.$$

The variational principles becomes

$$\delta \int_0^T \left(\langle \alpha_q, uq + \dot{q} \rangle + \ell(u) \right) dt = 0, \quad \text{resp.} \quad \delta \int_0^T \left(\langle \alpha_q, qu + \dot{q} \rangle + \ell(u) \right) dt = 0, \tag{3.5}$$

the optimal control u is given by

$$\frac{\delta \ell}{\delta u} = -\mathbf{J}(\alpha_q) = -i^* \left(\alpha_q q^{-1} \right), \quad \text{resp.} \quad \frac{\delta \ell}{\delta u} = -\mathbf{J}(\alpha_q) = -i^* \left(q^{-1} \alpha_q \right) \tag{3.6}$$

and the canonical Hamilton's equations are

$$\dot{\alpha} = -u\alpha, \quad \text{resp.} \quad \dot{\alpha} = -\alpha u. \tag{3.7}$$

However, in this case, if the initial condition α_0 lies at the identity, then the curves $q(t)$ and $g(t)$ of Theorem 2.3 are related by

$$q(t) = g(t)^{-1}.$$

This explains why condition (A)' in the Clebsch optimal control problem is referred to as the "inverse representation". As we will see below, this point of view is important for fluids. When u verifies the optimal control condition $\frac{\delta \ell}{\delta u} = -\mathbf{J}(\alpha)$, then equations (3.7) are the canonical Hamilton's equation of the collective Hamiltonian $H(\alpha) = h(-\mathbf{J}(\alpha))$, where h is the Hamiltonian associated to ℓ.

We now examine the statements of Theorem 2.7 in this particular case. As before, Hamilton's equations (3.7) restrict to canonical Hamilton's equations on any G-orbit $\mathcal{O}_q$. However, when the orbit is the subgroup G, these Hamilton's equations do not coincide with the usual canonical Hamilton's equations on T^*G associated to h by Lie-Poisson reconstruction. In fact we obtain what we call "Lie-Poisson reduction in inverse representation" that we describe in Remark 3.1 below.

In particular, when ℓ is the kinetic energy of γ, then the normal metric on orbit is given as before by

$$\gamma_{\mathcal{O}_q}(uf, vf) = \gamma(u, v) \quad \text{resp.} \quad \gamma_{\mathcal{O}_q}(fu, fv) = \gamma(u, v), \tag{3.8}$$

When the orbit is the subgroup G, the normal metric is the left (resp. right) invariant extension of γ to the group G. This differs from the Euler-Poincaré interpretation saying that $g(t)$ is a geodesic for the right (resp. left) invariant extension of γ, but is consistent since the geodesics are respectively given by $q(t)$ and $g(t)$, where $q(t) = g(t)^{-1}$ (modulo multiplication by a constant in G depending on the initial).

Remark 3.1 (Euler-Poincaré reduction using the inverse map) Consider the left (resp. right) Euler-Poincaré equations associated to a Lagrangian $\ell : \mathfrak{g} \to \mathbb{R}$:

$$\frac{d}{dt} \frac{\delta \ell}{\delta u} = \operatorname{ad}_u^* \frac{\delta \ell}{\delta u}, \quad \text{resp.} \quad \frac{d}{dt} \frac{\delta \ell}{\delta u} = -\operatorname{ad}_u^* \frac{\delta \ell}{\delta u}. \tag{3.9}$$

A physical example of an equation in this form is the Euler equation for the free rigid body (resp. the Euler equation for ideal flows). The natural approach to these equations is to consider them as induced by the Euler-Lagrange equations associated to the *left*-invariant (resp. *right*-invariant) Lagrangian L on TG defined by ℓ. As we have seen, $g(t)$ is a solution of L if and only if $u(t) = g(t)^{-1}\dot{g}(t)$ (resp. $u(t) = \dot{g}(t)g(t)^{-1}$) is a solution of ℓ.

We now force the Euler-Poincaré equations to come from a right (resp. left) invariant Lagrangian $\bar{L}$ on TG. For doing this, it suffices to make the change of variable $v := -u$ and $\bar{\ell}(v) := \ell(-v)$, and the left (resp. right) Euler-Poincaré equations (3.9) turn out to be equivalent to the right (resp. left) Euler-Poincaré equations

$$\frac{d}{dt}\frac{\delta\bar{\ell}}{\delta v} = -\operatorname{ad}_v^*\frac{\delta\bar{\ell}}{\delta v}, \quad \text{resp.} \quad \frac{d}{dt}\frac{\delta\bar{\ell}}{\delta v} = \operatorname{ad}_v^*\frac{\delta\bar{\ell}}{\delta v}.$$

These equations are now the reduced version of the Euler-Lagrange associated to the *right*-invariant (resp. *left*-invariant) Lagrangian $\bar{L}$ induced by $\bar{\ell}$ on G.

In conclusion, the solution u of (3.9) has the two following interpretations: $u = g^{-1}\dot{g}$ (resp. $u = \dot{g}g^{-1}$), where g is the solution of Euler-Lagrange equations for L; or $u = -\dot{l}l^{-1}$ (resp. $u = -l^{-1}\dot{l}$), where l is the solution of Euler-Lagrange equations for $\bar{L}$. If the curve g and l start at the identity e, then we have $l = g^{-1}$. In the particular case where ℓ is the kinetic energy of a positive definite inner product γ, then g is a geodesic for the left-invariant (resp. right-invariant) metric associated to γ whereas l is a geodesic for the corresponding right-invariant (resp. left-invariant) metric. Note that in this case we have $\bar{\ell} = \ell$.

As we will see below, this seemingly unnatural way to interpret the Euler-Poincaré equations turns out to be important for fluids, and allows one to work with the back-to-label map, instead of the usual Lagrangian map.

The N-dimensional free rigid body. We now apply the above results to the Lie groups $G = SO(N)$ and $H = GL(N)$, and recover the *symmetric representation of the N-rigid body*, see Bloch and Crouch [1996], Bloch, Brockett, and Crouch [1997], and Bloch, Crouch, Marsden, and Ratiu [1998]. One can also consider $G = SO(N)$ acting on $Q = \mathfrak{gl}(N)$ by matrix multiplication on the right.

A vector in $TSO(N) \subset TGL(N) = GL(N) \times \mathfrak{gl}(N)$ is of the form (Q, V), where $Q \in SO(N)$, $V = QU$, and $U \in \mathfrak{so}(N)$. We identify the cotangent and tangent bundles via the pairing

$$\langle P, V\rangle := \operatorname{Tr}(P^T V),$$

which turns out to be a bi-invariant Riemannian metric on $SO(N)$ (in fact the extension of the Killing form on $\mathfrak{so}(N)$). Given $Q_0, Q_T \in GL(N)$, the corresponding optimal control problem is

$$\min_{U(t)} \int_0^T \ell(U(t))dt \tag{3.10}$$

subject to the following conditions:

(A) $\dot{Q}(t) = Q(t)U(t)$

(B) $Q(0) = Q_0$ and $Q(T) = Q_T$.

Note that this corresponds to the *right* action of $SO(N)$ on $GL(N)$, although the rigid body is usually considered as *left* invariant. This is consistent with Theorem 2.2, where *right* cotangent lifted actions produce the *left* Euler-Poincaré equations. With respect to the pairing defined above, the cotangent bundle momentum map $\mathbf{J} : T^*GL(N) \to \mathfrak{so}(N)^*$ associated to right multiplication is

$$\mathbf{J}(Q,P) = \frac{1}{2}\left(Q^T P - P^T Q\right),$$

thus, in view of (3.2), the optimal control U is found by solving the equation

$$\frac{\delta \ell}{\delta U} = \frac{1}{2}\left(Q^T P - P^T Q\right).$$

Note that if $(Q,P) \in T^*SO(N)$, then we have $\mathbf{J}(Q,P) = \frac{1}{2}\left(Q^T P - P^T Q\right) = Q^{-1}P$, and we recover the usual momentum map associated to reduction on the left. Hamilton's equations (3.3) are given by

$$\dot{Q} = QU, \quad \dot{P} = PU, \tag{3.11}$$

and the optimal control U verifies the Euler-Poincaré equations

$$\frac{d}{dt}\frac{\delta \ell}{\delta U} = \left[\frac{\delta \ell}{\delta U}, U\right].$$

For the particular case of the N-dimensional rigid body, the Lagrangian ℓ is given by

$$\ell(U) := \frac{1}{2}\int_{\mathcal{B}} \rho(X)\|UX\|^2 d^N X,$$

where $\mathcal{B} \subset \mathbb{R}^N$, a compact set with nonempty interior, is the reference configuration of the body, and $\rho(X)$ is the mass density. This Lagrangian is of course obtained by a direct generalization of the Lagrangian of the three dimensional case. We can rewrite

$$\ell(U) = \frac{1}{4}\langle \mathcal{J}(U), U\rangle,$$

where $\mathcal{J} : \mathfrak{so}(N) \to \mathfrak{so}(N)$ is the symmetric and positive definite operator given by

$$\mathcal{J}(U) = JU + UJ, \quad \text{with} \quad J_{ij} = \int_{\mathcal{B}} X_i X_j \rho(X) d^N X.$$

In this case we have $\delta\ell/\delta U = \mathcal{J}(U)/2$ and the optimal control is given by $M = Q^T P - P^T Q$, where $M = \mathcal{J}(U)$ is the body angular momentum. In particular, the Euler-Poincaré equations recover the Lax equation $\dot{M} = [M,U]$ for the N-rigid body. From the general result given in Theorem 2.2, we know that if Q, P are solutions of Hamilton's equations (3.11), with $U = \mathcal{J}^{-1}\left(Q^T P - P^T Q\right)$, then U is a solution of the equations $\dot{M} = [M,U]$ for the N-dimensional rigid body. It is well known that the N-dimensional rigid body is a completely integrable Hamiltonian system on generic coadjoint orbits of $SO(N)$ (Manakov [1976]; Mishchenko and Fomenko [1976, 1982]; Ratiu [1980]).

When $N = 3$, we can identify $\mathfrak{so}(3)$ with $(\mathbb{R}^3, \times)$ via the map $U \mapsto \mathbf{U}$, defined by $U_{ij} = -\varepsilon_{ijk}\mathbf{U}_k$. In this case the vector associated to $\mathcal{J}(U) = JU + UJ$ is $\mathbf{IU}$, where the inertia tensor $\mathbf{I}$ is given by $\mathbf{I} = \mathrm{Tr}(J)I_3 - J$. We thus recover the Lagrangian $\ell(\mathbf{U}) = \frac{1}{2}\mathbf{IU}\cdot\mathbf{U}$ and the classical Euler equations $\dot{\mathbf{M}} = \mathbf{M}\times\mathbf{U}$, $\mathbf{M} = \mathbf{IU}$.

We now recall, using the Clebsch optimal control point of view, why equations (3.11) are called the *symmetric rigid body equations on* $SO(N) \times SO(N)$ (Bloch, Crouch, Marsden, and Ratiu [1998]). By the general result of Theorem 2.2, the solution of equations (3.11) lies on a $SO(N)$-orbit of the cotangent lifted action of $SO(N)$ on $T^*GL(N)$. Using the formula $(Q, P) \mapsto (Qg, Pg)$ for this cotangent lifted action, we note that if $Q(0), P(0) \in SO(N)$, then $Q, P \in SO(N)$ for all time, since $(Q, P) = (Q(0)g, P(0)g)$. Thus $SO(N) \times SO(N)$ is an invariant manifold.

From (2.5), the Hamiltonian for equations (3.11) on $T^*GL(N)$ reads $H(Q, P) = h(\mathbf{J}(Q, P)) = h\left(\frac{Q^TP-P^TQ}{2}\right)$. In the case of the rigid body, we get

$$H(P, Q) = \frac{1}{4}\left\langle \left(Q^TP - P^TQ\right), \mathcal{J}^{-1}\left(Q^TP - P^TQ\right)\right\rangle,$$

which agrees with (4.38) in Bloch, Brockett, and Crouch [1997].

From Theorem 2.7, we know that equations (3.11) are also Hamiltonian on the cotangent bundle of $SO(N)$-orbits in $GL(N)$, for any Lagrangian ℓ whose Legendre transform is a diffeomorphism. If Q_0, Q_T belong to the same orbit $\mathcal{O}$, then all extremal trajectories for the problem (3.10) are captured by the same problem restricted to the orbit $\mathcal{O}$.

If ℓ is given by a kinetic energy, which is the case for the N-rigid body, then Q is a geodesic on an $SO(N)$-orbit relative to the normal metric (3.4). When the $SO(N)$-orbit in $GL(N)$ is precisely the subgroup $SO(N)$ of $GL(N)$, this geodesic interpretation coincides with the usual Euler-Poincaré approach, since the normal metric coincides with the left-invariant metric induced by ℓ on $SO(N)$ (see (3.4)).

The results obtained in this paragraph coincide and generalize those of Bloch and Crouch [1996], Bloch, Brockett, and Crouch [1997], and Bloch, Crouch, Marsden, and Ratiu [1998], and are obtained here as a particular case of the general Clebsch optimal control problem studied in Theorem 2.2. The considerations for the N-dimensional rigid body can be easily extended to the the rigid body equations on semisimple Lie algebras considered by Mishchenko and Fomenko [1976, 1982].

The case of quadratic matrix Lie groups. We consider, as a generalization of $SO(N)$, quadratic matrix groups of the form

$$G := \{g \in GL(N) \mid g^TJg = J\},$$

where $J^2 = \alpha I_N$ and $J^T = \alpha J$, for $\alpha = \pm 1$. Optimal control on these groups were considered in Bloch, Crouch, Marsden, and Sanyal [2008] in order to generalize the symmetric representation of the rigid body flows. This class of groups includes the symplectic group and the group $O(p, q)$ of matrices that leave the nondegenerate, symmetric, bilinear form of signature (p, q) on $\mathbb{R}^n$ invariant. As before, we choose $Q = GL(N)$ and let the group G act on the right by translation. One can also choose $Q = \mathfrak{gl}(N)$ on which G acts on the right. Given $Q_0, Q_T \in GL(N)$, the corresponding Clebsch optimal control problem is

$$\min_{U(t)} \int_0^T \ell(U(t))dt$$

subject to the following conditions:

(A) $\dot{Q}(t) = Q(t)U(t)$;

(B) $Q(0) = Q_0$ and $Q(T) = Q_T$.

When ℓ is the kinetic energy of a positive definite inner product on $\mathfrak{g}$, this is precisely the optimal control problem studied in Bloch, Crouch, Marsden, and Sanyal [2008], formula (3.6). Using the orthogonal projection $\mathbb{P} : \mathfrak{gl}(N) \to \mathfrak{g}$, $\mathbb{P}(A) = \frac{1}{2}(A - \alpha J A^T J)$, the cotangent bundle momentum map $\mathbf{J} : T^*GL(N) \to \mathfrak{g}^*$ associated to right multiplication is

$$\mathbf{J}(Q,P) = \mathbb{P}(Q^T P) = \frac{1}{2}\left(Q^T P - \alpha J P^T Q J\right).$$

Thus, in view of (3.2), the optimal control U is found by solving the equation

$$\frac{\delta \ell}{\delta U} = \frac{1}{2}\left(Q^T P - \alpha J P^T Q J\right).$$

Using the expression $(Q,P) \mapsto \left(Qg, P(g^{-1})^T\right)$ for the cotangent lifted action of G on $GL(N)$, Hamilton's equations read

$$\dot{Q} = QU, \quad \dot{P} = -PU^T \tag{3.12}$$

and generalize the symmetric equations (3.11) of the N-rigid body. The solution is given by $(Q,P) = \left(Q(0)g, P(0)(g^{-1})^T\right)$. Since $g \in G$ it follows that $g^T \in G$ (see Lemma 2.1 in Bloch, Crouch, Marsden, and Sanyal [2008]), therefore, if $Q(0), P(0) \in G$, then $Q, P \in G$ for all times and the subset $G \times G$ is invariant under the flow of (3.12). From (2.5), the Hamiltonian for equations (3.12) is $H(Q,P) = h(\mathbf{J}(Q,P))$.

As for the N-rigid body, equations (3.12) are also Hamiltonian on the cotangent bundle of any G-orbit in $GL(N)$, and if Q_0 and Q_T belong to this orbit, then the optimal control problem can be restricted to it. However, the problem cannot be restricted to the invariant subset $G \times G$, since we may miss some of the extremal trajectories for the initial problem, as discussed in Bloch, Crouch, Marsden, and Sanyal [2008].

As before, when the Lagrangian ℓ is the kinetic energy of a positive definite inner product, then the flow of (3.12) describes geodesics on orbits, relative to the normal metric.

The case of matrix Lie groups. Let G be a Lie subgroup of $GL(N)$ and acting on the right by group multiplication. Given $Q_0, Q_T \in GL(N)$, the Clebsch optimal control problem is

$$\min_{U(t)} \int_0^T \ell(U(t))dt$$

subject to the following conditions:

(A) $\dot{Q}(t) = Q(t)U(t)$;

(B) $Q(0) = Q_0$ and $Q(T) = Q_T$.

This is the direct generalization of the cases mentioned before, to the case of an arbitrary matrix Lie group. Using the pairing $\langle A, B\rangle = \operatorname{Tr}(A^T B)$ and the orthogonal projection $\mathbb{P} : \mathfrak{gl}(N) \to \mathfrak{g}$, the cotangent momentum map $\mathbf{J} : T^*GL(N) \to \mathfrak{g}^*$ is given by

$$\mathbf{J}(Q,P) = \mathbb{P}(Q^T P).$$

Using the formula $(Q,P) \mapsto (Qg, P(g^{-1})^T)$ for the cotangent lift of right translation, we get Hamilton's equations

$$\dot{Q} = QU, \quad \dot{P} = -PU^T,$$

where the optimal control U is determined by $\delta\ell/\delta U = \mathbb{P}(Q^T P)$. The Hamiltonian is $H(Q,P) = h(\mathbb{P}(Q^T P))$ and the Euler-Poincaré equations for U are

$$\frac{d}{dt}\frac{\delta\ell}{\delta U} = \left[U^T, \frac{\delta\ell}{\delta U}\right].$$

Since the solution of Hamilton's equations is of the form $(Q,P) = (Q(0)g, P(0)(g^{-1})^T)$, we see that if the condition

$$g \in G \Rightarrow g^T \in G$$

holds for the matrix Lie group G, then $G \times G$ is preserved by the equations.

Optimal control for ideal flows. We apply the results above to the Lie group $H = \mathrm{Diff}(\mathcal{D})$ of all diffeomorphisms of the compact Riemannian manifold $\mathcal{D}$ with boundary and its subgroup $G = \mathrm{Diff}_{vol}(\mathcal{D})$ of volume preserving diffeomorphisms. We shall recover both the approaches given in Bloch, Crouch, Holm, and Marsden [2000] and Holm [2009] that appear here as particular cases of Theorem 2.2 and Theorem 2.3, respectively.

Recall that a curve $\eta_t \in \mathrm{Diff}_{vol}(\mathcal{D})$ represents the Lagrangian motion of an ideal fluid in the domain $\mathcal{D}$, that is, the curve $\eta_t(x)$ in $\mathcal{D}$ is the trajectory of the fluid particle located at x at time $t = 0$, assuming that η_0 is the identity; η_t is referred to as the *forward map.* The Lie algebra of G consists of divergence free vector fields on $\mathcal{D}$ parallel to the boundary and is denoted by $\mathfrak{g} = \mathfrak{X}_{vol}(\mathcal{D})$. The curve η_t is the flow of the Eulerian velocity $u_t \in \mathfrak{X}_{vol}(\mathcal{D})$, that is, we have $\dot{\eta}_t = u_t \circ \eta_t$. The curve $l_t := \eta_t^{-1}$ is called the *back-to-label map* and is related to the Eulerian velocity u_t via the relation $\dot{l}_t + Tl_t \cdot u_t = 0$. As is well known (Arnold [1966]), a curve $\eta_t \in \mathrm{Diff}_{vol}(\mathcal{D})$ is a geodesic with respect to the L^2 right invariant Riemannian metric if and only if u_t is a solution of the Euler fluid equations

$$\dot{u} + u \cdot \nabla u = -\operatorname{grad} p.$$

In other words, the Euler fluid equation is given by the Euler-Poincaré equation on $\mathfrak{X}_{vol}(\mathcal{D})$ associated to the Lagrangian $\ell(u) = \frac{1}{2}\int_{\mathcal{D}} \|u\|^2 = \frac{1}{2}\|u\|^2_{L^2}$.

First approach: We use Theorem 2.2 (left version) to obtain the optimal control formulation of Euler fluid equation using the forward map η_t. We let the group of volume preserving diffeomorphism act on the whole group of diffeomorphisms by composition on the left. The infinitesimal generator associated to $u \in \mathfrak{X}_{div}(\mathcal{D})$ is given by $u_{\mathrm{Diff}(\mathcal{D})}(\varphi) = u \circ \varphi$. Given two diffeomorphisms $\varphi_0, \varphi_T \in \mathrm{Diff}(\mathcal{D})$, the Clebsch optimal control problem is

$$\min_{u_t} \int_0^T \|u_t\|^2_{L^2} dt \tag{3.13}$$

subject to the following conditions:

(A) $\dot{\varphi}_t = u_t \circ \varphi_t$;

(B) $\varphi(0) = \varphi_0$ and $\varphi(T) = \varphi_T$.

This recovers the problem studied in Bloch, Crouch, Holm, and Marsden [2000]. We will use the L^2 pairing induced by the Riemannian metric on $\mathcal{D}$, to identify the tangent and cotangent bundles of the diffeomorphism groups. In this case, the cotangent momentum map $\mathbf{J} : T^* \operatorname{Diff}(\mathcal{D}) \to \mathfrak{X}_{vol}(\mathcal{D})^*$ associated to the cotangent-lift of left translation of $\operatorname{Diff}_{vol}(\mathcal{D})$ on $\operatorname{Diff}(\mathcal{D})$ is

$$\mathbf{J}(\varphi, \pi) = \mathbb{P}(J\varphi^{-1}(\pi \circ \varphi^{-1})) = J\varphi^{-1}(\pi \circ \varphi^{-1}) - \operatorname{grad} k,$$

where $\mathbb{P} : \mathfrak{X}(\mathcal{D}) \to \mathfrak{X}_{div}(\mathcal{D})$ is the (L^2-orthogonal) Helmholtz projector onto divergence free vector fields parallel to the boundary. The optimal control is thus given by $u = \mathbf{J}(\varphi, \pi) = \mathbb{P}(J\varphi^{-1}(\pi \circ \varphi^{-1}))$ and the canonical Hamilton's equations (3.3) on $T^* \operatorname{Diff}(\mathcal{D})$ are

$$\dot{\varphi} = u \circ \varphi, \quad \dot{\pi} = -(Tu \circ \varphi)^\dagger \pi, \tag{3.14}$$

where $\dagger$ means the transpose with respect to the Riemannian metric on $\mathcal{D}$. More precisely, if φ is a diffeomorphism, then $(T_x\varphi)^\dagger : T_{\varphi(x)}\mathcal{D} \to T_x\mathcal{D}$ is the transpose of $T_x\varphi$ with respect to the Riemannian metric; if u is a vector field, then $(Tu)^\dagger = \frac{d}{dt}\big|_{t=0} (T\eta_t)^\dagger$ where η_t is the flow of u. The Hamiltonian on $T^* \operatorname{Diff}(\mathcal{D})$ is

$$H(\varphi, \pi) = \frac{1}{2} \int_{\mathcal{D}} \left\| \mathbb{P}(J\varphi^{-1}(\pi \circ \varphi^{-1})) \right\|^2 \mu = \frac{1}{2} \int_{\mathcal{D}} \left\| \mathbb{P}(\pi) \right\|^2 (J\varphi)^{-1} \mu. \tag{3.15}$$

These equations can be obtained via the variational principle (3.1) which reads

$$\delta \int_0^T \left(\langle \pi, u \circ \varphi - \dot{\varphi} \rangle - \ell(u) \right) dt = 0$$

in the present case. The solution is given by the cotangent-lift acting on the initial condition (φ_0, π_0), that is, $(\varphi, \pi) = \big(\eta \circ \varphi_0, (T\eta^{-1})^\dagger \circ \pi_0\big)$, where $\eta \in \operatorname{Diff}_{vol}(\mathcal{D})$ verifies $\dot{\eta} \circ \eta^{-1} = u$. By the general theory (Theorem 2.2), u verifies the (right) Euler-Poincaré equations, producing here the ideal fluid motion $\dot{u} + u\nabla u = -\operatorname{grad} p$. This recovers the optimal control formulation of Euler equations given in Bloch, Crouch, Holm, and Marsden [2000].

In a similar way as for the N-rigid body, Hamilton's equations (3.14) are equivalent to the geodesic spray on the tangent bundle of a $\operatorname{Diff}_{vol}(\mathcal{D})$-orbit $\mathcal{O}$, relative to the normal metric given by

$$\gamma_{\mathcal{O}}(u \circ \varphi, v \circ \varphi) := \int_{\mathcal{D}} g(u, v) \mu, \quad \text{for all } u, v \in \mathfrak{X}_{div}(\mathcal{D}).$$

In the particular case when the $\operatorname{Diff}_{vol}(\mathcal{D})$-orbit is the subgroup $\operatorname{Diff}_{vol}(\mathcal{D})$, then we recover the usual Lagrangian approach to ideal flows. In particular, the normal metric coincides, on this special orbit, with the right-invariant L^2 metric, the Hamiltonian (3.15) recovers the usual kinetic energy expression $H(\varphi, \pi) = \frac{1}{2} \int_{\mathcal{D}} \|\pi\|^2 \mu$ on $T^* \operatorname{Diff}_{vol}(\mathcal{D})$, and the geodesic φ recovers the Lagrangian motion (or forward map) η.

Second approach: We now apply Theorem 2.3 (right version) in order to obtain the optimal control formulation for Euler fluid equations, via the back-to-label map $l_t = \eta_t^{-1}$. We let the group of volume preserving diffeomorphism act on the whole group of diffeomorphisms by composition on the right. The infinitesimal generator associated to $u \in \mathfrak{X}_{div}(\mathcal{D})$ is given by $u_{\mathrm{Diff}(\mathcal{D})}(\varphi) = Tq \circ u$. Given two diffeomorphisms $q_0, q_T \in \mathrm{Diff}(\mathcal{D})$, the corresponding Clebsch optimal control problem is

$$\min_{u_t} \int_0^T \|u_t\|_{L^2}^2 dt \tag{3.16}$$

subject to the following conditions:

(A) $\dot{q}_t + Tq_t \circ u_t = 0$;

(B) $q(0) = q_0$ and $q(T) = q_T$.

This recovers the problem studied in Holm [2009]. The cotangent bundle momentum map $\mathbf{J} : T^* \mathrm{Diff}(\mathcal{D}) \to \mathfrak{X}_{vol}(\mathcal{D})^*$ associated to the cotangent-lift of right translation is

$$\mathbf{J}(q, \pi) = \mathbb{P}(Tq^\dagger \circ \pi) = Tq^\dagger \circ \pi - \operatorname{grad} k.$$

The optimal control is thus given by $u = -Tq^\dagger \circ \pi + \operatorname{grad} k$ and the canonical Hamilton's equations (3.7) on $T^* \mathrm{Diff}(\mathcal{D})$ are

$$\dot{q} = -Tq \circ u, \quad \dot{\pi} = -T\pi \circ u. \tag{3.17}$$

The Hamiltonian for these equations on $T^* \mathrm{Diff}(\mathcal{D})$ is

$$H(q, \pi) = \frac{1}{2} \int_{\mathcal{D}} \|\mathbb{P}(Tq^\dagger \circ \pi)\|^2 \mu,$$

and the solutions are of the form $(q, \pi) = (q_0 \circ \eta^{-1}, \pi_0 \circ \eta^{-1})$ where $\eta \in \mathrm{Diff}_{vol}(\mathcal{D})$ is the flow of u. These equations can be obtained via the variational principle (3.5) which reads

$$\delta \int_0^T (\langle \pi, Tq \circ u + \dot{q} \rangle + \ell(u))\, dt = 0.$$

By the general theory (Theorem 2.3), u verifies the Euler-Poincaré equations, producing here the ideal fluid motion $\dot{u} + u \cdot \nabla u = -\operatorname{grad} p$. This recovers the optimal control formulation of Euler equations given in Holm [2009].

As before, the optimal control problem restricts to $\mathrm{Diff}_{vol}(\mathcal{D})$-orbits on which $q(t)$ describes a geodesic with respect to the normal metric given at $q \in \mathcal{O}$ by

$$\gamma_{\mathcal{O}}(Tq \circ u, Tq \circ v) := \int_{\mathcal{D}} g(u, v) \mu, \quad \text{for all } u, v \in \mathfrak{X}_{div}(\mathcal{D}).$$

On the special orbit given by the subgroup $\mathrm{Diff}_{vol}(\mathcal{D})$, the geodesic q coincides with the back-to-label map l, the normal metric coincides with the left-invariant extension of the L^2 Riemannian metric on the group of volume preserving diffeomorphisms, and we obtain the Euler-Poincaré description for fluids, in inverse representation, as described in Remark 3.1 for general Lie groups. In particular, the Lagrangian motion η_t and the back-to-label map $l_t = \eta_t^{-1}$ are both geodesics on the group of volume preserving diffeomorphisms; η_t is geodesic for the *right*-invariant extension to $\mathrm{Diff}_{vol}(\mathcal{D})$ of the L^2 inner product on $\mathfrak{X}_{div}(\mathcal{D})$ (this is the usual Lagrangian formulation given in Arnold [1966]); l_t is geodesic for the corresponding *left*-invariant extension. The Eulerian velocity is given by $u = \dot{\eta}_t \circ \eta_t^{-1} = -Tl_t^{-1} \circ \dot{l}_t$.

Note that equations (3.17) are completely analogous to Hamilton's equations $\dot{Q} = QU$ and $\dot{P} = PU$ associated to the N-rigid body and the respective optimal controls are given by

$$JU = \frac{1}{2}(Q^T P - P^T Q) = \mathbb{P}(Q^T P) \quad \text{and} \quad u = -\mathbb{P}(Tq^\dagger \circ \pi);$$

the projector $\mathbb{P}$ are respectively, the the projection $\mathfrak{gl}(N) \to \mathfrak{so}(N)$, and the Helmholtz projector $\mathbb{P} : \mathfrak{X}(\mathcal{D}) \to \mathfrak{X}_{vol}(\mathcal{D})$. For fluids, one can also replace the group $H = \mathrm{Diff}(\mathcal{D})$ by the manifold $Q = \mathrm{Emb}(\mathcal{D}, M)$ of all embeddings of M into a fixed manifold M. In fact, the Euler fluid equations, resp. the N-rigid body equations, can be obtained by a Clebsch optimal control problem on any manifold Q on which the group $G = \mathrm{Diff}_{vol}(\mathcal{D})$, resp. $G = SO(N)$ act.

Averaged hydrodynamics: It is known that replacing the L^2 metric by an H^1 metric, on the group of volume preserving diffeomorphisms (with appropriate boundary conditions), one gets the dynamics of averaged Euler (or Euler-α) equations given by

$$\dot{m} + u\cdot\nabla m - \alpha^2 \nabla u^T \cdot \Delta u = -\operatorname{grad} p, \quad m = (1 - \alpha^2 \Delta) u,$$

see Holm, Marsden and Ratiu [1998b]. Both the approaches described before for ideal fluids are directly applicable to averaged dynamics. It suffices to use the Lagrangian $\ell(u) = \frac{1}{2}\|u\|_{H^1}^2 = \frac{1}{2}\langle m, u\rangle$ associated to the Sobolev H^1 norm. The optimal control are respectively given by

$$(1 - \alpha^2\Delta)u = J\varphi^{-1}(\pi \circ \varphi^{-1}) \quad \text{and} \quad (1 - \alpha^2\Delta)u = -Tq^\dagger \circ \pi.$$

4 Optimal control for the N-Camassa-Holm equation and its singular solutions

In this section, we present two Clebsch optimal control problems that yield the N dimensional Camassa-Holm equations. One of them is related to the geometry of the singular solutions and gives a new interpretation of these singular solutions as geodesics. The other can be seen as a generalization of the approach using the back-to-label map. For N-Camassa-Holm, the appropriate choice for the manifold Q in the optimal control problem is not a group, it is a manifold of embeddings. Consequently, we are not in the situation described in the preceding section.

The N-Camassa-Holm equations

$$\dot{m} + u\cdot\nabla m + \nabla u^T \cdot m + m \operatorname{div} u = 0, \quad m = (1 - \alpha^2\Delta)u$$

are the spatial representation of geodesics on the group $\mathrm{Diff}(\mathcal{D})$ of all diffeomorphisms of $\mathcal{D}$, relative to a Sobolev H^1 metric, see Holm and Marsden [2004]. They are thus obtained by Euler-Poincaré reduction and are a particular case of the so called EPDiff equations, to which the approach described here generalizes easily. For simplicity, we assume that $\mathcal{D}$ has no boundary.

First approach: We use Theorem 2.2 (left version) with $G = \mathrm{Diff}(\mathcal{D})$ acting on the left on the manifold $Q = \mathrm{Emb}(S, \mathcal{D})$ of all embeddings of a given manifold S into M. Given two embeddings $\mathbf{Q}_0, \mathbf{Q}_T \in \mathrm{Emb}(S, \mathcal{D})$, the associated Clebsch optimal control problem is

$$\min_{u_t} \int_0^T \|u_t\|_{H^1}^2 dt \tag{4.1}$$

subject to the following conditions:

(A) $\dot{\mathbf{Q}}_t = u_t \circ \mathbf{Q}_t$;

(B) $\mathbf{Q}(0) = \mathbf{Q}_0$ and $\mathbf{Q}(T) = \mathbf{Q}_T$.

The cotangent bundle momentum map $\mathbf{J} : T^* \mathrm{Emb}(S, \mathcal{D}) \to \mathfrak{X}(\mathcal{D})^*$ is given by

$$\mathbf{J}(\mathbf{Q}, \mathbf{P}) = \int_S \mathbf{P}(s) \delta(x - \mathbf{Q}(s)) ds;$$

see Holm and Marsden [2004]. Since $\mathbf{J}$ produces singular solutions, the generalizations of the peakons of the one dimensional Camassa-Holm equation (see Camassa and Holm [1993]), $\mathbf{J}$ is referred to as the singular momentum map. The optimal control is thus given by

$$(1 - \alpha^2 \Delta) u = \int_S \mathbf{P}(s) \delta(x - \mathbf{Q}(s)) ds,$$

and the canonical Hamilton's equations are obtained via the variational principle

$$\delta \int_0^T (\langle \mathbf{P}, u \circ \mathbf{Q} - \dot{\mathbf{Q}} \rangle - \ell(u)) dt = 0.$$

Evaluated on the optimal control u, the Pontryagin Hamiltonian $\hat{H}$ produces the collective Hamiltonian

$$H(\mathbf{Q}, \mathbf{P}) = \frac{1}{2} \iint \mathbf{P}(s) G(\mathbf{Q}(s) - \mathbf{Q}(s')) \mathbf{P}(s') ds ds',$$

where G is the Green's function associated to the differential operator $(1 - \alpha^2 \Delta)$, see Holm and Marsden [2004]. By our general theory, the solution $(\mathbf{Q}, \mathbf{P})$ is obtained by letting the flow η_t of the optimal control act on the initial values $\mathbf{Q}(0), \mathbf{P}(0)$ by the cotangent-lifted action. The fact that optimal control u is solution of the N-Camassa-Holm equations, recovers the interpretation of the momentum map as a singular solution.

For $\mathbf{Q} \in \mathrm{Emb}(S, \mathcal{D})$, the isotropy subgroup of $\mathbf{Q}$ equals $\mathrm{Diff}(\mathcal{D})_{\mathbf{Q}} = \{\eta \mid \eta|_{\mathbf{Q}(S)} = id\}$. Therefore, the isotropy Lie algebra of $\mathbf{Q}$ is $\mathfrak{X}(\mathcal{D})_{\mathbf{Q}} = \{u \mid u|_{\mathbf{Q}(S)} = 0\}$. We write formally $\mathfrak{X}(\mathcal{D}) = \mathfrak{X}(\mathcal{D})_{\mathbf{Q}} \oplus (\mathfrak{X}(\mathcal{D})_{\mathbf{Q}})^{\perp}$, the perpendicular being taken relative to the H^1-metric on $\mathfrak{X}(\mathcal{D})$. Thus, given $u \in \mathfrak{X}(\mathcal{D})$, we can write $u = u_{\mathbf{Q}} + u^{\mathbf{Q}}$ where $u_{\mathbf{Q}} \in \mathfrak{X}(\mathcal{D})_{\mathbf{Q}}$ and $u^{\mathbf{Q}} \in (\mathfrak{X}(\mathcal{D})_{\mathbf{Q}})^{\perp}$.

By applying Theorem 2.7 to this case, we obtain the following new geometric interpretation of singular solutions.

Theorem 4.1 *The singular solutions $\delta\ell/\delta u = \mathbf{J}(\mathbf{Q},\mathbf{P})$ of the N-Camassa-Holm equations arise as geodesics on a* Diff$(\mathcal{D})$*-orbit*

$$\mathcal{O} = \{\eta \circ \mathbf{Q}_0 \mid \eta \in \mathrm{Diff}(\mathcal{D})\} \subset \mathrm{Emb}(S,\mathcal{D}),$$

with respect to the normal metric (2.6) *given in this case by*

$$\gamma_{\mathcal{O}}(u \circ \mathbf{Q}, v \circ \mathbf{Q}) := \left\langle u^{\mathbf{Q}}, v^{\mathbf{Q}} \right\rangle_{H^1},$$

for any $\mathbf{Q} \in \mathcal{O}$.

Note that choosing $S = \mathcal{D}$, we have $\mathrm{Emb}(S,\mathcal{D}) = \mathrm{Diff}(\mathcal{D})$ and we recover the dynamic of the strong (i.e. non singular) solutions. In this case, the momentum map $\mathbf{J}$ is the Lagrange-to-Euler map.

Second approach: As in the case of ideal fluids, we now apply Theorem 2.3 (right version) in order to obtain the optimal control formulation for Euler fluid equations, via a generalization of the back-to-label map.

Consider the group $G = \mathrm{Diff}(\mathcal{D})$ acting on the right on the manifold $\mathrm{Emb}(\mathcal{D}, M)$ for a fixed manifold M. As opposed to the previous approach, note that this right action is free. Given two embeddings $\mathbf{q}_0, \mathbf{q}_T \in \mathrm{Emb}(\mathcal{D}, M)$, the Clebsch optimal control problem is

$$\min_{u_t} \int_0^T \|u_t\|^2_{H^1} dt \tag{4.2}$$

subject to the following conditions:

(A) $\dot{\mathbf{q}}_t + T\mathbf{q}_t \circ u_t = 0$;

(B) $\mathbf{q}(0) = \mathbf{q}_0$ and $\mathbf{q}(T) = \mathbf{q}_T$.

The cotangent momentum map $\mathbf{J} : T^*\,\mathrm{Emb}(\mathcal{D}, M) \to \mathfrak{X}(\mathcal{D})^*$ is given by

$$\mathbf{J}(\mathbf{q},\mathbf{p}) = \mathbf{p}\cdot T\mathbf{q}.$$

Thus, the optimal control is given by

$$(1 - \alpha^2\Delta)u = -\mathbf{p}\cdot T\mathbf{q},$$

the canonical Hamilton's equations are obtained via the variational principle

$$\delta \int_0^T \left(\langle \mathbf{p}, T\mathbf{q} \circ u + \dot{\mathbf{q}} \rangle + \ell(u) \right) dt = 0,$$

and the collective Hamiltonian reads

$$H(\mathbf{q},\mathbf{p}) = h(-\mathbf{p}\cdot T\mathbf{q}) = \frac{1}{2} \iint \mathbf{p}(x)\cdot T\mathbf{q}(x) G(x - x')\mathbf{p}(x')\cdot T\mathbf{q}(x') dx dx'.$$

By Theorem 2.3 (right version), the solution $(\mathbf{q},\mathbf{p})$ is found by letting the back-to-label map (i.e. the inverse of the flow of u_t) act on the initial values $(\mathbf{q}(0), \mathbf{p}(0))$ by the cotangent lift of composition on the right. The optimal control u solves the N-Camassa-Holm equation. In this case, Theorem 2.7 yields the following interpretation of the solution $(\mathbf{q},\mathbf{p})$.

Theorem 4.2 *The solution* $(\mathbf{q}, \mathbf{p})$ *of Hamilton's equations defined by* H *projects to a geodesic on a* $\operatorname{Diff}(\mathcal{D})$*-orbit*

$$\mathcal{O} = \{\mathbf{q}_0 \circ \eta \mid \eta \in \operatorname{Diff}(\mathcal{D})\},$$

with respect to the normal metric

$$\gamma_{\mathcal{O}}(T\mathbf{q} \circ u, T\mathbf{q} \circ v) = \langle u, v \rangle_{H^1},$$

for $\mathbf{q} \in \mathcal{O}$.

Note that choosing $M = \mathcal{D}$, we have $\operatorname{Emb}(\mathcal{D}, M) = \operatorname{Diff}(\mathcal{D})$, and the previous theorem reduces to a geodesic interpretation of the back-to-label map. Indeed, in this particular case, the orbit $\mathcal{O}$ is the whole diffeomorphism group and the embedding $\mathbf{q}$ is the back-to-label map l. As it should, when $\mathcal{D} = M$, the normal metric $\gamma_{\mathcal{O}}$ coincides with the left-invariant extension of $\langle\, , \rangle_{H^1}$. If l is a geodesic on the diffeomorphism group relative to this metric, then by the above discussion, $u := -Tl^{-1} \circ \dot{l}$ solves the N-Camassa-Holm equations. Contrary to what happens if one chooses $S = \mathcal{D}$ in Theorem 4.1, we do not recover here the usual (right-invariant) geodesic interpretation for fluids. We obtain the Euler-Poincaré formulation in inverse representation, as described in Remark 3.1.

Remark 4.3 These two optimal control approaches to the Camassa-Holm equations are directly applicable to any fluid equation arising as the Euler-Poincaré equation on the diffeomorphism group (EPDiff equations, see Holm and Marsden [2004]), or on the automorphism group of a principal bundle (EPAut equations, see Gay-Balmaz, Tronci, and Vizman [2009]). This last family of equations includes for example, the two-component Camassa-Holm equation and its generalizations (Holm and Tronci [2008], Holm, Ó Náraigh, and Tronci [2009]).

5 Clebsch optimal control based on adjoint representation

In this section, we specialize the Clebsch optimal control problem to the case where the Lie group G acts on its Lie algebra $Q = \mathfrak{g}$ on the *right* with the adjoint representation, that is, $x \mapsto \operatorname{Ad}_{g^{-1}} x$. This will recover the double bracket equations of Bloch and Crouch [1996]. Beside this, we will also consider the case of an adjoint action arising naturally in the context symmetric spaces or, more generally, for reductive homogeneous spaces.

The infinitesimal generator of the right adjoint action is $u_{\mathfrak{g}}(x) = [x, u]$. In order to obtain Hamilton's equation in double bracket form, we suppose that the Lie algebra $\mathfrak{g}$ admits an Ad-invariant nondegenerate symmetric bilinear form γ. In particular we have

$$\gamma([x, u], v) = -\gamma(u, [x, v]).$$

In Medina and Revoy [1985] one can find the complete classification of such Lie algebras.

Given $x_0, x_T \in \mathfrak{g}$ and an arbitrary Lagrangian $\ell : \mathfrak{g} \to \mathbb{R}$, the associated Clebsch optimal control problem reads

$$\min_{u(t)} \int_0^T \ell(u(t))dt$$

subject to the following conditions:

(A) $\dot{x} = [x, u]$;

(B) $x(0) = x_0$ and $x(T) = x_T$.

We treat this problem using Theorem 2.2 with $Q = \mathfrak{g}$ and G acting on on the *right* on $\mathfrak{g}$ by $u \mapsto \mathrm{Ad}_{g^{-1}} u$. Identifying $\mathfrak{g}$ and $\mathfrak{g}^*$ via the the inner product γ, the Pontryagin function is $\hat{H}(x, p, u) = \langle p, [x, u] \rangle - \ell(u)$ and the momentum map $\mathbf{J} : T^*\mathfrak{g} \to \mathfrak{g}^*$ is $\mathbf{J}(x, p) = -[x, p]$. Thus the optimal control is given by $\frac{\delta \ell}{\delta u} = [p, x]$ and canonical Hamilton's equations are

$$\dot{x} = [x, u], \quad \dot{p} = [p, u].$$

In order to recover the result of Bloch and Crouch [1996], we consider the particular case where γ is positive definite and ℓ is the kinetic energy associated to γ. In this particular case, Hamilton's equations are given by the double bracket equations

$$\dot{x} = [x, [p, x]], \quad \dot{p} = [p, [p, x]].$$

By the general theory, these equations are Hamiltonian on $T^*\mathfrak{g}$, relative to the collective Hamiltonian

$$H(x, p) = \frac{1}{2} \| [p, x] \|^2.$$

By Theorem 2.2, the control u necessarily satisfies the Euler-Poincaré equations which in this case are $\dot{u} = -\mathrm{ad}_u u = 0$ since with the identification by the bi-invariant inner product γ we have $\mathrm{ad}_u^* = -\mathrm{ad}_u$. Hence the control u is constant along the flow of Hamilton's equations, that is, $u = [p, x] = u_0$ is a constant of the motion. Therefore, by Theorem 2.2, the solution of Hamilton's equation is simply given by

$$(x(t), p(t)) = \left(\mathrm{Ad}_{\exp(tu_0)^{-1}} x_0, \mathrm{Ad}_{\exp(tu_0)^{-1}} p_0 \right).$$

Indeed, the solution of the equation $g^{-1}\dot{g} = u_0$ is given by the exponential map $g = \exp(tu_0)$.

By Theorem 2.7, the double bracket equations are also the Hamiltonian description of geodesics on an adjoint orbit $\mathcal{O} = \{ \mathrm{Ad}_g \xi \mid g \in G \}$ relative to the normal metric $\gamma_{\mathcal{O}}$ defined in (2.6). This recovers a result of Bloch and Crouch [1996], proved there by a direct computation. Here we obtain the further result that geodesics are explicitly given by

$$x(t) = \mathrm{Ad}_{\exp(tu_0)^{-1}} x_0.$$

By (2.6), the normal metric is

$$\gamma_{\mathcal{O}}([x, p], [x, q]) = \gamma(p^x, q^x),$$

where the decomposition $p = p_x + p^x$ is made relative to the splitting $\mathfrak{g} = \mathfrak{g}_x \oplus \mathfrak{g}_x^\perp$. If $\mathfrak{g}$ is a compact semisimple Lie algebra endowed with minus the Killing form, then γ is the usual normal metric on adjoint orbits. Note that here we have

$$\mathfrak{g}_x = \ker(\mathrm{ad}_x) \quad \text{and} \quad \mathfrak{g}_x^\perp = \mathrm{im}(\mathrm{ad}_x).$$

Thus, the normal metric can be rewritten at $x \in \mathcal{O}$ as

$$\gamma_{\mathcal{O}}(\dot{x}_1, \dot{x}_2) = \gamma\left(\mathrm{ad}_x^{-1} \dot{x}_1, \mathrm{ad}_x^{-1} \dot{x}_1 \right),$$

where ad_x^{-1} denotes the inverse of the adjoint representation restricted to $\mathfrak{g}_x^\perp$. Note that, by our general theory, a similar result holds for any Lagrangian ℓ such that $u \mapsto \delta\ell/\delta u$ is a diffeomorphism. More precisely, canonical Hamilton's equations for $H(x,p) = h([p,x])$ on $T^*\mathfrak{g}$ restrict to canonical Hamilton's equation on $T^*\mathcal{O}$.

As for the rigid body, one can consider the kinetic energy associated to $\gamma(u, Jv)$, where $J : \mathfrak{g} \to \mathfrak{g}$ is a symmetric and positive definite operator. In this case, the optimal control is given by $u = J^{-1}[p,x]$, the Hamiltonian is $H(x,p) = \frac{1}{2}\gamma\left([x,p], J^{-1}[x,p]\right)$, and Hamilton's equations read

$$\dot{x} = \left[x, J^{-1}[p,x]\right], \quad \dot{p} = \left[p, J^{-1}[p,x]\right].$$

The inner product $\gamma(u, Jv)$ induces a normal metric on orbits in the same way as before. Geodesics of this metric are given by the above Hamilton's equations. In this case, the Euler-Poincaré equation for u reads

$$\dot{u} = J^{-1}[Ju, u],$$

and u is not constant in general.

More generally, in the case of an arbitrary Lagrangian $\ell : \mathfrak{g} \to \mathbb{R}$ such that $u \mapsto \delta\ell/\delta u$ is a diffeomorphism, the optimal control is given by $u = \mathbf{d}h([p,x])$, therefore one gets the *generalized double bracket equations*

$$\dot{x} = [x, \mathbf{d}h([p,x])], \quad \dot{p} = [p, \mathbf{d}h([p,x])], \tag{5.1}$$

where $h : \mathfrak{g}^* \to \mathbb{R}$ is the Hamiltonian associated to ℓ via the Legendre transformation. Note that we always identify $\mathfrak{g}^*$ with $\mathfrak{g}$, using the nondegenerate Ad-invariant bilinear symmetric form γ. For example, h is defined on $\mathfrak{g}^* \simeq \mathfrak{g}$ and the derivative is given in terms of γ by

$$\gamma(\mathbf{d}h(x), u) = \left.\frac{d}{dt}\right|_{t=0} h(x + tu).$$

Remark 5.1 For the example of adjoint actions, we have chosen to work with an Ad-invariant nondegenerate symmetric bilinear form γ that enabled us to identify the dual Lie algebra $\mathfrak{g}^*$ with $\mathfrak{g}$ and to obtain Hamilton's equation in the classical double bracket form. This was useful to recover known results, such as those of Bloch and Crouch [1996]. However, such a bilinear form γ is not needed in order to formulate the Clebsch optimal control problem. Indeed, it suffices to make use of the coadjoint representation. For example, in this case, the cotangent momentum map is $\mathbf{J}(x,p) = \operatorname{ad}_x^* p$ and the cotangent lifted action of G on $T^*\mathfrak{g}$ is $(x,p) \mapsto (\operatorname{Ad}_{g^{-1}} x, \operatorname{Ad}_g^* p)$. Thus, Hamilton's equations read

$$\dot{x} = -\operatorname{ad}_u x, \quad \dot{p} = \operatorname{ad}_u^* p,$$

and the optimal control is determined by $\delta\ell/\delta u = \operatorname{ad}_x^* p$. We can write Hamilton's equations in terms of the variables $(x,p) \in T^*\mathfrak{g}$ as

$$\dot{x} = -\operatorname{ad}_{\frac{\delta h}{\delta m}} x, \quad \dot{p} = \operatorname{ad}^*_{\frac{\delta h}{\delta m}} p, \quad \text{where} \quad m = \operatorname{ad}_x^* p.$$

These equations are readily seen to be equivalent to the generalized double bracket equations (5.1), when an Ad-invariant inner product is chosen.

Optimal control associated to symmetric spaces. In this paragraph, we study the double bracket equation associated to symmetric spaces (Bloch and Crouch [1996]) using the tools developed in this paper. Moreover, we obtain the explicit solution of the double bracket equations.

Let (M, g) be a Riemannian symmetric space. It is well known that M is a homogeneous space G/K where $G = I_0(M, g)$ is the connected component of the isometry group of (M, g) and the isotropy group K is compact. Moreover, there is a naturally induced involutive automorphism σ of the Lie algebra $\mathfrak{g}$ of G, to which is associated the decomposition into a direct sum $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, where $\mathfrak{k} = \{\xi \in \mathfrak{g} \mid \sigma(\xi) = \xi\}$ is the Lie algebra of K and the subspace $\mathfrak{m}$ is is defined by $\mathfrak{m} = \{\xi \in \mathfrak{g} \mid \sigma(\xi) = -\xi\}$. We thus have the inclusions

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}. \tag{5.2}$$

In fact, the adjoint representation on $\mathfrak{g}$ restricts to a representation of K on $\mathfrak{m}$, also denoted by $\mathrm{Ad}_k : \mathfrak{m} \to \mathfrak{m}$. This implies the second inclusion above. Let β be the Killing form on $\mathfrak{g}$. Then we have $\beta(\mathfrak{k}, \mathfrak{m}) = 0$ and $\beta|_{\mathfrak{k}}$ is negative definite.

In order to formulate the Clebsch optimal control relative to the adjoint action of K on $\mathfrak{m}$, we need to assume the existence of a nondegenerate (non necessarily positive definite) Ad-invariant symmetric bilinear form γ on $\mathfrak{g}$. This hypothesis is verified for symmetric spaces whose isometry group is semisimple since it suffices to choose $\gamma = -\beta$. Recall that any simply connected Riemannian symmetric space (M, g) is uniquely decomposed as the direct sum $(M, g) = (M_0, g_0) \times (M_1, g_1)$, where (M_0, g_0) is a Euclidean space and (M_1, g_1) is of semisimple type (Ise and Takeuchi [1991]). Symmetric spaces of compact or noncompact type are of semisimple type.

Given a Riemannian symmetric space of semisimple type (M, g), we consider the *right* adjoint action of K on $\mathfrak{m}$ and the Lagrangian $\ell : \mathfrak{k} \to \mathbb{R}$, $\ell(u) = \frac{1}{2}\|u\|^2$ associated to the Killing form β of $\mathfrak{g}$, restricted to $\mathfrak{k}$. Here we restrict to this type of Lagrangian to recover the results of Bloch and Crouch [1996]. The case of a general Lagrangian ℓ is treated below in the more general setting of reductive homogeneous spaces. Given $m_0, m_T \in \mathfrak{m}$, the associated Clebsch optimal control problem reads

$$\min_{u(t)} \int_0^T \frac{1}{2}\|u(t)\|^2 dt \tag{5.3}$$

subject to the following conditions:

(A) $\dot{m} = [m, u]$;

(B) $m(0) = m_0$ and $m(T) = m_T$.

We now compute the cotangent bundle momentum map $\mathbf{J} : T^*\mathfrak{m} \to \mathfrak{k}^*$. Using the Killing form on the semisimple Lie algebra $\mathfrak{g}$, we identify the dual $\mathfrak{g}^*$ with $\mathfrak{g}$. Since the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is orthogonal, we can use the the Killing form to identify the dual spaces to $\mathfrak{k}$, $\mathfrak{m}$ with themselves and obtain $(\mathfrak{k} \oplus \mathfrak{m})^* = \mathfrak{k}^* \oplus \mathfrak{m}^*$. Via these identifications, the momentum map is $\mathbf{J}(m, p) = -[m, p]$. The canonical Hamilton's equations on $T^*\mathfrak{m}$ are given by the coupled double bracket equations

$$\dot{m} = [m, [p, m]], \quad \dot{p} = [p, [p, m]].$$

These equations make sense on $T^*\mathfrak{m}$ thanks to the inclusions (5.2) and are associated to the Hamiltonian $H(m,p) = \frac{1}{2}\|[p,m]\|^2$. The associated Euler-Poincaré equation yields the condition $\dot{u} = 0$. Therefore, the solution of Hamilton's equations is explicitly given by

$$(m(t), p(t)) = \left(\mathrm{Ad}_{\exp(tu_0)^{-1}} m_0, \mathrm{Ad}_{\exp(tu_0)^{-1}} p_0\right).$$

By Theorem 2.7, these equations are also the Hamiltonian description of geodesics on the K-adjoint orbits $\mathcal{O} = \{\mathrm{Ad}_k m \mid k \in K\}$, for a fixed $m \in \mathfrak{m}$, relative to the normal metric $\gamma_{\mathcal{O}}$. In the present case, it is given by

$$\gamma_{\mathcal{O}}([m,k],[m,l]) = \gamma(k^m, l^m),$$

where the decomposition $k = k_m + k^m$ is relative to the splitting $\mathfrak{k} = \mathfrak{k}_m \oplus \mathfrak{k}_m^\perp$, where

$$\mathfrak{k}_m = \ker(\mathrm{ad}_m : \mathfrak{k} \to \mathfrak{m})$$

is the isotropy Lie algebra associated to the adjoint action of K on $\mathfrak{m}$. The normal metric at the point $m \in \mathcal{O}$ has the alternative expression

$$\gamma_{\mathcal{O}}(\dot{m}_1, \dot{m}_2) = \gamma\left(\mathrm{ad}_m^{-1} \dot{m}_1, \mathrm{ad}_m^{-1} \dot{m}_2\right),$$

where ad_m^{-1} denote the inverse of $\mathrm{ad}_m : (\mathfrak{k}_m)^\perp \to \mathrm{Im}(\mathrm{ad}_m |_{\mathfrak{k}})$. This recovers and extends to any symmetric space of semisimple type the results of Bloch and Crouch [1996]; Bloch, Brockett, and Crouch [1997]. Moreover, we obtain here the explicit expression of the geodesics of the normal metric. They are of the form

$$m(t) = \mathrm{Ad}_{\exp(tu_0)^{-1}} m_0.$$

A standard example of symmetric space is provided by the Grassmannians of oriented p-planes in $p+q$-space,

$$G_{p,p+q}(\mathbb{R}) = SO(p+q)/SO(p) \times SO(q).$$

In this particular case, elements in $\mathfrak{k}$ and $\mathfrak{m}$ are of the form

$$u = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} \quad \text{and} \quad m = \begin{pmatrix} 0 & Q \\ -Q^* & 0 \end{pmatrix},$$

where Q is a real $p \times q$ matrix, $U \in \mathfrak{so}(p)$, and $V \in \mathfrak{so}(q)$. The adjoint action of the isotropy group $K = SO(p) \times SO(q)$ on $\mathfrak{m}$ reads $Q \mapsto k_1^{-1} Q k_2$. Thus the Clebsch optimal control problem (5.3) reads

$$\min_{U(t),V(t)} \int_0^T \frac{1}{2}\left(\|U(t)\|^2 + \|V(t)\|^2\right) dt \tag{5.4}$$

subject to the following conditions:

(A) $\dot{Q} = QV - UQ$;

(B) $Q(0) = Q_0$ and $Q(T) = Q_T$.

Generalization to reductive homogeneous spaces. We now describe a more general situation for which the optimal control problem formulated previously makes sense. We treat the case of a general Lagrangian ℓ (verifying the usual convexity assumption) and do not restrict to the case when ℓ is just the kinetic energy.

Let G be a Lie group, $K \subset G$ a closed subgroup, and consider the homogeneous space G/K. Suppose that there exists a subspace $\mathfrak{m}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ and $\operatorname{Ad}_k \mathfrak{m} \subset \mathfrak{m}$, for all $k \in K$, where as before $\mathfrak{k}$ denotes the Lie algebra of K. If this condition is verified the homogeneous space is called *reductive*. This hypothesis is not very restrictive since any homogeneous space that admits a G-invariant metric is reductive (see e.g. Kowalski and Szenthe [2000]). Note that all the above hypotheses are verified for a symmetric space of semisimple type. In order to obtain Hamilton's equation in a double bracket form, we suppose that there exists a Ad-invariant nondegenerate bilinear symmetric form γ on $\mathfrak{g}$ such that $\gamma(\mathfrak{k}, \mathfrak{g}) = 0$. This will allow us to identify the spaces with their duals.

Given a Lagrangian $\ell : \mathfrak{k} \to \mathbb{R}$ and letting K act on $\mathfrak{m}$ via the *right* adjoint action, the Clebsch optimal control problem is

$$\min_{u(t)} \int_0^T \ell(u(t))dt \tag{5.5}$$

subject to the following conditions:

(A) $\dot{m} = [m, u]$;

(B) $m(0) = m_0$ and $m(T) = m_T$.

The cotangent bundle momentum map $\mathbf{J} : T^*\mathfrak{m} \to \mathfrak{k}^*$ is $\mathbf{J}(m, p) = [p, m]_{\mathfrak{k}}$, where the index $\mathfrak{k}$ indicates the projection onto $\mathfrak{k}$. The optimal control u is thus given by

$$\frac{\delta \ell}{\delta u} = [p, m]_{\mathfrak{k}}.$$

Assuming that $u \mapsto \delta\ell/\delta u$ is a diffeomorphism, the canonical Hamilton's equations are

$$\dot{m} = [m, \mathbf{d}h([p, m]_{\mathfrak{k}})], \quad \dot{p} = [p, \mathbf{d}h([p, m]_{\mathfrak{k}})],$$

where $h : \mathfrak{k}^* \to \mathbb{R}$ is the Hamiltonian associated to ℓ. The Hamiltonian on $T^*\mathfrak{m}$ is $H(m, p) = h([p, m]_{\mathfrak{k}})$. By Theorem 2.7 these equations are also canonically Hamiltonian on any K-orbit $\mathcal{O} = \{\operatorname{Ad}_k m \mid k \in K\}$. The Euler-Poincaré equations read

$$\frac{d}{dt}\frac{\delta \ell}{\delta u} = \left[\frac{\delta \ell}{\delta u}, u\right],$$

and the solution of Hamilton's equation is

$$(m(t), p(t)) = \left(\operatorname{Ad}_{k(t)^{-1}} m_0, \operatorname{Ad}_{k(t)^{-1}} p_0\right),$$

where $k(t) \in K$ solves the equation $k^{-1}\dot{k} = u$.

Geodesics on symmetric spaces. Let (M, g) be a symmetric space and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ the decomposition into the eigenspaces of the involution σ. Recall (see Ise and Takeuchi [1991], Theorem 2.4, Part II) that g is a G-invariant Riemannian metric on M, but the Riemannian connection (and therefore the geodesics) do not depend on a particular G-invariant metric. As we will see, there is a canonical choice for the G-invariant metric in the case of simply connected symmetric spaces of semisimple type. Indeed, it is known that such a symmetric space is isometric to the direct product $(M_+, g_+) \times (M_-, g_-)$ of a simply connected symmetric space of compact type (M_+, g_+) with a simply connected symmetric space of noncompact type (M_-, g_-).

For the compact case (M_+, g_+), the associated Lie algebra is compact and semisimple; therefore the Killing form β is negative definite. In this case, one can choose the Ad-invariant inner product $\gamma := -\beta$ on the Lie algebra and the associated normal metric is G-invariant on M_+. For the noncompact case (M_-, g_-), σ is a Cartan involution and we choose the Ad-invariant inner product $\gamma := -\beta^\sigma$, where $\beta^\sigma(\xi, \eta) := \beta(\xi, \sigma\eta)$.

We now compute the normal metric at $[e] \in G/K := M_\pm$. If $\xi \in \mathfrak{g}$, its infinitesimal generator given by the G-action on G/K is $\xi_{G/K}([e]) = T_e\pi(\xi)$ and the orthogonal decomposition $\mathfrak{g} = \mathfrak{g}_{[e]} \oplus \mathfrak{g}_{[e]}^\perp$ with respect to γ coincides with the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. Thus, for $\xi, \eta \in \mathfrak{m}$, the normal metric is given by $\gamma_{G/K}(T_e\pi(\xi), T_e\pi(\eta)) = \gamma(\xi, \eta) = -\beta(\xi, \eta)$ for the compact case, and $\gamma_{G/K}(T_e\pi(\xi), T_e\pi(\eta)) = \gamma(\xi, \eta) = \beta(\xi, \eta)$, for the noncompact case.

In term of the Clebsch optimal control problem, geodesics on a simply connected symmetric space (M, g) of semisimple type can be obtained as follows. Consider the decomposition $(M_+, g_+) \times (M_-, g_-)$ of (M, g) into symmetric spaces of compact and noncompact type, and let $G = G_+ \times G_-$ and $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ the associated Lie groups and Lie algebras. On $\mathfrak{g}$ we consider the inner product $(-\beta) \oplus (-\beta^\sigma)$. Given $m_0, m_T \in M$, the Clebsch optimal control problem relative to the G-action on M is

$$\min_{u(t)} \int_0^T \frac{1}{2} \|u(t)\|^2 dt$$

subject to the following conditions:

(A) $\dot{m} = u_M(m)$;

(B) $m(0) = m_0$ and $m(T) = m_T$.

Here $u(t) \in \mathfrak{g}$ and the norm is taken with respect to the inner product $(-\beta) \oplus (-\beta^\sigma)$. By Theorem 2.7, the extremal solutions $m(t)$ are geodesics on M relative to the normal metric induced by $(-\beta) \oplus (-\beta^\sigma)$. Since the inner product is Ad-invariant, the normal metric is G-invariant and thus its geodesics coincide with that of the original G-invariant metric g. Indeed, on a symmetric space, the Riemannian connection does not depend on the chosen G-invariant Riemannian metric. The Euler-Poincaré equations for the optimal control u read $\dot{u} = [u, u] = 0$; therefore the optimal control u is constant and the geodesics are of the form

$$m(t) = \Phi_{\exp(t\xi)}(m_0).$$

This recovers, by optimal control approach, a well-known result for symmetric spaces (see e.g. Ise and Takeuchi [1991], Theorem 2.4, Part II).

Remark 5.2 The above discussion generalizes to any homogeneous space in the following sense. Let G act transitively on a manifold M and endow the Lie algebra $\mathfrak{g}$ with a positive definite and Ad-invariant inner product γ. Then the geodesics of the normal metric on M associated to γ are of the form $m(t) = \Phi_{\exp(t\xi)}(m_0)$, where $\xi \in \mathfrak{g}$. These geodesics are the extremal solution of the Clebsch optimal control problem associated to the Lagrangian $\ell(u) = \frac{1}{2}\gamma(u, u)$.

6 Clebsch variables for ideal flows

The classical Clebsch variables for ideal flows are interpreted geometrically in Marsden and Weinstein [1983], §5. We review here their approach and formulate the associated Clebsch optimal control problem.

As explained before, the configuration manifold for ideal fluids is the group $G = \mathrm{Diff}_{vol}(\mathcal{D})$ of all volume preserving diffeomorphisms of the Riemannian manifold $\mathcal{D}$. For simplicity, we assume that $\mathcal{D}$ is compact and has no boundary. In order to obtain the classical Clebsch variables for fluids, it is convenient to work in vorticity representation, that is, we identify the dual space $\mathfrak{X}_{div}(\mathcal{D})^*$ with the quotient vector space $\Omega^1(\mathcal{D})/\mathbf{d}\mathcal{F}(\mathcal{D})$ of one-forms by exact one-forms. The elements are thus given by equivalence classes $[\alpha]$, where $\alpha \in \Omega^1(\mathcal{D})$ and the pairing uses the Riemannian volume on $\mathcal{D}$. To the equivalence class $[\alpha]$ we associate the exact two-form $\mathbf{d}\alpha \in \Omega^2_{ex}(\mathcal{D})$ and this map is an isomorphism if and only if the first cohomology $H^1(\mathcal{D})$ of $\mathcal{D}$ is trivial. Thus we identify the dual space $\mathfrak{X}_{div}(\mathcal{D})^*$ with exact two-forms $\omega \in \Omega^2_{ex}(\mathcal{D})$. The duality pairing is now given by

$$\langle \omega, u \rangle = \int_{\mathcal{D}} \alpha(u)\mu,$$

where α is any one-form such that $\mathbf{d}\alpha = \omega$ and μ is the Riemannian volume. Relative to this pairing, we have $\mathrm{ad}^*_u \omega = \boldsymbol{\pounds}_u \omega$, and the Euler-Poincaré equations associated to a Lagrangian $\ell : \mathfrak{X}_{div}(\mathcal{D}) \to \mathbb{R}$ read

$$\dot{\omega} + \boldsymbol{\pounds}_u \omega = 0, \quad \omega = \frac{\delta \ell}{\delta u} \in \Omega^2_{ex}(\mathcal{D}).$$

In particular, if ℓ is given by the kinetic energy of the L^2 metric, the functional derivative recovers the vorticity $\omega = \mathbf{d}u^\flat$ and we obtain the Euler equations for ideal flows in vorticity representation suitable for the study of point vortices in two dimensional flows.

In order to recover the classical Clebsch variables, we let $G = \mathrm{Diff}_{vol}(\mathcal{D})$ act on the vector space $\mathcal{F}(\mathcal{D})$ by the left action $\lambda \mapsto \lambda \circ \eta^{-1}$. The associated cotangent bundle momentum map is given by $\mathbf{J}(\lambda, \mu) = \mathbf{d}\lambda \wedge \mathbf{d}\mu = (\lambda, \mu)^* \Omega_{can}$, where Ω_{can} is the canonical symplectic form on $\mathbb{R}^2$. Note that if $\omega = \mathbf{d}\lambda \wedge \mathbf{d}\mu$, then $u^\flat - \lambda \mathbf{d}\mu$ is closed, therefore, we can write $u^\flat = \mathbf{d}\alpha + \lambda \mathbf{d}\mu$ which is the classical Clebsch representation for fluids (see Marsden and Weinstein [1983]).

We now formulate the Clebsch optimal control problem considered in Definition 2.1, for the particular case of Clebsch variables for ideal fluids. Given two functions $\lambda_0, \lambda_T \in \mathcal{F}(\mathcal{D})$, the Clebsch optimal control problem is

$$\min_{u_t} \int_0^T \|u_t\|^2_{L^2} dt \tag{6.1}$$

subject to the following conditions:

(A) $\dot{\lambda}_t + \mathbf{d}\lambda_t \cdot u_t = 0$;

(B) $\lambda(0) = \lambda_0$ and $\lambda(T) = \lambda_T$.

Note that we are in the situation of Theorem 2.2 (left version). The Pontryagin function is $\hat{H}(\lambda, \mu, u) = -\langle \mu, \mathbf{d}\lambda \cdot u\rangle - \frac{1}{2}\|u\|^2$ and the canonical Hamilton's equations read

$$\dot{\lambda} + \mathbf{d}\lambda \cdot u = 0, \quad \dot{\mu} + \mathbf{d}\mu \cdot u = 0$$

where we have used the expression $(\lambda, \mu) \mapsto (\lambda \circ \eta^{-1}, \mu \circ \eta^{-1})$ for the cotangent lifted action. These equations, as well as the optimal control $\mathbf{d}u^\flat = \mathbf{d}\lambda \wedge \mathbf{d}\mu$ can be obtained by the variational principle

$$\delta \int_0^T \left(\left\langle \mu, \mathbf{d}\lambda \cdot u + \dot{\lambda} \right\rangle + \frac{1}{2}\|u\|^2 \right) dt = 0.$$

By Theorem 2.7, if λ_0, λ_T belong to the same G-orbit in $\mathcal{F}(\mathcal{D})$, then the optimal control problem restricts to this G-orbit. Note that the Eulerian velocity $u \in \mathfrak{X}_{div}(\mathcal{D})$ is uniquely determined by the optimal control $\mathbf{d}u^\flat = \mathbf{d}\lambda \wedge \mathbf{d}\mu$. To see this, we observe that the exterior differential $\mathbf{d} : \{\alpha \in \Omega^1(\mathcal{D}) \mid \delta\alpha = 0\} \to \Omega^2_{ex}(\mathcal{D})$ is a linear isomorphism, by the Hodge decomposition. Here, δ is the codifferential defined by the Riemannian metric on $\mathcal{D}$. Thus, Hamilton's equation on $T^*\mathcal{F}(\mathcal{D})$ can be written as

$$\dot{\lambda} + \mathbf{d}\lambda \cdot \mathbf{d}^{-1}(\mathbf{d}\lambda \wedge \mathbf{d}\mu) = 0, \quad \dot{\mu} + \mathbf{d}\mu \cdot \mathbf{d}^{-1}(\mathbf{d}\lambda \wedge \mathbf{d}\mu) = 0,$$

which is reminiscent of the coupled double bracket equations.

Two dimensional flows and stream functions. If the manifold $\mathcal{D}$ has dimension two, the volume form μ can be thought of as a symplectic form, and we denote it by σ. In the two dimensional case, the identity $(\operatorname{div} u)\sigma = \boldsymbol{\pounds}_u\sigma = \mathbf{d}(\mathbf{i}_u\sigma)$ shows that the vector field u is divergence free if and only if it is locally Hamiltonian. If the first cohomology of the manifold vanishes, $H^1(\mathcal{D}) = 0$, then a divergence free vector field u is globally Hamiltonian and we can write $u = X_\psi$, relative to a *stream function* ψ defined up to an additive constant. The space $\mathfrak{X}_{div}(\mathcal{D})$ can thus be identified with the quotient space $\mathcal{F}(\mathcal{D})/\mathbb{R}$ and the dual is given by (generalized) functions ϖ on $\mathcal{D}$ such that $\int_\mathcal{D} \varpi\sigma = 0$. We denote by $\mathcal{F}(\mathcal{D})_0$ this space. The duality pairing is given by

$$\langle [\psi], \varpi\rangle = \int_\mathcal{D} \psi\varpi\sigma, \quad [\psi] \in \mathcal{F}(\mathcal{D})/\mathbb{R}, \ \varpi \in \mathcal{F}(\mathcal{D})_0.$$

Note that $\varpi\mu$ is an exact two-form since its integral over the boundaryless manifold $\mathcal{D}$ is zero. This description of the Lie algebra and its dual is compatible with the previous one in the following sense. First, we have the isomorphisms $u = X_\psi \in \mathfrak{X}_{div}(\mathcal{D}) \mapsto [\psi] \in \mathcal{F}(D)/\mathbb{R}$, and $\omega = \varpi\sigma \in \Omega^2_{ex}(\mathcal{D}) \mapsto \varpi \in \mathcal{F}(\mathcal{D})_0$. Second, the duality pairings verify

$$\langle \varpi\sigma, X_\psi\rangle = \int_\mathcal{D} \left(\mathbf{i}_{X_\psi}\alpha\right)\sigma = \int_\mathcal{D} \alpha \wedge \mathbf{i}_{X_\psi}\sigma = \int_\mathcal{D} \alpha \wedge \mathbf{d}\psi = \int_\mathcal{D} \psi\mathbf{d}\alpha = \int_\mathcal{D} \psi\varpi\sigma = \langle \varpi, [\psi]\rangle,$$

where $\alpha \in \Omega^1(\mathcal{D})$ is such that $\mathbf{d}\alpha = \varpi\sigma$. Note that the relation $\omega = \mathbf{d}u^\flat$ reads $\varpi = -\Delta\psi$. Indeed $\star\omega = \varpi$ and $\star\mathbf{d}u^\flat = \star\mathbf{d}X_\psi^\flat = \delta \star X_\psi^\flat = \delta(\mathbf{i}_{X_\psi}\sigma) = \delta\mathbf{d}\psi = -\operatorname{div}(\nabla\psi) = -\Delta\psi$.

Note that $\Delta : \mathcal{F}(\mathcal{D})/\mathbb{R} \to \mathcal{F}(\mathcal{D})_0$ is an isomorphism. Given $\varpi \in \mathcal{F}(\mathcal{D})_0$, by the Hodge decomposition there exists a unique one-form α such that $\delta\alpha = 0$ and $\mathbf{d}\alpha = \varpi\sigma$. Since $\delta\alpha = 0$, $\alpha^\sharp$ is locally (and hence globally) Hamiltonian with Hamiltonian function ψ and we have $\star\alpha = \mathbf{i}_{\alpha^\sharp}\sigma = \mathbf{d}\psi$. Thus $-\Delta\psi = \delta\mathbf{d}\psi = \delta \star \alpha = \star\mathbf{d}\alpha = \star\varpi\sigma = \varpi$.

With this interpretation of $\mathfrak{X}_{div}(\mathcal{D})$ and its dual, the infinitesimal generator associated to a stream function $[\psi] \in \mathcal{F}(\mathcal{D})/\mathbb{R}$ is $[\psi]_Q(\lambda) = -\mathbf{d}\lambda \cdot X_\psi = -\{\lambda, \psi\}$, where the bracket denotes the symplectic Poisson structure determined by σ. Using the formula

$$\int_{\mathcal{D}} f\{g, h\}\sigma = \int_{\mathcal{D}} \{f, g\} h\sigma,$$

where σ is the Liouville volume form (given here by the symplectic form, see Corollary 5.5.9 in Marsden and Ratiu [1999]), we obtain the expression $\mathbf{J}(\lambda, \mu) = \{\lambda, \mu\}$ for the cotangent bundle momentum map. Therefore, the optimal control $[\psi]$ is determined by $\delta\ell/\delta[\psi] = \{\lambda, \mu\}$ and Hamilton's equations become

$$\dot{\lambda} + \{\lambda, \psi\} = 0, \quad \dot{\mu} + \{\mu, \psi\} = 0. \tag{6.2}$$

By inserting the optimal control into these equations, we get

$$\dot{\lambda} + \left\{\lambda, \frac{\delta h}{\delta\{\lambda, \mu\}}\right\} = 0, \quad \dot{\mu} + \left\{\mu, \frac{\delta h}{\delta\{\lambda, \mu\}}\right\} = 0,$$

where $h : \mathcal{F}(\mathcal{D})_0 \to \mathbb{R}$ is the Hamiltonian associated to ℓ by Legendre transformation. The formulas $\mathrm{ad}_{[\psi_1]}[\psi_2] = \{\psi_1, \psi_2\}$ and $\mathrm{ad}^*_{[\psi]}\omega = \{\omega, \psi\}$, for the Lie bracket on stream functions and its dual, yield the Euler-Poincaré equation

$$\frac{d}{dt}\frac{\delta \ell}{\delta \psi} + \left\{\frac{\delta \ell}{\delta \psi}, \psi\right\} = 0.$$

Using the isomorphism $\mathfrak{X}_{div}(\mathcal{D}) \simeq \mathcal{F}(\mathcal{D})/\mathbb{R}$, a given Lagrangian ℓ can be expressed in terms of u or $[\psi]$. The corresponding functional derivatives of ℓ are related by the formula

$$\frac{\delta \ell}{\delta u} = \frac{\delta \ell}{\delta[\psi]}\sigma. \tag{6.3}$$

In the case of Euler equations, we have $\ell(u) = \frac{1}{2}\int_{\mathcal{D}} \|u\|^2\sigma = \frac{1}{2}\int_{\mathcal{D}} \|X_\psi\|^2\sigma = \ell(\psi)$ and $\frac{\delta\ell}{\delta[\psi]} = -\Delta\psi$, consistently with (6.3). In this case, the optimal control $[\psi]$ is determined by $[\psi] = \Delta^{-1}\{\mu, \lambda\}$ and the canonical Hamilton's equations on $T^*\mathcal{F}(\mathcal{D})$ take the double bracket form

$$\dot{\lambda} + \{\lambda, \Delta^{-1}\{\mu, \lambda\}\} = 0, \quad \dot{\mu} + \{\mu, \Delta^{-1}\{\mu, \lambda\}\} = 0,$$

the analogue to the double bracket equations appearing in Section 5, especially the equations $\dot{x} = [x, J^{-1}[p, x]]\,,\ \dot{p} = [p, J^{-1}[p, x]]$.

Relation with adjoint representation. As noted by Marsden and Weinstein [1983], for two dimensional flows the momentum mapping $(\lambda, \mu) \mapsto \{\lambda, \mu\}$ can be interpreted as a the momentum mapping associated to the adjoint action. To see this we recall that the adjoint representation of the Lie group $\mathrm{Diff}_{vol}(\mathcal{D})$ on its Lie algebra $\mathfrak{X}_{div}(\mathcal{D})$ is given by push-forward

$$u \mapsto \eta_* u.$$

In the particular case of two dimensional flows we can write $u = X_\psi$ for a stream function $\psi \in \mathcal{F}(\mathcal{D})/\mathbb{R}$, and the representation induced on stream functions is given by

$$\psi \mapsto \psi \circ \eta^{-1}.$$

Now it suffices to observe that this recovers the representation on Clebsch variables (up to a replacement of the space $\mathcal{F}(\mathcal{D})/\mathbb{R}$ by the space $\mathcal{F}(\mathcal{D})$). In conclusion, for two dimensional flows, the Clebsch optimal control problem using the classical Clebsch variables λ, μ and the one associated to the adjoint action, as described in §5, are the same. The Ad-invariant inner product is given by the L^2 pairing.

For higher dimensional flow, the Clebsch optimal control problem based on adjoint representation differs from that induced by the classical Clebsch variables. We now quickly describe it, following the abstract setting given in Remark 5.1. It is simpler to use the vorticity representation, that is, to identify the dual Lie algebra $\mathfrak{X}_{div}(\mathcal{D})^*$ with exact two-forms. Given $x_0, x_T \in \mathfrak{X}_{div}(\mathcal{D})$, and letting $\mathrm{Diff}_{vol}(\mathcal{D})$ act on its Lie algebra on the left by the adjoint representation $x \mapsto \eta_* x$, the Clebsch optimal control problem is

$$\min_{u_t} \int_0^T \|u_t\|^2_{L^2} dt \tag{6.4}$$

subject to the following conditions:

(A) $\dot{x} + \boldsymbol{\pounds}_u x = 0$;

(B) $x(0) = x_0$ and $x(T) = x_T$.

Using the expressions $(x, p) \mapsto (\eta_* x, \eta_* p)$ for the cotangent lifted representation and

$$\mathbf{J} : T^*\mathfrak{X}_{div}(\mathcal{D}) = \mathfrak{X}_{div}(\mathcal{D}) \times \mathbf{d}\Omega^1(\mathcal{D}) \to \mathfrak{X}_{div}(\mathcal{D})^* = \mathbf{d}\Omega^1(\mathcal{D}), \quad \mathbf{J}(x, p) = -\operatorname{ad}^*_x p = -\boldsymbol{\pounds}_x p$$

for its momentum map, we obtain the canonical Hamilton equations

$$\dot{x} + \boldsymbol{\pounds}_u x = 0, \quad \dot{p} + \boldsymbol{\pounds}_u p = 0, \tag{6.5}$$

where the optimal control $u \in \mathfrak{X}_{div}(\mathcal{D})$ is uniquely determined by the relation $\mathbf{d}u^\flat = -\boldsymbol{\pounds}_x p$. By the general theory, $\omega = \mathbf{d}u^\flat$ solves the Euler equations $\dot{\omega} + \boldsymbol{\pounds}_u \omega = 0$.

In the two dimensional case, writing $x = X_\lambda$, $p = \mu\sigma$, and $u = X_\psi$, where σ is the volume (and symplectic) form, consistently reproduces (6.2) from (6.5).

7 Generalization of the Clebsch optimal control

We now consider a slight generalization of the Clebsch optimal control given in Definition 2.1. Namely, we allow the Lagrangian ℓ to depend on $q \in Q$. The *Clebsch optimal control* is now

$$\min_{u(t)} \int_0^T \ell(u(t), q(t)) dt \tag{7.1}$$

subject to the following conditions:

(A) $\dot{q}(t) = u(t)_Q(q(t))$ or (A)′ $\dot{q}(t) = -u(t)_Q(q(t))$;

(B) $q(0) = q_0$ and $q(T) = q_T$.

In what follows we need the vertical lift operation. If $\alpha, \beta \in T_q^*Q$, the *vertical lift of* β *relative to* α is defined by

$$\operatorname{Ver}_\alpha \beta := \left.\frac{d}{ds}\right|_{s=0} (\alpha + s\beta) \in T_\alpha(T^*Q).$$

The analogue of Theorems 2.2, 2.3 in this case are the following.

Theorem 7.1 *Let* $\ell : \mathfrak{g} \times Q \to \mathbb{R}$ *be a function and assume that* $u \in \mathfrak{g} \mapsto \frac{\delta \ell}{\delta u} \in \mathfrak{g}^*$ *is a diffeomorphism. Let* G *act on the left (resp. on the right) on* Q. *Then, an extremal solution of the Clebsch optimal control problem* (2.4) *with condition* (A) *is a solution of*

$$\frac{\delta \ell}{\delta u} = \mathbf{J}(\alpha), \quad \dot{\alpha} = u_{T^*Q}(\alpha) + \operatorname{Ver}_\alpha \frac{\delta \ell}{\delta q}. \tag{7.2}$$

These equations imply (a generalization of) the Euler-Poincaré equations for the control u, *given by*

$$\frac{d}{dt}\frac{\delta \ell}{\delta u} = -\operatorname{ad}_u^* \frac{\delta \ell}{\delta u} + \mathbf{J}\left(\frac{\delta \ell}{\delta q}\right), \quad \textit{resp.} \quad \frac{d}{dt}\frac{\delta \ell}{\delta u} = \operatorname{ad}_u^* \frac{\delta \ell}{\delta u} + \mathbf{J}\left(\frac{\delta \ell}{\delta q}\right). \tag{7.3}$$

Proof. We give the proof in the case of a left action. The Pontryagin function is $\hat{H}(\alpha_q, u) = \langle \alpha_q, u_Q(q) \rangle - \ell(u, q)$. Thus, the first equation in (2.3) gives the condition

$$\frac{\delta \ell}{\delta u} = \mathbf{J}(\alpha)$$

on the optimal control u. The second equation is obtained by noting that the Hamiltonian vector field associated to $\hat{H}_u : T^*Q \to \mathbb{R}$ is $u_{T^*Q} + \operatorname{Ver}_\alpha \frac{\delta \ell}{\delta q}$. The Euler-Poincaré equations are computed as follows.

$$\begin{aligned}
\frac{d}{dt}\frac{\delta \ell}{\delta u} &= \frac{d}{dt}\mathbf{J}(\alpha) = T\mathbf{J}(u_{T^*Q}(\alpha)) + T\mathbf{J}\left(\operatorname{Ver}_\alpha \frac{\delta \ell}{\delta q}\right) \\
&= \left.\frac{d}{ds}\right|_{s=0} \mathbf{J}\left(\Phi^{T^*}_{\exp(su)}(\alpha)\right) + \left.\frac{d}{ds}\right|_{s=0} \mathbf{J}\left(\alpha + s\frac{\delta \ell}{\delta q}\right) \\
&= \left.\frac{d}{ds}\right|_{s=0} \operatorname{Ad}^*_{\exp(su)^{-1}} \mathbf{J}(\alpha) + \mathbf{J}\left(\frac{\delta \ell}{\delta q}\right) \\
&= -\operatorname{ad}_u^* \frac{\delta \ell}{\delta u} + \mathbf{J}\left(\frac{\delta \ell}{\delta q}\right). \qquad \blacksquare
\end{aligned}$$

Remark 7.2 The Euler-Poincaré equations (7.3) are a generalization of the Euler-Poincaré for semidirect products (see Holm, Marsden and Ratiu [1998a]) and of the affine Euler-Poincaré equations (see Gay-Balmaz and Ratiu [2008b]). They have been derived in Gay-Balmaz and Tronci [2009] by Lagrangian reduction. When Q is a dual vector space V^* on which G acts by a left representation, $a \in V^* \mapsto ga \in V^*$, the momentum map recover the usual expression

$$\mathbf{J}\left(\frac{\delta \ell}{\delta a}\right) = -\frac{\delta \ell}{\delta a} \diamond a$$

appearing in semidirect product theory. Here the diamond operator $\diamond : V \times V^* \to \mathfrak{g}$ is defined by $\langle v \diamond a, \xi \rangle = -\langle \xi a, v \rangle$, where ξa denotes the infinitesimal action of the Lie algebra element $\xi \in \mathfrak{g}$ on $a \in V^*$.

We now give the theorem in the case of the inverse representation, that is, when (A)$'$ is assumed instead of (A).

Theorem 7.3 (Inverse representation) *Let $\ell : \mathfrak{g} \times Q \to \mathbb{R}$ be a function and assume that $u \in \mathfrak{g} \mapsto \frac{\delta \ell}{\delta u} \in \mathfrak{g}^*$ is a diffeomorphism. Let G act on the left (resp. on the right) on Q. Then, an extremal solution of the Clebsch optimal control problem* (7.1) *with condition* (A)$'$ *is a solution of*

$$\frac{\delta \ell}{\delta u} = -\mathbf{J}(\alpha), \quad \dot{\alpha} = -u_{T^*Q}(\alpha) + \operatorname{Ver}_\alpha \frac{\delta \ell}{\delta q}.$$

These equations imply (a generalization of) the Euler-Poincaré equations for the control u, given by

$$\frac{d}{dt}\frac{\delta \ell}{\delta u} = \operatorname{ad}^*_u \frac{\delta \ell}{\delta u} - \mathbf{J}\left(\frac{\delta \ell}{\delta q}\right), \quad \textit{resp.} \quad \frac{d}{dt}\frac{\delta \ell}{\delta u} = -\operatorname{ad}^*_u \frac{\delta \ell}{\delta u} - \mathbf{J}\left(\frac{\delta \ell}{\delta q}\right). \tag{7.4}$$

The Hamiltonian H is defined by $H(\alpha_q) = \hat{H}(\alpha_q, u(\alpha_q))$, where $u(\alpha_q)$ is the optimal control. Thus $H(\alpha_q) = h(\mathbf{J}(\alpha_q), q)$, where $h : \mathfrak{g}^* \times Q \to \mathbb{R}$ is the Hamiltonian obtained from ℓ by Legendre transformation on $\mathfrak{g}$. When the constraint (A)$'$ is chosen instead of (A), the Hamiltonian is $H(\alpha_q) = h(-\mathbf{J}(\alpha_q), q)$. If ℓ is given by $\ell(u, q) = \frac{1}{2}\|u\|^2 - V(q)$ where $V : Q \to \mathbb{R}$ is a potential, we have

$$H(\alpha_q) = \frac{1}{2}\|\mathbf{J}(\alpha_q)\|^2 + V(q). \tag{7.5}$$

Restriction to G-orbits, simple mechanical systems associated to the normal metric. Note that in Theorem 7.1 and 7.3, one can replace the manifold Q acted on by the group G, by a G-orbit $\mathcal{O}$ in Q. To do this, it suffices to restrict the Lagrangian $\ell : \mathfrak{g} \times Q \to \mathbb{R}$ to $\ell^{\mathcal{O}} : \mathfrak{g} \times \mathcal{O} \to \mathbb{R}$. In this case we get the equations

$$\frac{\delta \ell}{\delta u} = -\mathbf{J}_{\mathcal{O}}(\alpha), \quad \dot{\alpha} = -u_{T^*\mathcal{O}}(\alpha) + \operatorname{Ver}_\alpha \frac{\delta \ell^{\mathcal{O}}}{\delta q}, \tag{7.6}$$

where $\mathbf{J}_{\mathcal{O}} : T^*\mathcal{O} \to \mathfrak{g}^*$ is the cotangent bundle momentum map associated to the G-action on the orbit. To prove that the system on Q restricts to an orbit $\mathcal{O}$, it remains to show that given a solution $\alpha \in T^*Q$ of equations (7.2), then $T^*i(\alpha) \in T^*\mathcal{O}$ is a solution of equations (7.6), where $i : \mathcal{O} \to Q$ denotes the inclusion and $T^*i : T^*Q \to T^*\mathcal{O}$ the cotangent map of i. This can be easily checked, by using the equalities

$$u_{T^*\mathcal{O}} \circ T^*i = TT^*i \circ u_{T^*Q}, \quad \operatorname{Ver}_{T^*i(\alpha)}\left(\frac{\delta l^{\mathcal{O}}}{\delta q}\right) = TT^*i\left(\operatorname{Ver}_\alpha\left(\frac{\delta l}{\delta q}\right)\right), \quad \text{and} \quad \mathbf{J}_{\mathcal{O}} \circ T^*i = \mathbf{J}.$$

We now consider the particular case when ℓ is of the form $\ell(u,q) = \frac{1}{2}\|u\|^2 - V(q)$, where the norm is associated to a positive definite inner product γ on $\mathfrak{g}$ and $V : \mathcal{O} \to \mathbb{R}$ is a potential. Using formula (7.5), the associated Hamiltonian on $T^*\mathcal{O}$ reads

$$H(\alpha_q) = \frac{1}{2}\gamma_{\mathcal{O}}(\alpha_q^\sharp, \alpha_q^\sharp) + V(q),$$

where $\gamma_{\mathcal{O}}$ is the normal metric on $\mathcal{O}$ induced by γ (see (2.6)). We obtain the Hamiltonian of a simple mechanical system on the orbit $\mathcal{O}$ endowed with the normal metric. We thus obtain the following result.

Theorem 7.4 *Let G act on the left (resp. on the right) on Q, let γ be a positive definite inner product on $\mathfrak{g}$, V a function on Q, and let ℓ be the Lagrangian given by kinetic minus potential energy. Then, an extremal solution of the Clebsch optimal control problem* (7.1) *with condition* (A) *is given by $u(t) = \mathbf{J}(\alpha(t))^\sharp$, where $\alpha(t)$ is the solution of the mechanical system on $\mathcal{O} \subset Q$ associated to the normal metric and to the potential V.*

The case of adjoint representation. As a first example for the Clebsch optimal control problem (7.1), we reconsider the case of adjoint orbits discussed before. The Lagrangian ℓ can now be taken to be of the form $\ell(u,x) = \frac{1}{2}\|u\| - V(x)$, where the norm is associated to a positive definite Ad-invariant inner product on $\mathfrak{g}$ and $V : \mathfrak{g} \to \mathbb{R}$. The associated Clebsch optimal control problem reads

$$\min_{u(t)} \int_0^T \left(\frac{1}{2}\|u(t)\|^2 - V(x(t))\right) dt$$

subject to the following conditions:

(A) $\dot{x} = [x, u]$;

(B) $x(0) = x_0$ and $x(T) = x_T$.

A particularly interesting special case of this problem is that of Brockett [1994] where we have $V(x) = -\frac{1}{2}\|[x,n]\|^2$ for a fixed $n \in \mathfrak{g}$. Using Theorem 7.1, and the relation $\delta\ell/\delta x = -\delta V/\delta x$, the optimal control is $u = [p, x]$ and the canonical Hamilton's equations are

$$\dot{x} = [x, u], \quad \dot{p} = [p, u] - \frac{\delta V}{\delta x}.$$

Thus, we get the equations

$$\dot{x} = [x, [p, x]], \quad \dot{p} = [p, [p, x]] - \frac{\delta V}{\delta x}.$$

Note that the Euler-Poincaré equation yields the relation $\dot{u} = \left[x, \frac{\delta V}{\delta x}\right]$. By Theorem 7.4, these equations are the Hamiltonian description of the mechanical system associated to the Lagrangian

$$L(x, \dot{x}) = \frac{1}{2}\|\operatorname{ad}_x^{-1}\dot{x}\|^2 - V(x)$$

on adjoint orbits.

In the special case of the potential $V(x) = -\frac{1}{2}\|[n,x]\|^2$, we get

$$\dot{x} = [x,[p,x]], \quad \dot{p} = [p,[p,x]] - [n,[n,x]]$$

and the Lagrangian is

$$L(x,\dot{x}) = \frac{1}{2}\|\operatorname{ad}_x^{-1}\dot{x}\|^2 + \frac{1}{2}\|[n,x]\|^2.$$

This recovers results of Bloch, Brockett, and Crouch [1997].

One can carry out the same generalizations for the case of symmetric spaces and homogeneous reductive spaces.

Optimal control on Stiefel manifolds. The Stiefel manifold $V(n,N) \subset \mathbb{R}^{nN}$ consists of orthogonal n frames in N dimensional real Euclidean space,

$$V(n,N) := \left\{Q \in \mathbb{R}^{nN} \mid QQ^T = I_n\right\}.$$

In the extremal cases $n = 1$ and $n = N$, the Stiefel manifold reduces to the sphere S^{N-1} and to the group $SO(N)$, respectively. An optimal control problem on Stiefel manifolds is introduced and studied in Bloch, Crouch, and Sanyal [2006], as a generalization of the geodesic flow on the sphere (case $n = 1$) and the motion of the N-rigid body (case $n = N$). In this paragraph we generalize this problem to arbitrary Lagrangians and recast it into a Clebsch optimal control problem of the form (7.1) and analyze some consequences of our approach.

The tangent space of the Stiefel manifold at Q is the subspace of $\mathbb{R}^{nN}$ given by $T_QV(n,N) = \left\{V \mid QV^T + VQ^T = 0\right\}$. We will identify the cotangent space $T_Q^*V(n,N)$ with the tangent space, by using the pairing $\langle P,V\rangle := \operatorname{Tr}(P^TV)$. Consider the group $G = SO(N)$ acting on the right on the Stiefel manifold $V(n,N)$ by matrix multiplication. This action is transitive and $V(n,N)$ is isomorphic to the homogeneous space $SO(N)/SO(N-n)$. The associated infinitesimal generator is $U_{V(n,N)}(Q) = QU \in T_QV(n,N)$. Given $Q_0, Q_T \in V(n,N)$, the Clebsch optimal control problem (7.1) reads

$$\min_{U(t)} \int_0^T \ell(U(t),Q(t))dt \tag{7.7}$$

subject to the following conditions:

(A) $\dot{Q}(t) = Q(t)U(t)$;

(B) $Q(0) = Q_0$ and $Q(T) = Q_T$.

The Pontryagin function reads $\hat{H}(Q,P,U) = \langle P,QU\rangle - \ell(U,Q)$ and the cotangent bundle momentum map $\mathbf{J} : T^*V(n,N) \to \mathfrak{so}(N)^*$ is given by

$$\mathbf{J}(Q,P) = \frac{1}{2}(Q^TP - P^TQ),$$

where, as before, the dual $\mathfrak{so}(N)^*$ is identified with the Lie algebra via the Killing form. The optimal control is thus given by $\delta\ell/\delta U = (Q^T P - P^T Q)/2$. The cotangent lifted action on $T^*V(n,N)$ reads $(Q,P) \mapsto (Qg, Pg)$ and hence Hamilton's equations (7.2) become

$$\dot{Q} = QU, \quad \dot{P} = PU + \frac{\delta\ell}{\delta Q}, \tag{7.8}$$

in our particular case. Recall that here $\delta\ell/\delta Q \in T^*_Q V(n,N)$ denotes the functional derivative of ℓ relative to the above defined pairing. The optimal control U is the solution of the Euler-Poincaré equation (7.3) (right version) given by

$$\frac{d}{dt}\frac{\delta\ell}{\delta U} = \left[\frac{\delta\ell}{\delta U}, U\right] + \frac{1}{2}\left(Q^T \frac{\delta\ell}{\delta Q} - \frac{\delta\ell}{\delta Q}^T Q\right) \tag{7.9}$$

in our particular case.

Let us specialize the above considerations to the Lagrangian of the problem studied in Bloch, Crouch, and Sanyal [2006], namely $\ell(U,Q) = \frac{1}{2}\langle QU\Lambda, QU\rangle$, where Λ is a given positive definite $N \times N$ matrix. The functional derivative of ℓ with respect to U is $\delta\ell/\delta U = \frac{1}{2}(Q^T QU\Lambda + \Lambda U Q^T Q)$. Therefore the optimal control U is determined by the condition

$$Q^T QU\Lambda + \Lambda U Q^T Q = Q^T P - P^T Q. \tag{7.10}$$

We now compute the functional derivative of ℓ with respect to Q. For a curve $Q_t \in V(n,N)$ such that $Q_0 = Q$, we have

$$\left\langle \frac{\delta\ell}{\delta Q}, \dot{Q} \right\rangle = \left.\frac{d}{dt}\right|_{t=0} \ell(U, Q_t) = \frac{1}{2}\langle \dot{Q}U\Lambda, QU\rangle + \frac{1}{2}\langle QU\Lambda, \dot{Q}U\rangle = -\langle QU\Lambda U, \dot{Q}\rangle.$$

Note that $QU\Lambda U$ does not belong to $T^*_Q V(n,N)$, in general. To obtain the expression of $\delta\ell/\delta Q \in T^*_Q V(n,N)$, we remark that $S := U\Lambda U$ is a symmetric matrix, and we get

$$\begin{aligned}\left\langle \frac{\delta\ell}{\delta Q}, \dot{Q} \right\rangle &= -\langle QS, \dot{Q}\rangle = -\langle QS - Q(QS)^T Q, \dot{Q}\rangle - \langle Q(QS)^T Q, \dot{Q}\rangle \\ &= -\langle QS - Q(QS)^T Q, \dot{Q}\rangle,\end{aligned}$$

where we used $\langle Q(QS)^T Q, \dot{Q}\rangle = \mathrm{Tr}(Q^T \dot{Q} Q^T QS) = 0$, since $Q^T \dot{Q} Q^T Q$ is a skew-symmetric matrix and S is symmetric. Now $QS - Q(QS)^T Q$ belongs to $T^*_Q V(n,N)$ and so the functional derivative equals

$$\frac{\delta\ell}{\delta Q} = -QS + Q(QS)^T Q = -QU\Lambda U + QU\Lambda U Q^T Q.$$

Thus, for this particular Lagrangian, the solution (7.8) of the Clebsch optimal control problem reads

$$\dot{Q} = QU, \quad \dot{P} = PU - QU\Lambda U + QU\Lambda U Q^T Q. \tag{7.11}$$

The corresponding Euler-Poincaré equations (7.9) reads

$$\frac{d}{dt}M = [M, U] + [U\Lambda U, Q^T Q], \quad M = 2\frac{\delta\ell}{\delta U} = Q^T QU\Lambda + \Lambda U Q^T Q.$$

This result, as well as Hamilton's equations (7.10) and the optimal control (7.11) are obtained here by applying Theorem 7.1. They coincide with those of Bloch, Crouch, and Sanyal [2006] (see equations (68, 69), (74), and (75)) obtained by ad hoc methods.

Optimal control formulation of (affine) Euler-Poincaré equations for semidirect products. Euler-Poincaré reduction for semidirect products is formulated in Holm, Marsden and Ratiu [1998a] and is useful to understand the Lagrangian dynamics of symmetric mechanical systems on Lie groups with advected quantities, such as the heavy top or compressible fluids. The basic ingredients are a Lie group G that acts by *right* representation on a vector space V and a G-invariant function $L : TG \times V^* \to \mathbb{R}$. This function L determines a family of Lagrangians $L_a : TG \to \mathbb{R}$, $L_a(v_g) := L(v_g, a)$, invariant under the action of the isotropy group G_a of a under the representation.

For $a_0 \in V^*$ held fixed, the Euler-Lagrange equations for L_{a_0} on TG are equivalent to the Euler-Poincaré equations

$$\frac{d}{dt}\frac{\delta \ell}{\delta u} = -\operatorname{ad}^*_u \frac{\delta \ell}{\delta u} + \frac{\delta \ell}{\delta a} \diamond a, \tag{7.12}$$

together with the advection equation $\dot{a} + au = 0$ with initial condition a_0. Here, for $u \in \mathfrak{g}$, $v \in V$, and $a \in V^*$, au denotes the infinitesimal action of $\mathfrak{g}$ on V^*, and $v \diamond a \in \mathfrak{g}^*$ is defined by $\langle v \diamond a, u\rangle = -\langle au, v\rangle$. The cotangent bundle momentum map for the representation is $\mathbf{J}(a, v) = -v \diamond a$. Therefore, the Euler-Poincaré equations (7.12) are a particular case of the equations (7.4) and can thus be obtained by the Clebsch optimal control problem (7.1) specialized to the case of a representation of the Lie group G on V^*:

$$\min_{u(t)} \int_0^T \ell(u(t), a(t))dt \tag{7.13}$$

subject to the following conditions:

(A) $\dot{a}(t) + a(t)u(t) = 0$;

(B) $a(0) = a_0$ and $a(T) = a_T$.

The optimal control u is determined by the relation $\delta\ell/\delta u = -v \diamond a$. This relation is reminiscent of the Clebsch approach mentioned in Marsden, Ratiu and Weinstein [1984] (see especially equation (4.2)). As we will see below, it is not always suitable in some examples.

As a consequence, we obtain an optimal control formulation for a wide class of physical systems arising as Euler-Poincaré equations with advected quantities such as heavy top, compressible fluid, magnetohydrodynamics (Holm, Marsden and Ratiu [1998a]) and Euler-Yang-Mills fluids (Gay-Balmaz and Ratiu [2008a]). For example, for compressible fluids, the group is $G = \operatorname{Diff}(\mathcal{D})$ and the advected parameter is the mass density $\rho \in \mathcal{F}(\mathcal{D})^*$. The Lagrangian is

$$\ell(u, \rho) = \frac{1}{2}\int_{\mathcal{D}} \rho \|u\|^2 \mu - \int_{\mathcal{D}} \rho e(\rho)\mu,$$

where e is the fluid's specific internal energy. This Lagrangian is strictly convex in u. The optimal control problems reads

$$\min_{u(t)} \int_0^T \left(\int_{\mathcal{D}} \rho \|u\|^2 \mu - \int_{\mathcal{D}} \rho e(\rho)\mu \right) dt \tag{7.14}$$

subject to the following conditions:

(A) $\dot{\rho} + \operatorname{div}(\rho u) = 0$;

(B) $\rho(0) = \rho_0$ and $\rho(T) = \rho_T$.

In this case the relation $\delta\ell/\delta u = -v \diamond a$ produces the condition $u = -\operatorname{grad} f$, thus the flow is required to be potential.

In order to eliminate this restriction, we introduce an additional variable acted on by the diffeomorphism group. From the abstract point of view, this consists in introducing, besides V, a second G-representation space W. In this case, the cotangent bundle momentum map associated to the G-representation on $V^* \times W^*$ reads $\mathbf{J}((a,b),(v,w)) = -v \diamond a - w \diamond b \in \mathfrak{g}^*$, $(a,b) \in V^* \times W^*$. For the example of the compressible fluid, we can consider the additional variable $\lambda \in \mathcal{F}(\mathcal{D})^*$. To the optimal control problem (7.14) we add the constraint

$$\dot{\lambda} + \operatorname{div}(\lambda u) = 0, \quad \lambda(0) = \lambda_0, \quad \lambda(T) = \lambda_T. \tag{7.15}$$

The abstract optimal control condition $\delta\ell/\delta u = -v \diamond a - w \diamond b$ becomes in this case $\rho u = -\rho \operatorname{grad} f - \lambda \operatorname{grad} g$ and hence u is not required to be potential anymore if the initial value $\lambda_0 \neq 0$. The previous expression of ρu in terms of the canonical variables (ρ, f, λ, g) recovers the classical Clebsch representation for compressible flows.

Remark 7.5 In this new optimal control problem the Lagrangian ℓ does not depend on the additional variable λ. In spite of this, the two optimization problems (7.14) and the new one with the constraint (7.15) added do not have the same extremum.

For magnetohydrodynamics (MHD), it is not needed to extend the representation space, since there is already an additional variable (the magnetic field $\mathbf{B}$) naturally present in the problem. The Lagrangian for MHD is

$$\ell(u, \rho, \mathbf{B}) = \frac{1}{2}\int_{\mathcal{D}} \rho \|u\|^2 \mu - \int_{\mathcal{D}} \rho e(\rho) \mu + \frac{1}{2}\int_{\mathcal{D}} \|\mathbf{B}\|^2 \mu,$$

and the motion equation can be obtained by the Clebsch optimal control problem with condition (A) given by

$$\dot{\rho} + \operatorname{div}(\rho u) = 0, \quad \dot{\mathbf{B}} + \operatorname{curl}(\mathbf{B} \times u) + u \operatorname{div} \mathbf{B} = 0.$$

The optimal control condition $\delta\ell/\delta u = -v \diamond a$ becomes

$$\rho u = -\rho \diamond f - \mathbf{b} \diamond \mathbf{B} = -\rho \operatorname{grad} f + \mathbf{b} \operatorname{div} \mathbf{B} - \mathbf{B} \times \operatorname{curl} \mathbf{b}.$$

This recover the usual Clebsch variables $(f, \mathbf{b}) \in \mathcal{F}(\mathcal{D}) \times \mathfrak{X}(\mathcal{D})$ in magnetohydrodynamics; see example 5C in Marsden, Ratiu and Weinstein [1984] and references therein.

As shown in Gay-Balmaz and Ratiu [2008b], the dynamics of complex fluids needs to consider affine representations of G on V^*. Such representations are given by $a \mapsto ag + c(g)$, where $a \mapsto ag$ is a representation of G on V^* and $c : G \to V^*$ is a group one-cocycle. In this case, equations (7.12) generalize to

$$\frac{d}{dt}\frac{\delta\ell}{\delta u} = -\operatorname{ad}^*_u \frac{\delta\ell}{\delta u} + \frac{\delta\ell}{\delta a} \diamond a - \mathbf{d}c^T\left(\frac{\delta\ell}{\delta a}\right), \tag{7.16}$$

together with the affine advection equation $\dot{a} + au + \mathbf{d}c(u) = 0$ with initial condition a_0. Here $\mathbf{d}c : \mathfrak{g} \to V^*$ denotes the derivative of c at the identity element of G, and $\mathbf{d}c^T : V \to \mathfrak{g}^*$ its transpose. One obtains the affine Euler-Poincaré equations (7.16) by the Clebsch optimal control problem (7.1), when specialized to the case of an affine representation of G on V^*:

$$\min_{u(t)} \int_0^T \ell(u(t), a(t)) dt \tag{7.17}$$

subject to the following conditions:

(A) $\dot{a}(t) + a(t)u(t) + \mathbf{d}c(u(t)) = 0$;

(B) $a(0) = a_0$ and $a(T) = a_T$.

This optimal control formulation is applicable to the examples of complex fluids studied in Gay-Balmaz and Ratiu [2008b], when the usual convexity assumption on the Lagrangian is verified. For application to complex fluids, the symmetry group G is the semidirect product $G = \mathrm{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathcal{O})$, where $\mathcal{O}$ is the order parameter group of the theory. The vector space V^* contains the usual linearly advected quantities such as the mass density and the entropy, that are acted on only by the diffeomorphism groups. Besides this, V^* contains also complex fluid variables such as directors or connections, that are acted on (possibly affinely) by the whole semidirect product group G. We do not pursue these investigations here since this subject is not directly related to optimal control. A complete treatment of complex fluids using the Clebsch optimal control problem linking the approach in Marsden, Ratiu and Weinstein [1984] and the derivation of the equations of motion using the classical Clebsch method in Holm [2002] will be carried out in future work.

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