Comment on "Finding finite-time invariant manifolds in two-dimensional velocity fields" [Chaos 10, 99 (2000)]

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This note serves as a commentary of the paper of Haller [Chaos **10**, 99 (2000)] on techniques for detecting invariant manifolds. Here we show that the criterion of Haller can be improved in two ways. First, by using the strain basis reference frame, a more efficient version of theorem 1 of Haller (2000) allows to better detect the manifolds. Second, we emphasize the need to nondimensionalize the estimate of *hyperbolic persistence*. These statements are illustrated by the example of the Kida ellipse. [DOI: 10.1063/1.1374241]

Stirring and mixing in fluid mechanics is often incomplete owing to a partition of the flow in subdomains separated by transport barriers. Recently Haller [Chaos 10, 99 (2000)] has proposed an algorithm based on analytic results to identify the invariant manifolds which are underlying the barriers. In this paper, we generalize this approach and provide an improved algorithm that optimizes the detection of manifolds.

I. HALLER'S APPROACH FOR HYPERBOLIC POINTS

Hyperbolic trajectories and invariant manifolds play a crucial role for transport and mixing in periodic and aperiodic flows. For aperiodic two-dimensional flows, several techniques for detecting these finite-time hyperbolic trajectories have been proposed.[1–4] Recently, Haller[2] has proven a mathematical theorem (his theorem 1) providing sufficient existence conditions for uniform hyperbolic trajectories defined as particles trajectories admitting finite-time stable and unstable manifolds.

Haller proposes a two-step numerical algorithm for detecting the hyperbolic trajectories. The first step consists in calculating the local maxima of *hyperbolic persistence* defined for forward and backward integration

$$d_T^+(x_0,t_0)=\max_{t\in[t_0,t_0+T]}\{(t-t_0)|\det D_x u(x(\tau,x_0),\tau)<0,\quad t_0\leqslant\tau<t\},\tag{1}$$

$$d_T^-(x_0,t_0)=\max_{t\in[t_0-T,t_0]}\{(t_0-t)|\det D_x u(x(\tau,x_0),\tau)<0,\quad t0<\tau\leqslant t_0\},\tag{2}$$

where $u(x,t)$ is the velocity and $D_x u(x(t),t)$ is the velocity gradient tensor on a Lagrangian trajectory $x(t,x_0)$. The second step consists in testing that the eigenvectors of $D_x u(x(t),t)$ do not rotate too fast along the Lagrangian trajectories selected by the first step [Eqs. (6) and (7) in Ref. 2)] over the duration of hyperbolic persistence.

Haller claims that the local maxima of d_T^+ and d_T^- are, respectively, mapping the stable manifold and the unstable manifold near the hyperbolic point. He further claims that this is generally sufficient to accurately identify the hyperbolic point as their intersection without need of the second step.

The algorithm is basically a Lagrangian version of Okubo–Weiss[5,6] criterion, separating strain dominated from vorticity dominated regions in the flow, where the separation is performed as a function of persistence along particle trajectories.

The Okubo–Weiss criterion has been, however, criticized[7–9] for not taking into account the strain axes rotation. In particular, it has been shown[9] that a more general criterion is provided by using as a reference frame the eigenvectors of the strain matrix

$$S(x(t),t)\equiv D_x u(x(t),t)+D_x u(x(t),t)^*,$$

where * denotes the transpose.

This suggests that weaker sufficient conditions can be found to detect hyperbolic trajectories and that Haller's algorithm can be improved.

We first show below a case for which the Haller's algorithm fails to identify unambiguously the hyperbolic point. We then introduce a modification of the algorithm based on the generalized criterion of Lapeyre, Klein, and Hua[9] and we show that it corresponds to a special case of a generalized version of Haller's theorem. We further propose a second modification rescaling the hyperbolic persistence according to the strain rate.

II. THE EXAMPLE OF THE KIDA ELLIPSE

We consider here the Kida ellipse[10,11] which is an exact solution of the two-dimensional Euler equations. This solution is an elliptic patch of constant vorticity ω within a uniform generalized strain $(u_{\rm ext}=-(\Omega_{\rm ext}+\sigma_{\rm ext})y, v_{\rm ext}=(\Omega_{\rm ext}-\sigma_{\rm ext})x)$. The elliptical shape is preserved by the temporal

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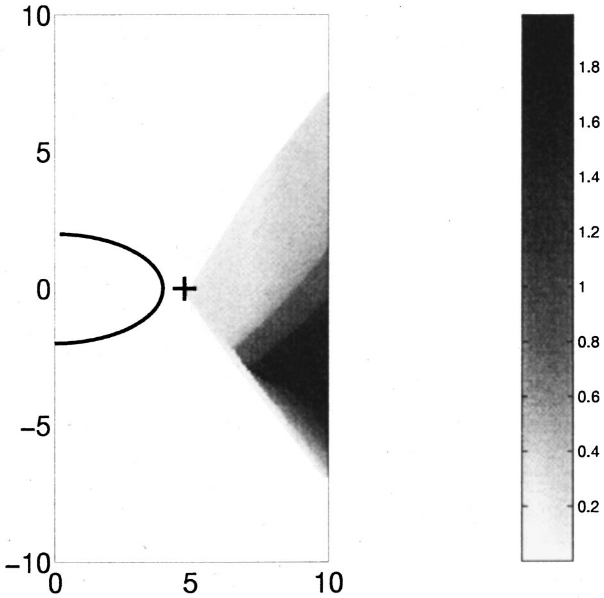

FIG. 1. Persistence $\tilde{d}_T^+(x_0)$ of the two conditions of Haller's theorem. The cross indicates the position of the hyperbolic point. The black contour shows the edge of the elliptical patch at $t=0$.

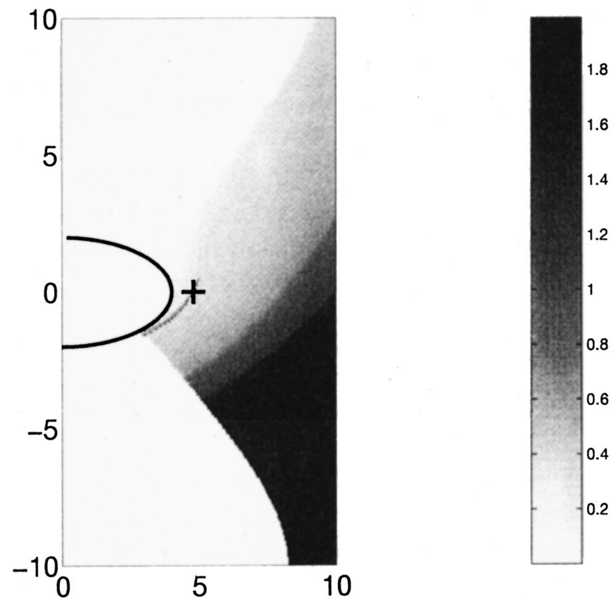

FIG. 2. Same as Fig. 1 but for the hyperbolicity persistence $d_T'^+(x_0)$ of the velocity gradient tensor expressed in strain basis.

evolution. The angle θ of the great axis with the x axis and the aspect ratio γ (great axis/small axis) vary according to

$$\frac{d\gamma}{dt}=-2\gamma\sigma_{\text{ext}}\sin(2\theta),$$

$$\frac{d\theta}{dt}=\Omega_{\text{ext}}+\frac{\omega\gamma}{(\gamma+1)^2}-\sigma_{\text{ext}}\frac{\gamma^2+1}{\gamma^2-1}\cos(2\theta).$$

This simple model is an approximation for the motion of a coherent vortex embedded in a turbulent field where the vortex feels the influence of other vortices through the strain field. Thus it can mimic the flow around vortices, and in particular the chaotic nature of the Lagrangian trajectories.[12,13]

Here we choose, $\omega=20$, $\Omega_{\text{ext}}=0$, $\sigma_{\text{ext}}=0.6$ and, at $t=0$, $\theta=0$ and $\gamma=2$ so that the ellipse is initially oriented along the x axis. The solution is periodic with period $T^*=1.38\cdots$, the aspect ratio oscillating between 1.54 and 2.

Because of this special initial condition, the backward time evolution of the ellipse is symmetric to the forward evolution with respect to the x axis. Consequently, we need only to perform forward integration.

In this example, the hyperbolic periodic orbit is easily localized on the x axis in $x=4.79\cdots$ by inspecting trajectories initiated on the x axis and can be used to test Haller's method. This location corresponds to the position of the orbit at the specific Poincare section. For each particle initially on a regular grid, we have computed the trajectory and diagnosed the strain properties using fourth order Runge–Kutta integrator. The total time of integration is $T=2$.

Since trajectories outside the ellipse experience strain but no vorticity, the first step of Haller's criterion is always satisfied. Using now the second step [second condition in Eq. (6) of Ref. 2], we show in Fig. 1 the distribution of persistence $\tilde{d}_T^+(x_0)$ of trajectories satisfying both conditions. The estimated hyperbolic region is limited to a part of the plane but clearly fails to identify the hyperbolic point as a local maximum of the hyperbolic persistence.

III. GENERALIZED CONDITIONS OF HYPERBOLICITY

We now use another approach, derived from Lapeyre *et al.*,[9] which takes into account the rotation of strain axes. Instead of considering the evolution of the line element y under the equation

$$\dot{y}=D_x u(x(t),t)y, \tag{3}$$

we consider here the evolution of the rotated line element $y=Ry'$, where R is the rotation matrix defined by the orthonormal basis of the strain matrix $S(x(t),t)$. This vector satisfies a new equation

$$\dot{y}'=(R^{-1}D_x u(x(t),t)R-R^{-1}\dot{R})y'\equiv[\nabla u]_{\text{strain}}y', \tag{4}$$

where the matrix $[\nabla u]_{\text{strain}}$ can be expressed[9] as a function of the strain rate and the *effective rotation* (rotation due to both vorticity and strain axes rotation).

New hyperbolic persistences $d_T'^{\pm}$ can be defined by replacing $D_x u$ by $[\nabla u]_{\text{strain}}$ in Eqs. (1) and (2). Figure 2 shows the distribution of d_T^+ for which strain is larger than effective rotation. We see that there is a well defined curve of local maxima passing through the hyperbolic point on the x axis. This curve is a piece of the stable manifold; the unstable manifold is obtained by symmetry with respect to the x axis.

This case shows that the fast rotation of the strain axes can mask the presence of the hyperbolic point. However, the use of the reference frame of the strain axes allows to detect the hyperbolic point: Such axes are the principal axes that optimize the growth and decay in amplitude of the tracer gradient.

The rationale of this result lies in the possibility to modify Haller's theorem with weaker conditions. We can generalize Eq. (3) to any field $R_\phi(x,t)$ applying a rotation of angle ϕ to the material element y in x, leading to a new evolution equation

$$\dot{y}' = D'_x u(x(t),t) y' + O(|y'|^2), \tag{5}$$

where

$$D'_x u(x(t),t) = R_{-\phi} D_x u(x(t),t) R_\phi - \dot{\phi} R_{\pi/2}.$$

The proof and the results entirely follow those of Haller but for replacing D_x and its eigenvalues by D'_x and its eigenvalues which are obviously different because of the $\dot{\phi}$ term.

Since the rotation ϕ is largely arbitrary, it can lead to weaker as well to stronger conditions for hyperbolicity. Its choice can be dictated by physical consideration and we show above, on the example of the Kida ellipse, that using the strain axes as a reference frame provides weaker conditions, at least on this case.

Notice that we have only used in this example the first condition of the improved theorem, not the condition of axes rotation in the reference frame. It turns out that using this condition does not improve the localization of the hyperbolic point as a persistence maximum.

IV. FURTHER IMPROVEMENT

The stable manifold in Fig. 2 is only visible as a line of local maxima in the persistence field. The map exhibits also a broad region of high persistence, actually saturating at the value of the integration time, much larger than that obtained near the hyperbolic point. It has been checked by increasing the integration time that the maxima do not appear in this region but are rejected at infinity.

The reason of this artifact is that the hyperbolic persistence is a dimensional quantity. When we compare its values in various regions of the flow, we implicitly assume that the Lagrangian time scales are similar. If this is not the case, the interesting hyperbolic features may be blurred out, as above.

In order to remedy this effect, we propose to use the characteristic time scale provided by the strain rate to adimensionalize the hyperbolic persistence. Thus we replace

$$d'^{+}_T(x_0) = \int_{\{0 \leq t < T | \det([\nabla u]_{\text{strain}}) < 0\}} dt,$$

by

$$e'^{+}_T(x_0) = \int_{\{0 \leq t < T | \det([\nabla u]_{\text{strain}}) < 0\}} \sigma(t) dt, \tag{6}$$

where σ is the strain rate.

Figure 3 shows the distribution of $e'_T(x_0)$. We see that the hyperbolic point is still captured by this method and that the high-persistence region seen in Fig. 2 has disappeared. Moreover, the stable manifold is now a global maximum of $e'_T(x_0)$ in this case. The conjunction of the two approaches (opting for a strain basis reference frame and using a nondimensional time scale) allows to capture very efficiently the hyperbolic point.

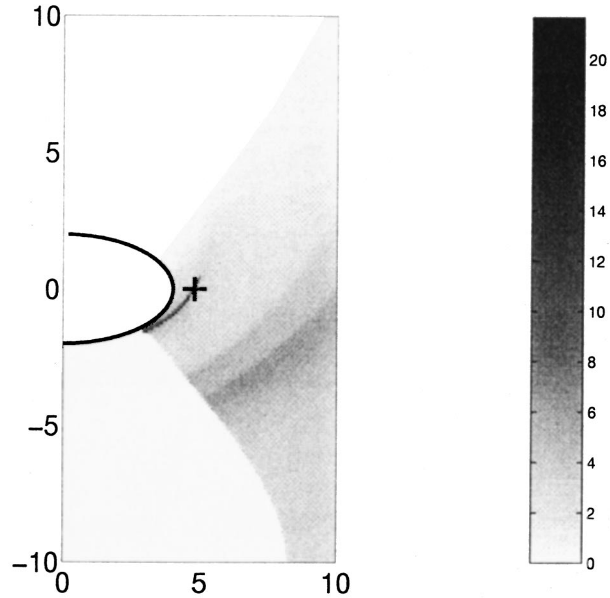

FIG. 3. Same as Fig. 1 but for the normalized hyperbolicity persistence $e'^{+}_T(x_0)$.

V. CONCLUSION

We have seen that in the simple case of a Kida ellipse, the Haller's algorithm to determine hyperbolic trajectories has to be modified. This is due to the rotation of the strain axes which is important in this flow. We propose two types of modifications. The first one, which is based on a generalized version of Haller's theorem, is to use a condition based on strain and ''effective rotation,'' taking into account the rotation of the strain axes. The second modification, which is more heuristic, is to adimensionalize the hyperbolic persistence by the strain rate. As the Kida ellipse is a good prototype for the large family of flows with long-lived coherent structures, it is likely that our modifications will improve the detection of hyperbolic trajectories in many cases.

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