Black-body law

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<< PhysicalConstants`
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Classical statistical physics says that in an atom, the accelerated movement of the electron, which is a charged particle, must be accompanied by emission or absorption of electromagnetic energy. All levels are accessible (in the limit where the electron is not in such a low orbit that it would “touch” the nucleus). The most energetic levels correspond to X-rays, discovered by Wilhelm Roentgen in 1895.

Boltzmann’s law of thermal equilibrium is based on the fact that all states of the system for a given total energy are equally accessible. It is then shown that the number of states allowing a parcel of the system to have an energy ϵ is ~ Exp[-(ϵ/(k T))] where k is the Boltzmann constant.
Applied to the kinetics of gases, this law makes it possible to obtain that the energy of a perfect monatomic gas is U= 3/2nkT where n is the number of atoms, i.e. U=3/2RT for a mole of gas with R=N k where N is Avogadro’s number. The degrees of freedom are here the three components of the speed of the atoms.

Let us apply the law to the electrons surrounding molecules or atoms, seen as many independent oscillators. The average energy associated with an oscillator is then

$$\epsilon = \text{Assuming}\left[k\,T > 0, \frac{\int_0^\infty \epsilon\,\text{Exp}\left[-\frac{\epsilon}{k\,T}\right] d\epsilon}{\int_0^\infty \text{Exp}\left[-\frac{\epsilon}{k\,T}\right] d\epsilon}\right]$$

supposant

k T

When a body is heated to high temperature, it is excited and its electrons store energy. They give it to the environment by emitting energy in the form of photons, which makes the body luminous. By losing energy, the body cools down.
The same phenomenon also occurs at room temperature but, as the law of the black body will confirm, the emission takes place outside the visible range, in the infrared.
We consider here that matter only exchanges energy in radiative form and we neglect all the other effects (conduction, convection, etc.).

In its normal state, a body is bathed in a flow of energy (thermal exchange with other neighboring bodies and exchange with radiation). It is this exchange with radiation that is dealt with by the law of the black body.

A prototype black body may consist of a dense gas at high temperature T, opaque in all radiation, thermally insulated and surrounded by perfect mirrors which reflect all radiations. Another (more practical) model is a perfectly insulated oven with an internal material brought to temperature T, exchanging nothing with the outside of the oven and surrounding an empty cavity where the electromagnetic radiations propagate. An observer can in both cases measure the radiation by drilling a small diameter hole in the wall generating little disturbance to the system.

In this case, classical physics says that the energy of the radiation at the frequency ν is

$$J[\nu\_] := \frac{4\,\nu^2}{c^2}\,k\,T$$

This result is obtained by balancing the cooling by radiation of the electrons oscillating at the frequency ν which is ~ν^2 k T with the absorption by this oscillator of the ambient radiation which is ~ J[\[Nu]]. Precise calculation implies knowing the laws of electromagnetism and calculating the radiation diagram of an oscillating dipole, but this is ignored here.

The problem is that this law predicts that radiation in equilibrium grows as ν^2, and therefore any heated body should emit high doses of X-rays, which is not the case. Therefore, there must be a fundamental error in this reasoning and its correction must provide a cut-off factor at large ν for J[ν].

This is more or less the reasoning path followed by Planck who proved that such a cut-off factor could be produced by assuming that the energy exchanges are quantified in multiples of hν.

Thus the possible states of the oscillator are no longer continuous but discrete in the form ϵ_0, ϵ_0+hν, ϵ_0 + 2 hν, ϵ_0+3 hν, ... where ϵ_0 is the fundamental.

If we apply Boltzmann's law to obtain the average energy of oscillators at frequency ν, we then obtain it as a ratio of two sums rather than two integrals, that is

$$\epsilon = \frac{\sum_{n=0}^{\infty} (\epsilon_0 + n\,h\,\nu)\,\mathrm{Exp}\left[-\frac{\epsilon_0 + n\,h\,\nu}{k\,T}\right]}{\sum_{n=0}^{\infty} \mathrm{Exp}\left[-\frac{\epsilon_0 + n\,h\,\nu}{k\,T}\right]}$$

By denoting $p = \mathrm{Exp}\left[\frac{-h\,\nu}{k\,T}\right]$, we have

$$\epsilon = \epsilon_0 + h\nu\,\frac{\sum_{n=0}^{\infty} n\,p^n}{\sum_{n=0}^{\infty} p^n} = \epsilon_0 + h\nu\,\frac{\frac{p}{(1-p)^2}}{\frac{1}{1-p}} = \epsilon_0 + h\nu\,\frac{1}{p^{-1}-1}$$

$$\epsilon = \epsilon_0 + h\nu\,\frac{1}{\mathrm{Exp}\left[\frac{h\,\nu}{k\,T}\right]-1}$$

In the limit where ν is very small in front of $\frac{k\,T}{h}$, we obtain ϵ-ϵ_0 = kT, thus recovering the classical formula.

As only the part of the energy above the fondamental level can be exchanged, kT must be replaced in equation 1 by $\frac{h\,\nu}{\mathrm{Exp}\left[\frac{h\,\nu}{k\,T}\right]-1}$ and we finally obtain

$$Jp[\nu\_, T\_] := \frac{4\,h\,\nu^3}{c^2\left(\mathrm{Exp}\left[\frac{h\,\nu}{k\,T}\right]-1\right)}$$

This is the Planck law which is very accurately checked by experimental spectrometry.

By replacing ν par $\lambda = \frac{c}{\nu}$ and accounting that $\left|d\nu\right| = \frac{c}{\nu}\left|d\lambda\right|$,

we obtain the expression of the Planck law in terms of the wavenumber

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Jp2[λ_, T_] = Jp[ν, T] c/λ^2 /. ν → c/λ
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$$\frac{4 c^2 h}{\left(-1 + e^{\frac{c h}{k T \lambda}}\right) \lambda^5}$$

Let us define the constants involved in the Planck law

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Const = {k → BoltzmannConstant, h → PlanckConstant, c → SpeedOfLight}
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$$\left\{k \to \frac{1.38065 \times 10^{-23}\ \text{Joule}}{\text{Kelvin}},\ h \to 6.62607 \times 10^{-34}\ \text{Joule Second},\ c \to \frac{299\,792\,458\ \text{Meter}}{\text{Second}}\right\}$$

Plot of Planck law as a function of wavenumber for the temperature T = 255 K

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Plot[Jp2[λ Meter, 255 Kelvin] /. Const /. Joule → Meter^3 Second , {λ, 0 , 30 × 10^-6}]
tracé
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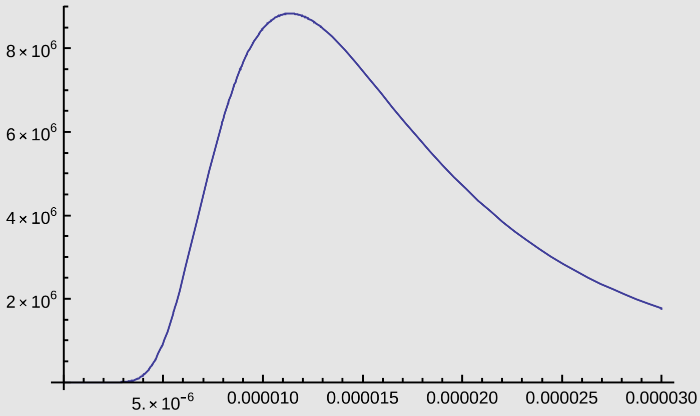

Stephan law

The Stephan law provides the total energy emitted at a given temperature with the remarkable result that it is proportional to T^4

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Assuming[c h/(k T) > 0, ∫_0^∞ Jp2[λ, T] dλ]
supposant
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$$\frac{4 k^4 \pi^4 T^4}{15 c^2 h^3}$$