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Large-scale Kolmogorov flow on the beta-plane and resonant wave interactions

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Abstract

The large-scale dynamics of the Kolmogorov flow near its threshold of instability is studied in the presence of the β-effect (Rossby waves). The governing equation, obtained by a multiscale technique, fails the Painlevé test of integrability when $\beta \neq 0$. This "β-Cahn–Hilliard" equation with cubic nonlinearity is simulated numerically in various régimes. The dispersive action of the waves modifies the inverse cascade associated with the Kolmogorov flow (She, Phys. Lett. A 124 (1987) 161). For small values of β the inverse cascade is interrupted at a wavenumber which increases with β. For large values of β only resonant wave interactions (RWI) survive. An original approach to RWI is developed, based on a reduction to normal form, of the sort used in celestial mechanics. Otherwise, wavenumber discreteness effects, which are dramatic in the present case, are not captured. (The method is extendable to arbitrary RWI problems.) The only four-wave resonances present involve two pairs of opposite wavenumbers. This allows leading-order decoupling of moduli and phases of the various Fourier modes, so that an exact kinetic equation is obtained for the energies of the modes. It has a Lyapunov (gradient flow) functional formulation and multiple attracting steady-states, each with a single mode excited.

1. Introduction

This paper is centered around a one-dimensional toy model for studying an instance of the interaction of turbulence and waves. The model, called the β-Cahn–Hilliard equation, describes the large-scale dynamics of the Kolmogorov flow in the presence of Rossby waves.

The Kolmogorov flow is obtained by subjecting two-dimensional incompressible flow with kinematic viscosity ν to a time-independent spatially periodic force

$$\mathbf{f} = \nu(-\sin y, 0). \tag{1}$$

This ensures that the parallel flow $\mathbf{u} = (-\sin y, 0)$ is *a* time-independent solution of the Navier–Stokes equation. The basic flow develops a negative eddy viscosity when $\nu < (1/2)^{1/2}$ and thereby becomes unstable to large-scale perturbations perpendicular to the basic flow [1–4]. Near the threshold, the large-scale secondary flow is, to leading order, a function only of a suitably rescaled large-scale X-coordinate and is governed by a one-dimensional Cahn–Hilliard equation. This equation is integrable in the sense that the steady-state solutions are expressible by elliptic functions. With periodic boundary conditions and for large times, the solution always goes to a steady state

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which minimizes a certain Lyapunov functional. Still, the intermediate temporal dynamics can be nontrivial: the steady state may be attained via an *arithmetic inverse cascade* in which excitation migrates to larger and larger scales through a succession of long-lasting quasi-equilibrium states [5].

The aim of this paper is to study the modification of the large-scale dynamics of the Kolmogorov flow when planetary rotation, in the form of the strongly dispersive β-effect, is incorporated. As is known, the Rossby waves produced by the β-effect are highly anisotropic: the generation of vorticity is proportional to the North-South (poloidal) component of the velocity; by taking the basic Kolmogorov flow to be in the East-West (toroidal) direction we ensure that it remains unaffected by the waves, while the large-scale secondary flow may be strongly affected. Actually, the larger the scale, the more important the β-effect becomes, since Rossby waves possess the somewhat unusual feature that their period is inversely proportional to their wavelength [1]. Thus, at sufficiently large scales, the wave period becomes shorter than any other characteristic time, and strong phase mixing suppresses all but *resonant* wave interactions for which the phase factors cancel out.

Resonant wave interaction theory (RWI) has numerous applications in geophysical fluid dynamics, plasma physics and solid state physics (see, e.g., Refs. [6,7]). Work in the early 1960s on the subject made use of a Gaussian assumption, introduced in a heuristic way (see, e.g., Refs. [8-10]), to derive 'kinetic' equations for the mean square Fourier amplitudes. Thanks to the smallness of the ratio of characteristic times, various asymptotic expansions can be carried out. A systematic asymptotic theory, in which the small parameter is the ratio of the wave period to the nonlinear characteristic time, was developed in the late 1960s by Benney, Newell and Saffman, called here the BNS method [11,12]. This theory assumes a continuum of wave vectors, and may run into difficulties when discreteness of the wave vectors becomes an important feature, for example when studying the largest scales of a bounded (or spatially periodic) system, which is precisely the goal of our investigation. One instance where discreteness leads to terms not captured by the BNS method is the so-called S-theory of Zakharov, L'vov and Starobinets [7,13].

Our alternative approach to RWI makes use of normal form techniques borrowed from celestial mechanics which are directly applicable to the discrete problem.

The paper is organized as follows. In Section 2 we formulate the problem leading to the β-Cahn-Hilliard equation. The derivation uses a multiscale method. Only the essential scaling arguments are given, technical details being relegated to Appendix A. In Section 3 we show that the β-Cahn-Hilliard equation, contrary to the Cahn-Hilliard equation, is not integrable, in the sense that it does not have the Painlevé property. Here, again, the emphasis is on concepts, with more technical steps relegated to Appendix B. Section 4 is devoted to numerical exploration of the β-Cahn-Hilliard equation. In Section 5 we discuss the asymptotics for large values of β, using a normal form technique to decouple the dynamics of amplitudes and phases of spatial Fourier modes. In Section 5.1 we explain why some terms are missed by the BNS method in the discrete case. We also comment on the work of the Russian school which did incorporate such terms, albeit in a somewhat heuristic way. For the β-Cahn-Hilliard equation, we obtain a "resonant interaction Cahn-Hilliard" (RICH) equation, some of the key properties of which are presented in Section 5.2. Concluding remarks are made in Section 6.

2. The β-Cahn-Hilliard equation for the large-scale dynamics

We consider a two-dimensional incompressible flow subject to an external force $\mathbf{f}$ in the presence of a β-effect [2]. The velocity $\mathbf{u} = (u_1, u_2)$ can be written in terms of a stream function [3]

[1] Until a wavelength comparable to the 'deformation radius' is reached; see Section 6.

[2] For background on the β-plane approximation and its limitations, see, e.g., Refs. [14,15].

[3] We use the fluid dynamicist's definition of the stream-function. In the geophysical community, the opposite sign is generally

$$u_i = \varepsilon_{ij}\partial_j\Psi, \quad i,j = 1,2. \tag{2}$$

Here, ε_{ij} is the fundamental antisymmetric tensor ($\varepsilon_{12} = -\varepsilon_{21} = 1$, zero otherwise), ∂_j stands for $\partial/\partial x_j$ and ∂^2 for the Laplacian operator[4].

In terms of the stream function the Navier–Stokes equation reads

$$\partial_t\partial^2\Psi + J(\partial^2\Psi,\Psi) = \nu\partial^2\partial^2\Psi - \varepsilon_{ij}\partial_i f_j - \beta_1\partial_1\Psi. \tag{3}$$

Here, $J(f,g) = \varepsilon_{ij}(\partial_i f)(\partial_j g)$ is the Jacobian, ν is the (kinematic molecular) viscosity and β_1 is the Rossby parameter[5].

We now observe that when the external force is given by (1), the Navier–Stokes equation admits the solution $\Psi = \cos y$, called the *Kolmogorov flow*, for which only the viscous and forcing terms are nonvanishing.

When the Reynolds number of the Kolmogorov flow, defined as $R = 1/\nu$, exceeds the critical value $R_c = \sqrt{2}$, x-dependent large-scale perturbations experience a negative eddy viscosity $\nu_E = \nu - 1/(2\nu)$ (see, e.g., Ref. [4]). In the neighborhood of R_c, multiscale techniques can be used to derive an equation for the large-scale dynamics. For the case of the Kolmogorov flow without the β-effect, this was done in Refs. [2] and [3]. The β-effect introduces only relatively minor modifications in this derivation. Let us here just state the main result and give some heuristic insight, leaving details for Appendix A.

Let us, in the Navier–Stokes (3), replace the Kolmogorov flow Ψ, called the (small-scale) basic flow, by $\Psi + \psi$, where the perturbation ψ is assumed to depend on x, y and t, the dependence in x and t being 'slow' in a sense we shall now define. We assume that the Reynolds number is slightly in excess of the critical value,

$$R = R_c(1 + \epsilon^2). \tag{4}$$

We now show how the various scalings and the form of the large-scale equation may be obtained heuristically. Consider a large-scale x-dependent perturbation of wavenumber $k \ll 1$. Momentarily ignoring the β-effect which is dispersive, the linear growth rate of this perturbation is of the form $\gamma(k) = -\nu_E k^2 + \sigma_4 k^4 + O(k^6)$. We know that the eddy viscosity ν_E is negative and $O(\epsilon^2)$ while σ_4 has no reason to vanish at $R = R_c$ and is actually negative. Hence perturbations with wavenumbers $O(\epsilon)$, up to $k_c = (-\nu_E/\sigma_4)^{1/2}$, are linearly unstable and have a maximum growth rate $O(\epsilon^4)$. This suggests the introduction, in addition to the 'fast' variable y, of the following 'slow' variables:

$$X = \epsilon x, \quad T = \epsilon^4 t. \tag{5}$$

Consider now the β-effect, which acts nontrivially on x-dependent perturbations. The frequency associated to the wavenumber k is β_1/k. If we require that, for $k = O(\epsilon)$, this frequency be comparable to the aforementioned growth rate, we must take $\beta_1 = O(\epsilon^5)$, that is, set

$$\beta_1 = \epsilon^5\beta. \tag{6}$$

Finally, we must find the scaling in ϵ for the amplitude of the perturbation. This is determined by the form of the nonlinearity which saturates the exponential growth of large-scale perturbations predicted by the linear theory. Since the leading-order large-scale motion depends only on X, there can be no advective nonlinearity (the Jacobian of $\psi(X)$ and of $\partial_X^2\psi(X)$ vanishes). Other types of quadratic nonlinearities are ruled out by momentum conservation or parity. The lowest order admissible nonlinearity is a cubic Cahn–Hilliard term which may be viewed as an additive correction to the eddy viscosity, proportional to the square of the large-scale velocity, that is, to $(\partial_X\psi)^2 = \epsilon^2(\partial_x\psi)^2$. Since the eddy viscosity is $O(\epsilon^2)$, nonlinear saturation should lead to an amplitude of ψ which is $O(\epsilon^0)$.

Once the proper scaling has been identified, standard multiscale techniques can be used to derive the leading order large-scale equation (see Appendix A). The resulting equation, which emerges technically as a solvability condition to order ϵ^6, reads

assumed.

[4] Instead of x_1 and x_2 we shall often use x and y and denote the space derivatives by ∂_x and ∂_y.

[5] The notation β will be reserved for a suitably rescaled version of the Rossby parameter.

$$\partial_T \partial_X \psi(X,T) = \partial_X \left(\left(\lambda_1 (\partial_X \psi)^2 - \lambda_2 \right) \partial_X^2 \psi \right) - \lambda_3 \partial_X^5 \psi - \beta(\psi - C). \quad (7)$$

Here, the notation ψ is shorthand for $\langle \psi^{(0)} \rangle$, the leading order of an expansion in powers of ϵ, averaged over the fast variable [6]; the constants in (7) are given by

$$\lambda_1 = 2\sqrt{2}, \quad \lambda_2 = \sqrt{2}, \quad \lambda_3 = \tfrac{3}{2}\sqrt{2}. \quad (8)$$

The constant C depends on the boundary conditions. For example, if ψ is periodic in the X-variable, C is the average of ψ over one spatial period. We observe that when the operator ∂_X is applied to the two sides of (7), the l.h.s. and the β term are just rescaled versions of the corresponding terms in the original Navier–Stokes equation (3). As for the λ_2, λ_3 and λ_1 terms, they are respectively the (negative) eddy viscosity term and the correction thereto involving higher order derivatives and nonlinearities.

Eq. (7) can also be written in terms of the (large-scale) velocity

$$v(X,T) \equiv -\partial_X \psi(X,T), \quad (9)$$

as

$$\partial_T v(X,T) = \partial_X \left(\left(\lambda_1 v^2 - \lambda_2 \right) \partial_X v \right) - \lambda_3 \partial_X^4 v - \beta \partial_X^{-1} v, \quad (10)$$

where ∂_X^{-1} denotes the inverse of the X-derivative (with suitable boundary conditions).

Eqs. (7)–(10) will be called the β-Cahn–Hilliard equation (β-CH). For $\beta = 0$, the usual Cahn–Hilliard (CH) equation for the Kolmogorov flow is recovered [2,3] [7]. Let us recall a few facts about this equation. After removal of one X-derivative [8], the Cahn–Hilliard equation has a variational formulation in terms of a Lyapunov functional (gradient flow) [3,16],

$$\partial_T \psi(X,T) = -\frac{\delta F}{\delta \psi},$$

$$F[\psi] \equiv \int dx \left[\tfrac{1}{12}\lambda_1 (\partial_X \psi)^4 - \tfrac{1}{2}\lambda_2 (\partial_X \psi)^2 + \tfrac{1}{2}\lambda_3 (\partial_X^2 \psi)^2 \right]. \quad (11)$$

When periodic boundary conditions with period L are assumed in the variable X, the solution tends for large times to the steady state which minimizes the functional $F[\psi]$. This state is unique (up to a translation) and may be expressed in terms of elliptic functions. However, when the period L is very large, that is, when there are many linearly unstable modes, there are many different steady states which, locally, have a kink/antikink structure,

$$v = \pm \left(\frac{3\lambda_2}{\lambda_1} \right)^{1/2} \tanh \left[\left(\frac{\lambda_2}{2\lambda_3} \right)^{1/2} (X - X_0) \right]. \quad (12)$$

The succession of quasi-equilibria with period L/q (where q is a positive integer taking the values q_{max}, $q_{max} - 1$, $q_{max} - 2, \ldots$) observed in numerical simulations by She [5] can be interpreted in terms of kink dynamics with successive annihilations of pairs of kinks–antikinks [22,17].

3. Painlevé analysis of the β-Cahn–Hilliard equation

At the end of the last century it was realized by S. Kowalesvki and P. Painlevé that there could be a connection between the integrability of a nonlinear differential equation and its analytic structure for complex values of the independent variable [18–20]. Indeed, equations which possess the so-called Painlevé property are often found to be integrable. This property means that, in the complex domain, all the movable singularities of the solutions are poles. Movable singularities are such that their locations depend on initial and/or boundary conditions. By Painlevé analysis, one understands the testing of an ODE to see if it has the Painlevé property. This is done by trying to perform a Laurent expansion of the solutions around an arbitrary pole, the order of which is determined

[6] This average is actually trivial because the leading-order perturbation of the stream function is independent of y.

[7] The equation, in a generalized form with arbitrary rather than cubic nonlinearity, occurs in studies of the spinodal decomposition (see Ref. [16] and references therein).

[8] The addition to the stream function of a function depending solely on the time variable leaves the velocity unchanged.

by dominant balance (for details and extension of the technique to PDE's, see, e.g., Ref. [21]).

It is quite clear that the Cahn–Hilliard equation has the Painlevé property, since its steady state solutions are elliptic functions and the latter are meromorphic. We shall now show that the β-Cahn–Hilliard fails the Painlevé test and could therefore have considerably more involved dynamics.

It is enough to consider the time-independent β-CH equation. After suitable rescaling of the independent and dependent variables and the use of a complex space variable z, we then obtain the following ODE:

$$\partial_z{}^3\left(u^3/3 - u - \lambda\partial_z{}^2 u\right) - \beta u = 0, \tag{13}$$

with a positive constant λ. Let z_* be a movable singularity of this ODE. Dominant balance, using the most singular terms near z_*, shows that the leading-order singularity should be a simple pole. Hence, we try the following Laurent expansion in the neighborhood of z_*:

$$u(z) = (z - z_*)^{-1}\sum_{j=0}^{\infty} u_j (z - z_*)^j, \tag{14}$$

where the complex Laurent coefficients u_j are to be determined (if possible) by substitution into (13). Equating the coefficients of the most singular terms, we obtain $u_0 = \pm\sqrt{6\lambda}$. Higher order Laurent coefficients satisfy equations of the form

$$a_j u_j = b_j, \quad j = 1, 2, \ldots . \tag{15}$$

The coefficients a_j and b_j are determined recursively. If all the a_js turned out to be nonvanishing, the u_js would be well determined and the solution given by the Laurent expansion (assuming it converges) would have only one ee parameter, namely z_*. The ODE (13) being however of fifth order, this is a very restrictive class of solutions. Actually, there are several values of j for which $a_j = 0$. These are called *resonances*. They fall into two classes. If $b_j = 0$ for such a j, the resonance is called *compatible*: the corresponding u_j is then arbitrary and gives an additional free parameter. If $b_j \neq 0$, the resonance is *noncompatible* in the sense that (15) has no solution. It is then impossible to construct a Laurent expansion and the Painlevé test is said to fail.

In Appendix B we obtain the following results for the CH and β-CH equations. First, when $\beta = 0$ and we drop the three derivatives on the l.h.s. of (13), so as to obtain a second order equation, the Painlevé test holds, as expected. Second, when $\beta = 0$ and we keep the three derivatives, we have a kind of weak Painlevé property. Third, when $\beta \neq 0$, a noncompatible resonance occurs for $j = 5$ and the Painlevé test fails. This, of course, tells us nothing about the integrability of the time-dependent problem.

4. Numerical simulations of the β-Cahn–Hilliard equation

The numerical results presented in this section are for the β-CH equation with spatial periodicity. Periodicity is not just a convenient way of doing the numerics: in the absence of the β-effect, the presence of an infrared cutoff is essential for the solutions to attain a steady state, since otherwise the arithmetic inverse cascade would proceed for ever. In the presence of the β-effect, the infrared cutoff bounds the frequency of Rossby waves.

The only two parameters of the problem are the spatial period and the (rescaled) Rossby parameter β. For numerical purposes, it is convenient to keep the spatial period fixed, say, equal to 2π. This is achieved through rescaling of the space variable: $X \to pX$. In terms of the velocity, the β-CH equation then reads

$$\partial_T v = \frac{\lambda_1}{3p^2}\partial_X^2 v^3 - \frac{\lambda_2}{p^2}\partial_X^2 v - \frac{\lambda_3}{p^4}\partial_X^4 v - p\beta\partial_X{}^{-1} v. \tag{16}$$

Note that the number n of linearly unstable modes is $p(2/3)^{1/2}$ (more precisely, its integer part). This is not modified by the presence of the β-effect.

The numerical integration of (16) makes use of a standard pseudo-spectral method in which spatial derivatives and inverse derivatives are calculated in k-space (Fourier space) while the cubic term is calculated in X-space (physical space). Alias removal, resulting from the use of a finite number of Fourier modes (from $k = -M$ to $k = M$), is done by using $4M$ points in physical space. Time-stepping is done by an Adams–Bashforth scheme [9] for all terms except the fourth derivative damping term, which is treated by an exponential scheme.

Particular attention has to be paid to the choice of the time step δT when β is large. It is not enough to require that δT be small compared to the various characteristic times such as the Rossby period $T_{\mathrm{Rossby}} \sim k/(p\beta)$, the damping time $T_{\mathrm{damp}} \sim p^4/k^4$ and the growth time due to the negative eddy viscosity $T_E \sim p^2/k^2$. Indeed, the Adams–Bashforth scheme, when applied to an oscillator of frequency $\omega = p\beta/k$, leads to a very slow instability on a time scale $T_{\mathrm{sp}} \sim \omega^{-4}(\delta T)^{-3}$. When β is large and $\delta T \ll T_{\mathrm{Rossby}}$ this spurious time scale may still be comparable to T_E, unless $(\delta T)^3\beta^4(p/k)^6 \ll 1$. This condition is most stringent for the gravest mode ($k = 1$). Failure to satisfy

$$(\delta T)^3\beta^4 p^6 \ll 1 \tag{17}$$

produces wrong numerical results.

Most of the simulations reported hereafter are done with random initial conditions in the spatial domain with a white spectrum. Each run is then characterized mathematically by the amplitude of the initial condition, the seed for the pseudo-random generator, the number of unstable modes $n = p(2/3)^{1/2}$ and the value of the Rossby parameter β. The purely numerical parameters are the truncation wavenumber M (which takes the value 64 in all the runs) and the time step δt which must be suitably adjusted as explained above.

Fig. 1 corresponds to $n = 7$ and $\beta = 0$. It is meant to illustrate the arithmetic inverse cascade which takes place in the pure Cahn–Hilliard case and which has already been reported in Ref. [5]. This figure and all subsequent ones display the temporal variation of the energies of the various Fourier modes, as labeled. Observe that very sharp transitions take place at which the dominant mode changes to a smaller wavenumber. These transitions correspond to kink–antikink annihilations [2,5]. This inverse cascade proceeds until a steady state is reached in which the gravest mode (here, $k = 1$) dominates [10].

Fig. 2 has $n = 7$ and $\beta = 10^{-4}$ and differs from Fig. 1 by the presence of conspicuous oscillations [11]. The steady state values (for the energies) are essentially the same as for $\beta = 0$.

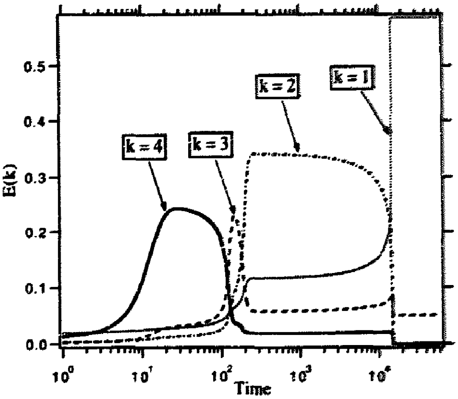

Fig. 1. Inverse arithmetic cascade. Time dependence of the energy of various Fourier modes for the solution of the Cahn–Hilliard equation ($\beta = 0$) with $n = 7$ unstable modes. This is basically the result obtained by She [5]. Note the very sudden changes in the dominant mode, which correspond to kink–antikink annihilations.

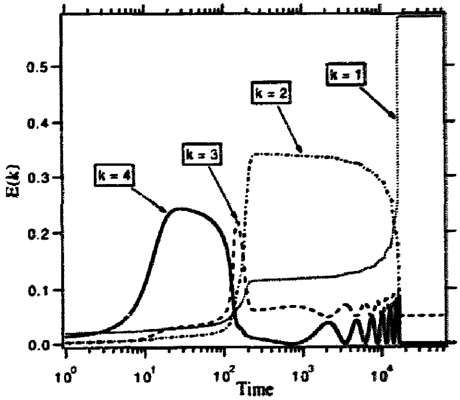

Fig. 2. Modification of the arithmetic inverse cascade by inclusion of a very small β-effect with $n = 7$ and $\beta = 10^{-4}$; otherwise, same conditions as in Fig. 1. Note the oscillations.

[9] Except for the first time step which is a second order Runge–Kutta step.

[10] This solution is expressible in terms of elliptic integrals and has nonvanishing harmonics for all odd wavenumbers.

[11] The actual times of kink–antikink recombinations have changed by about five per cent for $k = 1, 2, 3$, an amount too small to be visible on Fig. 2.

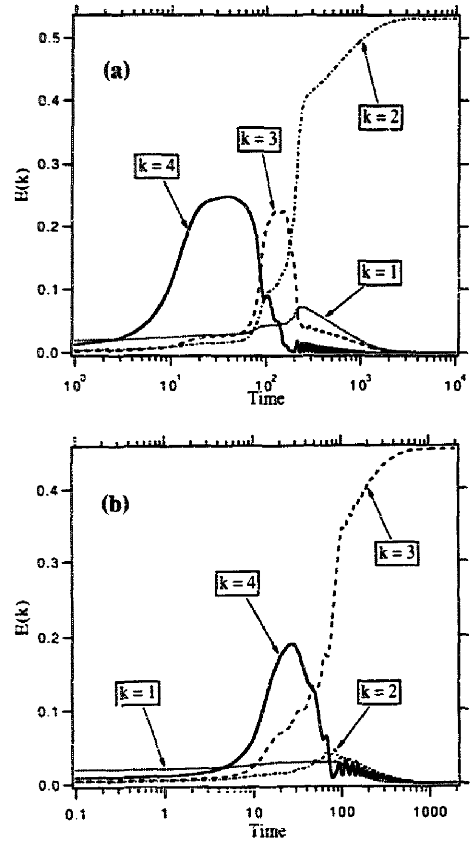

Fig. 3. Interruption of the arithmetic inverse cascade by the β-effect with $n = 7$ unstable modes. (a) $\beta = 0.01$; (b) $\beta = 0.1$. Note that the wavenumber of the mode dominating at large times increases with β.

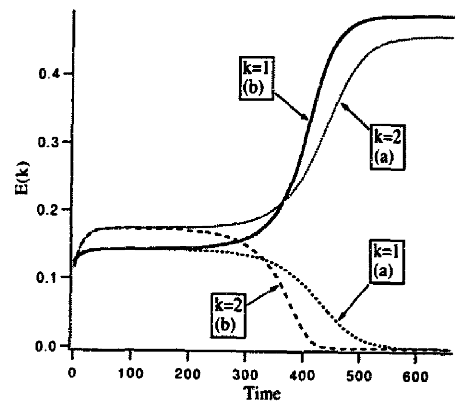

Fig. 4. Simulation of the β-Cahn-Hilliard equation with strong Rossby term: $\beta = 10$ and $n = 7$ unstable modes. At long times the Fourier amplitudes go to a steady state with a single Fourier mode excited, as predicted by the resonant interaction Cahn-Hilliard equation (53). Several single-Fourier mode attractors are competing, as indicated by (a) and (b) which correspond to two slightly different initial conditions.

As β is increased, more significant changes take place. Figs. 3a and 3b have $\beta = 0.01$ and $\beta = 0.1$, respectively, but the same initial conditions. They illustrate the phenomenon of *interrupted inverse cascade*: as β is increased, the wavenumber at which the period of Rossby waves becomes comparable to a characteristic time of the Cahn-Hilliard equation increases itself and the inverse cascade stops as phase-mixing becomes increasingly important. Note that for $\beta = 0.01$ and $\beta = 0.1$ the wavenumber which dominates at large times is $k = 2$ and $k = 3$, respectively. With the same initial conditions and $\beta = 1$ (simulation not shown) the dominant wavenumber is $k = 4$. For such 'intermediate' values of β, the temporal behavior can be rather complex and we do not rule out some weak form of chaos. Note that the increasing with β of the wavenumber which dominates at large times is a typical result when random initial conditions are used; we shall see later that, even for large values of β, small wavenumbers (e.g., $k = 1$) may dominate at large times, but the corresponding basins of attractions are probably quite small (Section 5 and the end of Section 6).

As β becomes even larger, Rossby waves become important at all wavenumbers within the linearly unstable band. The dynamics should then be dominated by resonant wave interactions. Application of standard resonant wave interaction theory, of the sort discussed in Section 5.1, led us to expect that, at large times, all the energy would be concentrated in the gravest mode. Actually, we found by performing a large number of numerical experiments with different initial conditions that there are several competing attractors. They all have a single Fourier mode excited, but not necessarily the gravest. The examples shown in Fig. 4 correspond to $n = 7$ and $\beta = 10$ with two slightly different initial conditions (chosen near the separatrix of two basins of attraction by a dichotomic procedure). The solution marked (a) has all the energy going for large times into the mode $k = 2$, while for the solution marked (b), it is the mode $k = 1$.

This (apparent) contradiction, which we first observed in our simulations, has led us to revisit the theory of resonant wave interactions.

5. A theory of discrete resonant wave interactions

In Section 4 we found that, for large values of β, very simple asymptotic behavior emerges as $T \to +\infty$.

When β is large, the period of Rossby waves is much shorter than any other characteristic time arising from Cahn–Hilliard dynamics. This is the kind of situation for which the *resonant waves interaction* (RWI) theory has been developed. It turns out that, in the form in which RWI has been mostly used so far, it is not applicable to our problem because a continuum of wavevectors is assumed.

As pointed out by Benney and Saffman [11], early work on RWI, such as may be found in Refs. [8,9], assumed a discrete spectrum, that is, a representation of a spatially periodic field in term of its Fourier series. It also used *ad hoc* closure assumptions, such as a Gaussian distribution of the complex amplitudes. Subsequent work on the continuum case by Benney and Saffman and Benney and Newell [11,12] led to a systematic justification of closed amplitude equations by the use of a multiple time method combined with a cumulant expansion.

In the opinion of Benney and Saffman [11], the continuum results can be obtained from the discrete results in the limit of infinitely close wavenumbers. As we shall see, the solutions for the discrete and continuous cases will stay close only for moderately long times. We found that a convenient way to tackle the discrete problem is through the use of averaging and normal form techniques (see, e.g., Ref. [23], Chapter 5). The idea is to consider the β-CH equation as a perturbation of a system of oscillators, which can be handled by techniques frequently used in celestial mechanics, but without the restriction to conservative systems.

We start from the β-CH equation (7) and expand the solution in a spatial Fourier series,

$$v(T,X) = \sum_{k=-\infty}^{\infty} \hat{v}_k(T) \exp(ikX). \tag{18}$$

It is traditional, in the averaging formalism, to choose a unit of time such that the unperturbed frequencies are order one. Here, it is simpler to choose the unit of time associated to the Cahn–Hilliard dynamics and thus to have a *small* Rossby time.

Hence, we set $\eta = 1/(p\beta)$ and use η as expansion parameter. (The notation ϵ was already used in the multiscale approach of Section 2.)

Using (8), (16) and (18), the β-CH equation can be written as follows in the Fourier representation:

$$\begin{aligned}\partial_T \hat{v}_k = &-\frac{2\sqrt{2}}{3p^2} k^2 \sum_{k_1+k_2+k_3=k} \hat{v}_{k_1}\hat{v}_{k_2}\hat{v}_{k_3} \\ &+\left(\frac{\sqrt{2}}{p^2}k^2 - \frac{3\sqrt{2}}{2p^4}k^4\right)\hat{v}_k + \frac{1}{\eta}\frac{i}{k}\hat{v}_k, \\ &k = \pm 1, \pm 2, \pm 3, \ldots\end{aligned} \tag{19}$$

Note that the Fourier coefficients $\hat{v}_k$ satisfy Hermitian symmetry,

$$\hat{v}_{-k} = \hat{v}_k^*. \tag{20}$$

We now introduce amplitude and phases (which will here play the same role as action and angle variable in conservative systems),

$$\hat{v}_k = a_k \exp i\theta_k,$$

$$a_k \geq 0, \quad a_{-k} = a_k, \quad \theta_{-k} = -\theta_k. \tag{21}$$

The a_ks and θ_ks satisfy

$$\partial_T a_k = f_k(a_\bullet, \theta_\bullet), \tag{22}$$

$$\partial_T \theta_k = \frac{1}{\eta}\omega_k + g_k(a_\bullet, \theta_\bullet), \tag{23}$$

where $\omega_k = 1/k$ is the nondimensionalised Rossby frequency [12], $a_\bullet$ and $\theta_\bullet$ stand for the full set of a_ks and θ_ks and

[12] For convenience, we define the *frequency* ω_k as being associated to a time-dependence of the form $e^{i\omega_k t}$ rather than $e^{-i\omega_k t}$. Hence, the *phase-speed* of the Rossby waves (not used in this paper) should be $-\omega_k/k$ rather than ω_k/k.

$$f_k = (\bar{\lambda}_2 k^2 - \bar{\lambda}_3 k^4) a_k - \bar{\lambda}_1 k^2 \,\mathrm{Re} \sum_{k_1+k_2+k_3=k} a_{k_1} a_{k_2} a_{k_3} \times \exp i(\theta_{k_1} + \theta_{k_2} + \theta_{k_3} - \theta_k), \tag{24}$$

$$g_k = -\bar{\lambda}_1 \frac{k^2}{a_k} \,\mathrm{Im} \sum_{k_1+k_2+k_3=k} a_{k_1} a_{k_2} a_{k_3} \times \exp i(\theta_{k_1} + \theta_{k_2} + \theta_{k_3} - \theta_k). \tag{25}$$

Here,

$$\bar{\lambda}_1 = \frac{2\sqrt{2}}{3p^2}, \quad \bar{\lambda}_2 = \frac{\sqrt{2}}{p^2}, \quad \bar{\lambda}_3 = \frac{3\sqrt{2}}{2p^4}. \tag{26}$$

Let us now explain how the averaging method is applied to the system (22),(23) (for details, see Ref. [23]). We treat (22),(23) as if it were a system of finitely many ordinary differential equations. In fact, there are infinitely many Fourier components and, thus, infinitely many equations. However, because of the strong dissipation at high wavenumbers the system (22),(23) can be approached very closely by systems having only a finite number N of Fourier components. Anyway, our goal here is only to present a systematic theory, not a rigorous one [13]. In the spirit of normal form theory, we seek a change of the *independent* variables a_k and θ_k, reducing to the identity for $\eta = 0$, and such that the transformed equations do not involve any phases in their right hand sides, if possible.

Let us denote the new amplitudes and phases by ρ_k and φ_k, respectively. The change of variables is sought in the form of a power series in η,

$$a_k = \rho_k + \eta \rho_k^{(1)}(\rho_\bullet, \varphi_\bullet) + \eta^2 \rho_k^{(2)}(\rho_\bullet, \varphi_\bullet) + \dots, \tag{27}$$

$$\theta_k = \varphi_k + \eta \varphi_k^{(1)}(\rho_\bullet, \varphi_\bullet) + \eta^2 \varphi_k^{(2)}(\rho_\bullet, \varphi_\bullet) + \dots, \tag{28}$$

where, again, $\rho_\bullet$ and $\varphi_\bullet$ stand for the full set of ρ_ks and φ_ks and the functions $\rho^{(1)}, \rho^{(2)}, \dots; \varphi^{(1)}, \varphi^{(2)}, \dots$ are 2π-periodic in all their phase arguments. They are to be chosen, if possible, in such a way that in the new variables the equations take the form

$$\partial_T \rho_k = F_k^{(0)}(\rho_\bullet) + \eta F_k^{(1)}(\rho_\bullet) + \dots, \tag{29}$$

$$\partial_T \varphi_k = \frac{1}{\eta}\, \omega_k + G_k^{(0)}(\rho_\bullet) + \eta\, G_k^{(1)}(\rho_\bullet) + \dots. \tag{30}$$

Differentiating (27),(28) with respect to T, we obtain (repeated indices are summed over)

$$\partial_T a_k = \partial_T \rho_k + \eta \frac{\partial \rho_k^{(1)}}{\partial \rho_\ell} \partial_T \rho_\ell + \eta \frac{\partial \rho_k^{(1)}}{\partial \varphi_\ell} \partial_T \varphi_\ell + \eta^2 \frac{\partial \rho_k^{(2)}}{\partial \rho_\ell} \partial_T \rho_\ell + \eta^2 \frac{\partial \rho_k^{(2)}}{\partial \varphi_\ell} \partial_T \varphi_\ell + \dots \tag{31}$$

and

$$\partial_T \theta_k = \partial_T \varphi_k + \eta \frac{\partial \varphi_k^{(1)}}{\partial \rho_\ell} \partial_T \rho_\ell + \eta \frac{\partial \varphi_k^{(1)}}{\partial \varphi_\ell} \partial_T \varphi_\ell + \eta^2 \frac{\partial \varphi_k^{(2)}}{\partial \rho_\ell} \partial_T \rho_\ell + \eta^2 \frac{\partial \varphi_k^{(2)}}{\partial \varphi_\ell} \partial_T \varphi_\ell + \dots \tag{32}$$

Substituting (27),(28) in (22),(23) and performing a Taylor expansion, we obtain

$$\partial_T a_k = f_k + \eta \frac{\partial f_k}{\partial \rho_\ell} \rho_\ell^{(1)} + \eta \frac{\partial f_k}{\partial \varphi_\ell} \varphi_\ell^{(1)} + O(\eta^2), \tag{33}$$

$$\partial_T \theta_k = \frac{1}{\eta} \omega_k + g_k + \eta \frac{\partial g_k}{\partial \rho_\ell} \rho_\ell^{(1)} + \eta \frac{\partial g_k}{\partial \varphi_\ell} \varphi_\ell^{(1)} + O(\eta^2), \tag{34}$$

where all the functions are evaluated at the arguments $(\rho_\bullet, \varphi_\bullet)$. Upon using (29) and (30) in (31) and (32), we obtain

$$\partial_T a_k = F_k^{(0)} + \frac{\partial \rho_k^{(1)}}{\partial \varphi_\ell} \omega_\ell + \eta \frac{\partial \rho_k^{(1)}}{\partial \rho_\ell} F_\ell^{(0)} + \eta F_k^{(1)} + \eta \frac{\partial \rho_k^{(1)}}{\partial \varphi_\ell} G_\ell^{(0)} + \eta \frac{\partial \rho_k^{(2)}}{\partial \varphi_\ell} \omega_\ell + O(\eta^2) \tag{35}$$

and

$$\partial_T \theta_k = \frac{1}{\eta} \omega_k + \frac{\partial \varphi_k^{(1)}}{\partial \varphi_\ell} \omega_\ell + G_k^{(0)} + \eta G_k^{(1)} + \eta \frac{\partial \varphi_k^{(1)}}{\partial \rho_\ell} F_\ell^{(0)} + \eta \frac{\partial \varphi_k^{(1)}}{\partial \varphi_\ell} G_\ell^{(0)} + \eta \frac{\partial \varphi_k^{(2)}}{\partial \varphi_\ell} \omega_\ell + O(\eta^2). \tag{36}$$

We now identify (33) with (35) and (34) with (36) order by order in η. The only terms $O(\eta^{-1})$ are in (34) and (36) and are already identical. To order η^0, we obtain

[13] An example of a rigorous infinite-dimensional perturbation theory for Hamiltonian systems may be found in Ref. [24].

$$\frac{\partial \rho_k^{(1)}}{\partial \varphi_\ell}\omega_\ell = f_k - F_k^{(0)}, \tag{37}$$

$$\frac{\partial \varphi_k^{(1)}}{\partial \varphi_\ell}\omega_\ell = g_k - G_k^{(0)}. \tag{38}$$

To order η^1, we obtain

$$\frac{\partial \rho_k^{(2)}}{\partial \varphi_\ell}\omega_\ell = R_k^{(2)}, \tag{39}$$

$$\frac{\partial \varphi_k^{(2)}}{\partial \varphi_\ell}\omega_\ell = S_k^{(2)}, \tag{40}$$

with

$$R_k^{(2)} = \frac{\partial f_k}{\partial \rho_\ell}\rho_\ell^{(1)} + \frac{\partial f_k}{\partial \varphi_\ell}\varphi_\ell^{(1)} - F_k^{(1)} - \frac{\partial \rho_k^{(1)}}{\partial \varphi_\ell}G_\ell^{(0)} - \frac{\partial \rho_k^{(1)}}{\partial \rho_\ell}F_\ell^{(0)}, \tag{41}$$

$$S_k^{(2)} = \frac{\partial g_k}{\partial \rho_\ell}\rho_\ell^{(1)} + \frac{\partial g_k}{\partial \varphi_\ell}\varphi_\ell^{(1)} - G_k^{(1)} - \frac{\partial \varphi_k^{(1)}}{\partial \varphi_\ell}G_\ell^{(0)} - \frac{\partial \varphi_k^{(1)}}{\partial \rho_\ell}F_\ell^{(0)}. \tag{42}$$

We observe that all four equations (37), (38), (39) and (40) have the same structure: a combination of derivatives with respect to the φs equal to a right hand side.

We now denote by angular brackets the operation of averaging over all phases,

$$\langle h(\varphi_1, \varphi_2, \ldots)\rangle = \int_0^{2\pi} \frac{d\varphi_1}{2\pi} \times \int_0^{2\pi} \frac{d\varphi_2}{2\pi} \ldots h(\varphi_1, \varphi_2, \ldots). \tag{43}$$

Consider first (37). It is clear that the term $\partial \rho_k^{(1)}/\partial \varphi_\ell$ in (37) has a vanishing average. Therefore, a *necessary* condition for (37) to be solvable is

$$\langle f_k(\rho_\bullet, \varphi_\bullet)\rangle = F_k^{(0)}(\rho_\bullet). \tag{44}$$

Thus, to leading order in η, (29) reads

$$\partial_T \rho_k = F_k^0(\rho_\bullet) = \langle f_k(\rho_\bullet, \varphi_\bullet)\rangle. \tag{45}$$

In other words, the leading order equation is just obtained by averaging the original equation (22) over the phases. However, (44) is only a necessary condition. To actually try and solve (37) for the $\rho_k^{(1)}$s, we expand f_k and $\rho_k^{(1)}$ in a Fourier series in all the phases,

$$f_k(\rho_\bullet, \varphi_\bullet) = \sum_{\xi} f_{k,\xi}(\rho_\bullet)\exp(i\varphi_\ell \xi_\ell), \tag{46}$$

$$\rho_k^{(1)}(\rho_\bullet, \varphi_\bullet) = \sum_{\xi} \rho_{k,\xi}^{(1)}(\rho_\bullet)\exp(i\varphi_\ell \xi_\ell). \tag{47}$$

Here, the Fourier variable $\xi = \{\xi_\ell\}$ has signed integer components. Note that the Fourier series in the phases is unrelated to the spatial Fourier decomposition used at the beginning of this section.

Eq. (37) is equivalent to the following relation among phase Fourier components:

$$\langle f_k(\rho_\bullet, \varphi_\bullet)\rangle = F_k^{(0)}(\rho_\bullet) \quad (\xi = 0), \tag{48}$$

$$i\xi_\ell\, \omega_\ell\, \rho_{k,\xi}^{(1)} = f_{k,\xi}(\rho_\bullet) \quad (\xi \neq 0). \tag{49}$$

Eq. (48) is just the solvability condition already written (44). Eq. (49) is immediately solved as

$$\rho_{k,\xi}^{(1)} = \frac{f_{k,\xi}(\rho_\bullet)}{i\xi_\ell\, \omega_\ell}, \tag{50}$$

provided that $\xi_\ell\, \omega_\ell$ does not vanish for those $f_{k,\xi}(\rho_\bullet)$s which are nonzero. The condition

$$\xi_\ell\, \omega_\ell = 0, \tag{51}$$

which is called 'the resonance condition', prevents the existence of a solution unless

$$f_{k,\xi} = 0, \tag{52}$$

in which case one has a 'compatible resonance'.

As shown in Ref. [23], when resonances are present, it is in general necessary to modify the asymptotic expansion and to include a certain number of integer combinations of the original phases among the independent variables of the averaged equation.

So far our formalism has been quite general with no particular use made of the specific form of the dynamical equation. In Appendix C we show that for the β-Cahn–Hilliard equation four-wave resonances are present in (37) which determines the leading-order asymptotics. These resonances are, however, 'decomposed', in the sense that they are made of two pairs of opposite wavenumbers. This implies compatibility.

Hence, to leading order, standard averaging, namely (45), applies. The final equation for the amplitudes, the *resonant interaction Cahn–Hilliard* (RICH) equation, reads

$$\partial_T \rho_k = \left(\frac{\sqrt{2}}{p^2}k^2 - \frac{3\sqrt{2}}{2p^4}k^4\right)\rho_k$$
$$-\frac{2\sqrt{2}}{p^2}k^2\left[2\sum_{\ell=1}^{\infty}\rho_\ell^2 - \rho_k^2\right]\rho_k. \qquad (53)$$

Here, no summation is implied on k. For the details, the reader is referred to Appendix C.

Equations for higher order terms, such as (39) involve resonances among six waves and more. As shown in Appendix D such resonances are not necessarily decomposed. It may be shown that this implies the existence of noncompatible resonances. Thus, the phases cannot be completely eliminated when studying the corrections to the leading-order asymptotics and some kind of slow chaos (on time scales $O(1/\eta)$ or larger) could be present.

We note that situations with compatible leading-order resonances and noncompatible higher order resonances are often encountered in celestial mechanics, the best examples coming from the theory of secular motion of asteroids [25–27].

5.1. Comparison with continuous resonant wave interaction theory

To facilitate the comparison with the Benney–Newell–Saffman (BNS) theory of resonant wave interactions [11,12], we rewrite (53) in terms of the (discrete) kinetic energy spectrum $E(k) \equiv \rho_k^2$ as

$$\partial_T E(k) = 2\left(\frac{\sqrt{2}}{p^2}k^2 - \frac{3\sqrt{2}}{2p^4}k^4\right)E(k)$$
$$-\frac{4\sqrt{2}}{p^2}k^2\left[2\sum_{\ell=1}^{\infty}E(\ell) - E(k)\right]E(k). \qquad (54)$$

Here, again, no summation is implied on k. Suppose we now go to the continuous limit. This can be done by either of the following methods: (i) by changing the assumed 2π-periodicity into an L-periodicity and then letting $L \to \infty$; (ii) by letting the number of linearly unstable modes $n \to \infty$, spreading the energy over a very large number of modes, in such a way that each individual $E(k) \to 0$, but the sum $\sum_{\ell=1}^{\infty} E(\ell)$ remains finite and goes over into the integral $\int_0^\infty E_{\rm cont}(q)dq$. In this continuous limit, (54) goes over into

$$\partial_T E_{\rm cont}(k) = 2\left(\frac{\sqrt{2}}{p^2}k^2 - \frac{3\sqrt{2}}{2p^4}k^4\right)E_{\rm cont}(k)$$
$$-\frac{8\sqrt{2}}{p^2}k^2 E_{\rm cont}(k)\int_0^\infty E_{\rm cont}(q)\,dq. \qquad (55)$$

Observe that the term proportional to $E^2(k)$ on the r.h.s. of (54) has dropped out.

Eq. (55) may be derived directly from the β-CH equation (10) by making the *quasi-normal approximation*, i.e. by discarding the fourth-order cumulant. The BNS method gives a rationale for this approximation. The argument goes roughly as follows. One writes the cumulant hierarchy derived from the β-CH Eq. (7). One then observes that for homogeneous random functions *in the continuous limit*, the fourth order Fourier-space cumulant $\langle\hat{v}(k_1)\hat{v}(k_2)\hat{v}(k_3)\hat{v}(k_4)\rangle_c$, a distribution with support in the hyperplane $k_1+k_2+k_3+k_4=0$, involves quartets of wavevectors (k_1,k_2,k_3,k_4) such that their full sum vanishes but no *partial* sum vanishes. It follows that such quartets cannot be resonant for Rossby waves (see Appendix D). Hence, for large βs, the fourth order cumulant is phase mixed.

The preceding argument does not work in the discrete case. Indeed, there exist then discrete fourth-order cumulants with *vanishing partial sums of wavevector arguments*, for example $\langle \hat{v}_k\hat{v}_{-k}\hat{v}_k\hat{v}_{-k}\rangle_c$. With such cumulants are associated resonant wave interactions and, hence, no phase mixing. This is a (cumbersome) way to understand why the term proportional to $E^2(k)$ survives in the discrete case.

For our problem, the long-time behavior of (54) and (55) can be very different. If (55) is used with a minimum wavenumber $k_{\rm min}$, eventually a steady state is obtained with all the energy concentrated at $k_{\rm min}$ [14].

[14] This is an instance of a general result presented in Ref. [28].

In contrast, as we shall see in the next section, the solutions of (54) can go to any of many possible stable steady state solutions.

We also mention that discreteness effects similar to those encountered here are present in the so-called S-theory for the parametric excitation of spin waves in the presence of time-periodic pumping, a problem which also has cubic nonlinearity [13]. It was realized in this reference that it is not correct to just discard the fourth order cumulant. The appropriate modification was however introduced in a somewhat heuristic way. Later, a more systematic theory was developed, which uses diagrammatic expansions [29]. It appears that the normal form approach (which here reduces basically to averaging) gives more insight. In this context it is of interest to note that the Russian school was aware of some connection between resonant wave interaction theory (called by them 'wave turbulence' or 'weak turbulence') and analytical mechanics. For example, on p. 11 of Ref. [7] at the end of a section on the elimination of nonresonant terms from a wave Hamiltonian, the following is observed: "Note that the above-described transformation is analogous to the transformation of the Hamiltonians to their normal forms, i.e. in the vicinity of fixed points in classical analytical mechanics". Similar remarks were frequently made by V.E. Zakharov (V.S. L'vov, private communication).

5.2. Solutions of RICH

We now study the resonant interaction equation RICH, written in the form (54). We first show that RICH has a Lyapunov functional formulation. We set

$$E(k) \equiv k^2 b_k^2. \tag{56}$$

Note that he b_ks are just the moduli of the Fourier components of the stream function. It is elementary to check that the equations for the b_ks may be written as

$$\partial_T b_k = -\frac{1}{4}\frac{\partial \mathcal{G}}{\partial b_k}, \tag{57}$$

where the Lyapunov functional $\mathcal{G}$ is given by

$$\mathcal{G} \equiv \frac{4\sqrt{2}}{p^2}\left[\left(\sum_{k=1}^{\infty} k^2 b_k^2\right)^2 - \tfrac{1}{2}\sum_{k=1}^{\infty} k^4 b_k^4\right] - \frac{2\sqrt{2}}{p^2}\sum_{k=1}^{\infty} k^2 b_k^2 + \frac{3\sqrt{2}}{p^4}\sum_{k=1}^{\infty} k^4 b_k^2. \tag{58}$$

The presence of a Lyapunov functional may seem surprising, since the variational formulation for the original Cahn–Hilliard equation could be lost by the presence of the dispersive β-effect. Actually, in the limit $\beta \to \infty$, there is no dispersive term left in RICH, but only a selection of a subset of all possible nonlinear interactions present in the Cahn–Hilliard equation, namely the resonant ones.

An alternative formulation, in terms of the $E(k)$s, is to rewrite (54) as

$$\frac{1}{k^2}\partial_T \ln(E(k)) = -\frac{\partial \mathcal{G}}{\partial E(k)}, \tag{59}$$

where $\mathcal{G}$ has now only linear and quadratic terms in the $E(k)$s,

$$\mathcal{G} = \frac{4\sqrt{2}}{p^2}\left(\left(\sum_{k=1}^{\infty} E(k)\right)^2 - \tfrac{1}{2}\sum_{k=1}^{\infty} E^2(k)\right) - \frac{2\sqrt{2}}{p^2}\sum_{k=1}^{\infty} E(k) + \frac{3\sqrt{2}}{p^4}\sum_{k=1}^{\infty} k^2 E(k). \tag{60}$$

It follows from either of the Lyapunov formulations (57) and (59) that the solutions of RICH tend, at large times, to any of the stable steady states corresponding to a local minimum of the functional $\mathcal{G}$.

We shall now show that, when p is large, i.e. when there are many linearly unstable modes, there are many stable steady-state solutions.

Clearly, RICH has single-mode steady-state solutions. Indeed, it is seen that (54) is satisfied, if for a particular mode k the energy $E(k)$ has the value

$$\bar{E}(k) = \tfrac{1}{2} - \frac{3}{4}\frac{k^2}{p^2}, \tag{61}$$

while it vanishes for all other modes $k' \neq k$. Multiple-mode steady-state solutions for a set $K = \{k_1, k_2, \ldots, k_m\}$ of m modes can be obtained as

$$\bar{E}(k_i) = 2\bar{E} - \tfrac{1}{2} + \frac{3}{4}\frac{k_i^2}{p^2}, \tag{62}$$

where the total energy is

$$\bar{E} = \frac{1}{2(2m-1)}\left(m - \frac{3}{2p^2}\sum_{k_i \in K} k_i^2\right). \tag{63}$$

The existence of these steady-state solutions is constrained by the realisability condition, i.e. $\bar{E}(k_i) \geq 0$ for all $k_i \in K$. Hence, the number of m-mode stationary solutions is finite for a given p and scales with p^m when p is large.

The stability of the steady-state solutions is studied by introducing a small perturbation in (54), that is, we set

$$E(k) = \bar{E}(k) + E'(k), \quad |E'(k)| \ll \bar{E}(k). \tag{64}$$

This leads to stability equations which separate into two subsets, one for the wavenumbers k_i belonging to K, namely

$$\frac{p^2}{4\sqrt{2}k_i^2}\partial_T E'(k_i) = -\bar{E}(k)\left[2\sum_{\ell=1}^{\infty} E'(\ell) - E'(k_i)\right], \tag{65}$$

and one for the exterior wavenumbers q_j not in K, namely

$$\frac{p^2}{4\sqrt{2}q_j^2}\partial_T E'(q_j) = -\left[2\bar{E} - \tfrac{1}{2} + \frac{3}{4}\frac{q_j^2}{p^2}\right]E'(q_j). \tag{66}$$

From (66), we see that the stability to exterior modes is most restrictive for the smallest wavenumber q_0 not in K. Hence, a necessary stability condition is

$$\tfrac{1}{3}p^2 + (2m-1)q_0^2 - \sum_{k_i \in K} k_i^2 > 0. \tag{67}$$

Let us first consider single-mode solutions with wavenumber k. If $k = 1$, then $q_0 = 2$ and stability to exterior modes follows from (67). If $k > 1$ stability to exterior modes requires

$$k^2 < \tfrac{1}{2} + \tfrac{1}{3}p^2. \tag{68}$$

When stability to exterior perturbations holds, we can replace $\sum_{\ell=1}^{\infty} E'(\ell)$ by $\sum_{l\in K} E'(l)$ in (65) without loss of generality; that is, we restrict the stability problem to the subspace spanned by the excited modes. For single-mode steady-state solutions, the eigenvalue is then always negative. This means that condition (68) is then sufficient for stability. We have thus established that RICH, unlike the Cahn–Hilliard equation, has multiple stable steady-state solutions with a single mode excited. By (68), the number of such solutions scales with p when p is large.

We now consider 2-mode steady-state solutions, the (interior) stability equations are

$$\partial_T E'(k_1) = -A_1 E'(k_1) - 2A_2 E'(k_2), \tag{69}$$

$$\partial_T E'(k_2) = -2A_2 E'(k_1) - A_1 E'(k_1), \tag{70}$$

where

$$A_i = \frac{4\sqrt{2}k_i^2}{p^2}\bar{E}(k_i). \tag{71}$$

The product and the sum of the eigenvalues being negative, the eigenvalues are real, one of them being strictly positive. Hence 2-mode steady-state solutions are always unstable.

For 3-mode steady-state solutions, the characteristic equation is

$$\sigma^3 + \sigma^2(A_1 + A_2 + A_3) - 3\sigma(A_1A_2 + A_2A_3 + A_3A_1) + 4A_1A_2A_3 = 0. \tag{72}$$

Hence the sum of the eigenvalues, the sum of pair products and their product are all negative numbers. This implies that two eigenvalues have positive real parts and thus instability.

More generally, we conjecture that m-mode steady-state solutions with $m > 1$ are always unstable to interior perturbations.

Steady-state solutions are the extrema of the Lyapunov functional $\mathcal{G}$. The stability analysis shows that these extrema are saddle points of $\mathcal{G}$ for 2-mode and 3-mode steady-state solutions (and probably for m-mode solutions with $m > 3$). The long-time behavior of solutions to RICH is therefore quite simple: the solutions are attracted to any of the finitely many minima corresponding to the stable steady-state single-mode solutions. This number grows with p as the integer part of $((1/2) + (p^2/3))^{1/2}$. Note that $(p^2/3)^{1/2}$ is the wavenumber which maximizes the linear rate-of-growth for RICH. The existence of several competing basins of attraction was already revealed in our numerical experiments on the β-Cahn–Hilliard equation (Section 4, Fig. 4). The basins of attractions of each of the stable solutions may be rather complex and have not yet been mapped out. The presence, for large ps, of numerous unstable steady-state solutions, which attract to their vicinity trajectories located near their stable manifold, suggests that transients with numerous excited modes may persist for long durations.

6. Concluding remarks

In this paper there are some results and open questions which are specific of the β-Cahn–Hilliard (β-CH) equation and others which can appear in a broad class of problems involving resonant wave interactions (RWI). Let us first discuss issues of the latter type.

Our normal form approach to RWI, as described in Section 5, can in principle be applied to any problem with discrete wavenumbers. For the β-CH equation the main conclusion is that, in the limit $\beta \to \infty$, the leading-order asymptotics is governed by a kinetic equation for the wave moduli (or their squares, the wave energies), called the resonant interaction Cahn–Hilliard (RICH) equation. The decoupling of moduli and phases has its origin in the compatibility of four-wave resonances. This, in turn, is so because the resonances are 'decomposed', i.e. made of two pairs of opposite wavenumbers. Similar decoupling holds for equations other than Cahn-Hilliard, provided they are one-dimensional and have a cubic nonlinearity. Such a simple 'kinetic' situation can be upset in at least two ways.

First, we observed that, beyond the leading order, there are noncompatible resonances, involving for example six or more waves. These inhibit the decoupling of moduli and phases. It would be of interest to derive and study the corresponding normal form which describes the slow and weak, possibly chaotic, deviations from the long-time asymptotic behavior predicted by RICH.

Second, if the dispersion relation is changed from $\omega_k \propto 1/k$ to a more general functional form, four-wave resonances may not be decomposed and, hence, will in general not be compatible, so that no kinetic equation is obtained to leading order. Actually, it may be shown that four-wave resonances are still decomposed as long as $\omega_k = f(k)$, where $f(k)$ is (i) odd-convex: $f(-k) = -f(k)$ and $f''(k) > 0$ for $k > 0$ and (ii) superadditive (resp. subadditive): $f(k_1 + k_2) > f(k_1) + f(k_2)$ (resp. $f(k_1 + k_2) < f(k_1) + f(k_2)$) for $k_1 > 0$ and $k_2 > 0$. The function $f(k) = -25k^3 + k^5$, which does not satisfy these conditions, has the four-wave resonance $k_1 = 1$, $k_2 = 4$, $k_3 = -2$ and $k_4 = -3$ which is clearly not decomposed (M. Vergassola, private communication).

Noncompatible resonances can occur in a more natural way if we depart from our toy model (3) to incorporate physical effects which are present in realistic geophysical flow and in laboratory experiments on two-dimensional flow. The most important ones are a bottom friction by Ekmann pumping and a finite radius of deformation L_R arising from free surface effects [14,15]. In (3) we must then add a term $-\nu_e\partial^2\Psi$ in the r.h.s. and replace ∂^2 by $\partial^2 - L_R^2$ in the l.h.s..

Bottom friction with realistic values shifts the critical Reynolds number for negative eddy viscosity from $\sqrt{2}$ to much larger values [3]. Also, by damping the low-k modes, it induces a large-scale cut-off for the instability which may slow down or stop the inverse cascade.

In the presence of a finite radius of deformation the dispersion relation becomes $\omega_k \propto k/(k^2 + 1/L_R^2)$. When $kL_R \ll 1$, the phase speed saturates and the Rossby waves are only weakly dispersive. Therefore, the validity of RWI requires much larger values of β than in the absence of deformation radius. Furthermore, the dispersion relation is not odd-convex, so that noncompatible resonances do occur.

Let us turn to issues more specific of the β-Cahn–Hilliard equation. When starting from the Navier–Stokes equation (3), the derivation of RICH involves successively two asymptotics: (i) the multiscale expansion in which the ratio of scales is ϵ and the slight excess of the Reynolds number over its critical value is $O(\epsilon^2)$ (cf. (4)), (ii) the RWI expansion in which the small parameter is $1/\beta$. Since $\beta = \beta_1/\epsilon^5$, where

β_1 is the parameter appearing in the Navier–Stokes Eq. (3), we conjecture that RICH holds when $\beta_1 = \epsilon^5 g(\epsilon)$, where $g(\epsilon) \to \infty$ at a rate slower than $1/\epsilon$. In other words, even for very small values of the physical Rossby parameter β_1, RICH should be the relevant equation at sufficiently large scales when the Reynolds number is close to its critical value.

Simulations of the β-Cahn–Hilliard equation have revealed that the solutions can be very close to what is predicted by RICH, say within one tenth of a percent, for values such as $\beta \approx 10$. Comparison of the negative eddy viscosity term with the Rossby term in (16) suggests that the large expansion parameter which controls the validity of RICH is $\beta(p/k)^3$ rather than β. (The number of linearly unstable mode is $p(2/3)^{1/2}$.) If we apply this argument solely to the gravest modes for which $k = O(1)$, we find the condition $\beta p^3 \gg 1$, which is much less restrictive than $\beta \gg 1$ for the kind of values of p used in our numerical explorations ($p \approx 9$). However, if we require the condition to be satisfied for all the wavenumbers in the linearly unstable band, which extends up to $k = O(p)$, we recover the condition $\beta \gg 1$. An interesting asymptotic problem is to let $p \to \infty$, $\beta \to 0$ and $\beta p^3 \to \infty$. RICH could then be valid at large scales while small scales exhibit a standard Cahn–Hilliard inverse cascade.

Finally, we mention the problem of understanding the structure of the multiple basins of attraction of single-mode solutions to RICH, particularly when p is large. Numerical experiments on the β-Cahn–Hilliard equation with large values of β (10–100) and Gaussian initial conditions with low amplitude and a white (flat) spectrum indicate that single-mode solutions which are attained have mostly their wavenumbers near $(p^2/3)^{1/2}$, the value which maximizes the linear rate-of-growth. Since there is a quartic Lyapunov functional (58), this challenging problem could be amenable to geometrical methods.

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Appendix A. Derivation of the β-Cahn–Hilliard equation by a multiscale technique

Our purpose is to give some details on the derivation of the β-CH equation (7). It is assumed that the reader is familiar with multiscale techniques [15]. Derivations of the CH equation in the absence of the β-effect may be found in Refs. [2,3].

The Navier Stokes equations with Kolmogorov-flow forcing $\mathbf{f} = \nu(-\sin y, 0)$ and the β-effect is rewritten here for convenience,

$$\partial_t \partial^2 \Psi + J(\partial^2 \Psi, \Psi) = \nu \partial^2 \partial^2 \Psi - \nu \cos y - \beta_1 \partial_x \Psi. \tag{A.1}$$

We are interested in long-wavelength perturbations ψ to the basic solution of (A.1), namely

$$\Psi = \cos y. \tag{A.2}$$

When $\beta_1 = 0$, it is known that the threshold of instability is for $R = 1/\nu = R_c = \sqrt{2}$ and that, just above this threshold, the most unstable mode has a large-scale dependence only on the coordinate x [2–4]. It may be checked that these features are unaffected by the presence of the dispersive β-term.

As in Ref. [3], we set

$$\nu = \nu_c(1 - \epsilon^2) \tag{A.3}$$

and introduce *slow* space and time variables,

$$X = \epsilon x, \quad T = \epsilon^4 \partial_T. \tag{A.4}$$

The multiscale technique treats fast and slow variables as being independent. Since the basic flow depends only on y and is time-independent, derivatives appearing in (A.1) are now given by the following rules:

$$\partial_t \to \epsilon^4 \partial_T, \quad \partial_x \to \epsilon \nabla_X, \quad \partial_y \to \partial_y. \tag{A.5}$$

[15] A brief introduction may be found in Ref. [30] (Section 9.6.2).

(We used the notation ∇_X rather than ∂_X to make the difference between slow and fast variables more conspicuous.) For the reasons explained in Section 2, we take

$$\beta_1 = \epsilon^5 \beta \tag{A.6}$$

and the amplitude of the perturbation ψ of order ϵ^0, namely

$$\psi(y, X, T) = \psi^{(0)} + \epsilon\psi^{(1)} + \epsilon^2\psi^{(2)} + \dots . \tag{A.7}$$

Here, the functions $\psi^{(0)}, \psi^{(1)}, \dots$ depend both on fast and slow variables. Substituting the perturbed stream function $\Psi + \psi$ for Ψ in (A.1), we obtain

$$\begin{aligned} &\partial_t\partial^2\psi + J(\partial^2\Psi, \psi) + J(\partial^2\psi, \Psi) + J(\partial^2\psi, \psi) \\ &\quad -\nu\partial^2\partial^2\psi + \beta_1\partial_x\psi = 0. \end{aligned} \tag{A.8}$$

We now evaluate the various terms appearing in (A.8), using (A.3), (A.5) and (A.7) and stop at the highest order in ϵ which will turn out to stay relevant for obtaining the final large-scale equation. In the following equations we omitted all the terms involving fast derivatives of $\psi^{(0)}$. The solvability condition at order ϵ^0 implies indeed that $\psi^{(0)}$ depends only on slow variables. We thus obtain

$$\partial_t\partial^2\psi \rightsquigarrow \epsilon^5\partial_T\partial_y^2\psi^{(1)} + \epsilon^6\partial_T\nabla_X^2\psi^{(0)} + \epsilon^6\partial_T\partial_y^2\psi^{(2)}, \tag{A.9}$$

$$\begin{aligned} J(\partial^2\Psi, \psi) &\rightsquigarrow \epsilon[J(\partial^2\Psi, \psi^{(1)}) - (\partial_y^3\Psi)(\nabla_X\psi^{(0)})] \\ &\quad -\epsilon^2(\partial_y^3\Psi)(\nabla_X\psi^{(1)}) - \epsilon^3(\partial_y^3\Psi)(\nabla_X\psi^{(2)}), \end{aligned} \tag{A.10}$$

$$\begin{aligned} J(\partial^2\psi, \Psi) &\rightsquigarrow \epsilon J(\partial^2\psi^{(1)}, \Psi) \\ &\quad +\epsilon^2(\nabla_X\partial_y^2\psi^{(1)})(\partial_y\Psi) + \epsilon^3(\nabla_x\partial_y^2\psi^{(2)} \\ &\quad +\nabla_X^3\psi^{(0)})(\partial_y\Psi), \end{aligned} \tag{A.11}$$

$$\begin{aligned} J(\partial^2\psi, \psi) &\rightsquigarrow -\epsilon^2(\partial_y^3\psi^{(1)})(\nabla_X\psi^{(0)}) \\ &\quad +\epsilon^3[(\nabla_X\partial_y^2\psi^{(1)})(\partial_y\psi^{(1)})\partial_y^3\psi^{(1)}(\nabla_X\psi^{(1)}) \\ &\quad -(\partial_y^3\psi^{(2)})(\nabla_X\psi^{(0)})], \end{aligned} \tag{A.12}$$

$$\begin{aligned} \nu\partial^2\partial^2\psi &\rightsquigarrow \nu_c[\epsilon\partial_y^4\psi^{(1)} + \epsilon^2\partial_y^4\psi^{(2)} \\ &\quad +\epsilon^3(\partial_y^4\psi^{(3)} + 2\partial_y^2\nabla_X^2\psi^{(1)} - \partial_y^4\psi^{(1)})], \end{aligned} \tag{A.13}$$

$$-\beta_1\partial_x\psi \rightsquigarrow -\beta\epsilon^6\nabla_X\psi^{(0)}. \tag{A.14}$$

In the r.h.s of these equations, $J(\cdot,\cdot)$ denotes the Jacobian in the fast variables. Since y is the only fast variable, such Jacobians are zero. Upon using (A.9)–(A.14) in (A.8) and equating terms having equal powers in ϵ, we obtain a hierarchy of equations which all have the general form

$$Af = g, \tag{A.15}$$

where the operator

$$A \equiv -\nu_c\partial^4 \tag{A.16}$$

acts only on the fast variable y. Because of the y-periodicity, the operator A has as its null-space the 'constants', i.e. the functions which do not depend on the fast variable y. For an equation of the form (A.15) to be solvable, the r.h.s. must be orthogonal to such constants, i.e. $\langle g\rangle = 0$, where the angular brackets denote averages over the basic 2π period in y.

The final equation for the large-scale dynamics emerges as a solvability condition to oder ϵ^6. In principle, we should therefore write the hierarchy up to that order. However, as noted in Ref. [31], solvability conditions can be generated more quickly by just decomposing derivatives in (A.8), using (A.5) and taking averages on y without yet expanding ψ. In this fashion we obtain

$$\begin{aligned} &\epsilon^6\partial_T\nabla_X^2\langle\psi\rangle - \epsilon^3\langle(\nabla_X^3\psi)\sin y\rangle + \epsilon^3\langle(\nabla_X^3\psi)(\partial_y\psi)\rangle \\ &\quad -\epsilon^3\langle(\partial_y\nabla_X^2\psi)(\nabla_X\psi)\rangle - \nu_c\epsilon^4(1-\epsilon^2)\nabla_X^4\langle\psi\rangle \\ &\quad +\beta\epsilon^6\nabla_X\langle\psi\rangle = 0, \end{aligned} \tag{A.17}$$

from which the whole set of solvability conditions can be generated by expanding ψ in powers in ϵ. It may then be checked that all the solvability conditions up to order ϵ^5 are satisfied (for the last two orders this requires the use of the condition $R = R_c$). The solvability condition to order ϵ^6 reads

$$\begin{aligned} &\partial_T\nabla_X^2\psi^{(0)} - \langle(\nabla_X^3\psi^{(3)})\sin y\rangle \\ &\quad +\langle(\nabla_X^3\psi^{(1)})\partial_y\psi^{(2)}\rangle + \langle(\nabla_X^3\psi^{(2)})\partial_y\psi^{(1)}\rangle \\ &\quad -\langle(\partial_y\nabla_X^2\psi^{(1)})\nabla_X\psi^{(2)}\rangle - \langle(\partial_y\nabla_X^2\psi^{(2)})\nabla_X\psi^{(1)}\rangle \\ &\quad -\nu_c\nabla_X^4\langle\psi^{(2)}\rangle + \nu_c\nabla_X^4\psi^{(0)} + \beta\nabla_X\psi^{(0)} = 0. \end{aligned} \tag{A.18}$$

This will become a closed equation for $\psi^{(0)}$ only after $\psi^{(1)}$, $\psi^{(2)}$ and $\psi^{(3)}$ have been calculated in terms of $\psi^{(0)}$.

For this we write the equations arising at order ϵ^0, ϵ^1, ϵ^2 and ϵ^3,

$$A\psi^{(0)} = 0, \tag{A.19}$$

$$A\psi^{(1)} = \sin y \nabla_X \psi^{(0)}, \tag{A.20}$$

$$A\psi^{(2)} = \sin y(1+\partial_y^2)\nabla_X\psi^{(1)} + \partial_y^3\psi^{(1)}\nabla_X\psi^{(0)}, \tag{A.21}$$

$$\begin{aligned} A\psi^{(3)} &= \sin y\,(1+\partial_y^2)\nabla_X\psi^{(2)} + \sin y \nabla_X^3\psi^{(0)} \\ &\quad -(\partial_y\psi^{(1)}\partial_y^2 - \partial_y^3\psi^{(1)})\nabla_X\psi^{(1)} + \partial_y^3\psi^{(2)}\nabla_X\psi^{(0)} \\ &\quad +2\nu_c\partial_y^2\nabla_X^2\psi^{(1)} - \nu_c\partial_y^4\psi^{(1)}. \end{aligned} \tag{A.22}$$

From (A.19), it follows that $\psi^{(0)}$ depends only on slow variables. Eqs. (A.20)-(A.22) may be solved explicitly, up to arbitrary additive constants, i.e. functions depending only on the slow variables. The results are given hereafter,

$$\psi^{(1)} = -R_c \sin y \nabla\psi^{(0)} + \langle\psi^{(1)}\rangle, \tag{A.23}$$

$$\begin{aligned}\psi^{(2)} &= -R_c^2\cos y(\nabla_X\psi^{(0)})^2 - R_c\sin y\nabla_X\langle\psi^{(1)}\rangle \\ &\quad +\langle\psi^{(2)}\rangle,\end{aligned} \tag{A.24}$$

$$\begin{aligned}\psi^{(3)} &= -3R_c\sin y\nabla_X^3\psi^{(0)} + R_c^3\sin y(\nabla_X\psi^{(0)})^3 \\ &\quad -R_c\sin y\nabla_X\psi^{(0)} - R_c\sin y\nabla_X\langle\psi^{(2)}\rangle \\ &\quad -2R_c^2\cos y(\nabla_X\psi^{(0)})(\nabla_X\langle\psi^{(1)}\rangle) + \langle\psi^{(3)}\rangle.\end{aligned} \tag{A.25}$$

Substitution of (A.23), (A.24) and (A.25) into (A.18) and use of $R_c = \sqrt{2}$ produces

$$\begin{aligned}\partial_T\nabla_X\psi(X,T) &= \nabla_X\left(\left(\lambda_1(\nabla_X\psi)^2 - \lambda_2\right)\nabla_X^2\psi\right) \\ &\quad -\lambda_3\nabla_X^5\psi - \beta(\psi - C),\end{aligned} \tag{A.26}$$

with

$$\lambda_1 = 2\sqrt{2},\quad \lambda_2 = \sqrt{2},\quad \lambda_3 = \frac{3}{\sqrt{2}}. \tag{A.27}$$

The constant C, which may depend on the time T, stems from the integration with respect to the X-variable (otherwise, there is an additional ∇_X acting on the left of each term). Except for the change of notation (∇_X rather than ∂_X), (A.26) is the β-Cahn-Hilliard equation (7).

Appendix B. Details of the Painlevé analysis

We show how to perform the Painlevé test on the steady-state β-CH equation, rewritten here for convenience as

$$\partial_z{}^3\left(u^3/3 - u - \lambda\partial_z{}^2u\right) - \beta u = 0. \tag{B.1}$$

We set

$$A(z) \equiv u^3/3 - u - \lambda\partial_z{}^2 u, \tag{B.2}$$

and we shall consider the three following problems:

$$A(z) = 0, \tag{B.3}$$

$$\partial_z^3 A(z) = 0, \tag{B.4}$$

$$\partial_z^3 A(z) - \beta u = 0. \tag{B.5}$$

Eqs. (B.3) and (B.5) are respectively the (steady-state) CH and β-CH equations. The solutions of (B.4) include those of the CH equation. We do not specify any boundary conditions, since the Painlevé test involves only the *local* structure of the equation.

The Painlevé test of an equations consists in trying to construct meromorphic solutions, i.e. solutions having only poles as movable singularities. Near such a singularity z_*, assumed to be a pole of (integer) order ρ, a meromorphic function has a Laurent expansion of the form

$$u(z) = \frac{1}{(z-z_*)^\rho}(u_0 + u_1 z + u_2 z^2 + \ldots). \tag{B.6}$$

The order of the pole ρ and the coefficient u_0 are determined by dominant balance after substitution of (B.6) into the equation under consideration. By examining (B.3), we find that the contributions from the $u^3/3$ term and the $-\lambda\partial_z^2 u$ term are always more singular than the $-u$ term. Balancing these terms gives

$$\rho = 1,\quad u_0 = \pm\sqrt{6\lambda}. \tag{B.7}$$

The same dominant balance works also for (B.4) and (B.5).

Since all the equations under consideration are autonomous, no generality is lost by assuming $z_* = 0$. Hence, we are seeking solutions of the form

$$u(z) = \frac{1}{z}(u_0 + u_1 z + u_2 z^2 + \ldots). \tag{B.8}$$

The successive Laurent coefficients u_1, u_2, ... are determined by identification of the various powers of z. It is important to note that all these coefficients (except u_0), when they appear for the first time in this identification process, appear *linearly*. Hence, the u_js ($j \geq 1$) satisfy equations of the form

$$a_j u_j = b_j, \tag{B.9}$$

where the b_js are functions of $u_0, u_1, \ldots, u_{j-1}$. Simple algebra gives, for the case of (B.3),

$$a_j = \lambda(4+3j-j^2) = \lambda(1+j)(4-j), \quad j = 1,2,\ldots, \tag{B.10}$$

while, for the cases of (B.4) and (B.5), we obtain

$$a_j = \lambda(j-3)(j-4)(j-5)(1+j)(4-j), \quad j = 1,2,\ldots. \tag{B.11}$$

Let us first examine the case of (B.3). We observe that a_j vanishes when j equals four[16]. This is called a *resonance*. Hence, (B.9) has no solution for $j = 4$ unless $b_4 = 0$. For $j > 4$ the a_js do not vanish and (B.9) has always a well-defined solution. In order to examine the possibility of constructing a Laurent expansion, it suffices to calculate the b_js up to $j = 4$.

Substitution of (B.8) into (B.3) gives a sequence of relations, the first four of which read

$$u_0^2 u_1 = 0, \tag{B.12}$$

$$u_0^2 u_2 = -u_1^2 u_0 + u_0, \tag{B.13}$$

$$(u_0^2 - 2\lambda) u_3 = -2u_0 u_1 u_2 + u_1, \tag{B.14}$$

$$(u_0^2 - 6\lambda) u_4 = -u_1^2 u_2 - u_2^2 u_0 - 2u_0 u_1 u_3 + u_2. \tag{B.15}$$

Successively solving (B.12) to (B.15) and using (B.7), we obtain

$$u_2 = 1/\sqrt{6\lambda}, \tag{B.16}$$

$$u_1 = u_3 = 0, \tag{B.17}$$

$$0 u_4 = 0. \tag{B.18}$$

[16] It also vanishes when j equals -1, but this reflects just the arbitrary location of the pole.

For u_4 we thus have $b_4 = 0$, that is, a compatible resonance and, hence, a second free (complex) parameter, in addition to z_*. Having thus constructed a two-parameter family of Laurent series solutions of (B.3), we have proven that it has the Painlevé property. Actually, (B.3) may be explicitly integrated in terms of elliptic functions.

We now turn to (B.4), which may be rewritten as

$$A(z) = \alpha + \gamma z + \delta z^2, \tag{B.19}$$

with arbitrary constants α, γ and δ. Going through the same procedure as above, we find that compatibility holds if and only if $\gamma = 0$. It may be checked that the corresponding equation $A(z) = \alpha + \delta z^2$ is in the class of Painlevé transcendents which does have meromorphic solutions [32].

Alternatively, we can work directly with the fifth order equation (B.4). We observe that the coefficients a_j, given by (B.11), vanishes for $j = 3, 4, 5$. It may be checked that all three resonances are compatible. Together with z_* this gives four arbitrary constants, not enough for a fifth order equation. So, we only have a kind of weak Painlevé property.

Finally, we turn to the β-CH equation (B.5). The coefficients a_j are still given by (B.11). Hence, there are the same three resonances $j = 3, 4, 5$ as for (B.4). The $j = 3$ and $j = 4$ resonances are compatible, but for $j = 5$ we find

$$0 u_5 = \beta u_0. \tag{B.20}$$

This is a noncompatible resonance. Hence, the Painlevé test fails for the β-CH equation.

Appendix C. Details of the theory of resonant interactions

We show here how the general averaging formalism presented in Section 5 can be applied to the β-Cahn–Hilliard equation. We want to solve Eqs. (37)–(40).

We begin with (37). The function f_k for the β-Cahn–Hilliard case is given by (24), which may be rewritten in more symmetrical form as

$$f_k(\rho_\bullet, \varphi_\bullet) = (\tilde{\lambda}_2 k^2 - \tilde{\lambda}_3 k^4)\rho_k - \tilde{\lambda}_1 k^2$$

$$\times \mathrm{Re} \sum_{k_1+k_2+k_3+k_4=0} \rho_{k_1}\rho_{k_2}\rho_{k_3} e^{i(\varphi_{k_1}+\varphi_{k_2}+\varphi_{k_3}+\varphi_{k_4})}, \tag{C.1}$$

where $k_4 \equiv -k$ and we have used the Hermitian symmetry ($\varphi_{k_4} = -\varphi_k$).

As explained in Section 5, the terms in (37) are expanded in multiple Fourier series in all the phases. From (C.1) it is seen that the Fourier coefficient $f_{k,\xi}(\rho_\bullet)$ differs from zero if and only if $\xi = 0$ or

$$\xi = \pm X(k_1, k_2, k_3, k_4), \tag{C.2}$$

where $X = \{X_i\}$ has only four nonvanishing components, all equal to $+1$, corresponding to a tetrad (k_1, k_2, k_3, k_4) of indices i such that (i)

$$k_1 + k_2 + k_3 + k_4 = 0, \tag{C.3}$$

and (ii) no partial sum of two wavenumbers, e.g., $k_1 + k_2$ vanishes; otherwise the Hermitian symmetry implies the vanishing of $\varphi_{k_1} + \varphi_{k_2} + \varphi_{k_3} + \varphi_{k_4}$ so that the corresponding terms pertain to $f_{k,\xi=0}$.

Let us now consider (49), rewritten for the convenience of the reader,

$$i\xi_\ell\, \omega_\ell\, \rho^{(1)}_{k,\xi} = f_{k,\xi}(\rho_\bullet) \quad (\xi \neq 0). \tag{C.4}$$

The existence of solutions to (C.4) requires that, whenever $\xi_\ell\, \omega_\ell$ vanishes, the r.h.s. should also vanish (Fredholm alternative). We have seen that the $f_{k,\xi}$s are nonvanishing only for those ξs given by (C.2). We then have

$$\xi_\ell\, \omega_\ell = \omega_{k_1} + \omega_{k_2} + \omega_{k_3} + \omega_{k_4}. \tag{C.5}$$

It is shown in Appendix D that in the Rossby case, four-wave resonances, that is, the simultaneous vanishing of $k_1 + k_2 + k_3 + k_4$ and of $\omega_{k_1} + \omega_{k_2} + \omega_{k_3} + \omega_{k_4}$, require the wavevectors to be opposite in pair, e.g., $k_1 + k_2 = 0$ and $k_3 + k_4 = 0$ or permutations. This situation is ruled out by the condition (ii) on tetrads. Hence, we have established that, whenever there is a resonance, the r.h.s of (C.4) vanishes, i.e., that there are only *compatible resonances*. Therefore the leading-order asymptotics is given just by standard averaging, namely (45). To write this equation in explicit form for the β-Cahn–Hilliard case, we must evaluate

$$\langle f_k(\rho_\bullet, \varphi_\bullet)\rangle = f_{k,\xi=0}(\rho_\bullet). \tag{C.6}$$

Using (C.1), we obtain

$$f_{k,\xi=0}(\rho_\bullet) = (\tilde\lambda_2 k^2 - \tilde\lambda_3 k^4)\rho_k - 6\tilde\lambda_1 k^2 \sum_{\ell \neq k} \rho_\ell^2 \rho_k - 3\tilde\lambda_1 k^2 \rho_k^3. \tag{C.7}$$

Using (26), (C.6) and (C.7) in (45) we obtain the resonant interaction Cahn–Hillard (RICH) equation (53).

The solution of (38) is handled in essentially the same way as for (37). Again there are only compatible resonances. The main difference with the previous case is that the average of (25) vanishes because of the presence of the imaginary part. Hence, to leading order there is no modification of the frequency of the Rossby waves. When considering the Eqs. (39) and (40), which determine the next-to-leading-order corrections, the matters become more complex: the right hand sides of these equations contain expressions quadratic in the f_ks and g_ks. It may be checked that sixth order noncompatible resonances are present. Hence, straightforward averaging fails beyond the leading order.

Appendix D. Resonances: a diophantine problem

We investigate here the conditions for resonances among $n \geq 2$ waves. The problem will be solved completely for $n = 2, 3, 4$, the only values needed for the leading-order behavior for large values of β. Special solutions will be given for $n = 5$ and $n = 6$.

The problem may be formulated as follows. Let there be given $n - 1$ (signed) integer wavenumbers $k_1, k_2, \ldots, k_{n-1}$; define

$$k \equiv k_1 + k_2 + \cdots + k_{n-1}. \tag{D.1}$$

Find those wavenumbers satisfying the condition of *resonance*, that is,

$$\omega_k = \omega_{k_1} + \omega_{k_2} + \cdots + \omega_{k_{n-1}}, \tag{D.2}$$

where

$$\omega_k \equiv \frac{\beta}{k}. \tag{D.3}$$

This constitutes a *diophantine* problem, that is, a system of equations for which one seeks *integer* solutions.

Without loss of generality, we may assume $\beta = 1$. The problem may be recast in more symmetrical form, namely

$$k_1 + k_2 + \cdots + k_n = 0, \tag{D.4}$$

$$\frac{1}{k_1} + \frac{1}{k_2} + \cdots + \frac{1}{k_n} = 0, \tag{D.5}$$

where $k_n \equiv -k$. It follows from (D.4) and (D.5) that the k_is are the roots of an nth order polynomial equation of the form

$$x^n + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_0 = 0, \tag{D.6}$$

where the coefficients $a_0, a_2, \ldots, a_{n-2}$ are integers.

For $n = 2$, (D.6) has the form

$$x^2 + a_0 = 0. \tag{D.7}$$

It follows that the most general solution is that k_1 and k_2 form a pair of opposite wavenumbers.

For $n = 3$, Eq. (D.6) has the form

$$x^3 + a_0 = 0, \tag{D.8}$$

which has only trivial (real) solutions: $k_1 = k_2 = k_3 = 0$.

For $n = 4$, Eq. (D.6) has the form

$$x^4 + a_2x^2 + a_0 = 0, \tag{D.9}$$

which is the most general biquadratic equation with integer coefficients. It follows that the general solution is made of two pairs of opposite wavenumbers,

$$k_1 + k_2 = 0, \quad k_3 + k_4 = 0 \quad \text{(or permutations)}. \tag{D.10}$$

Such solutions may be called 'decomposed', in the sense that they decompose into the solutions of two lower-order ($n = 2$) problems.

For $n = 5$ and $n = 6$, Eq. (D.6) has the form

$$x^5 + a_3x^3 + a_2x^2 + a_0 = 0, \tag{D.11}$$

$$x^6 + a_4x^4 + a_3x^3 + a_2x^2 + a_0 = 0. \tag{D.12}$$

We do not know if these diophantine problems can be solved completely. We did a numerical search by varying $k_1, k_2, \ldots, k_{n-1}$ between prescribed bounds (from $-N$ to N) with no bounding constraint on $k_n = -k_1 - k_2 - \ldots - k_{n-1}$ and looked for all instances which satisfy the resonance condition. For $n = 5$ and $N = 30$, we found about 200 solutions, possessing no particular symmetry. An example is

$$-5, \quad 3, \quad -8, \quad -30, \quad 40. \tag{D.13}$$

For $n = 6$ we found a huge proliferation of solutions. They include a substantial number of solutions with two wavenumbers having multiplicity two, that is,

$$k_1, \quad k_2, \quad k_1, \quad k_2, \quad k_5, \quad k_6, \tag{D.14}$$

an example being

$$-5, \quad 8, \quad -5, \quad 8, \quad 4, \quad -10. \tag{D.15}$$

It is actually possible to find all solutions of the form (D.14). Indeed, if in (D.5), with $n = 6$, we assume $k_3 = k_1$ and $k_4 = k_2$, we obtain

$$k_5 + k_6 = -2(k_1 + k_2),$$
$$k_5k_6 = k_1k_2. \tag{D.16}$$

It follows that

$$k_5 = -(k_1 + k_2) + \sqrt{k_1^2 + k_1k_2 + k_2^2},$$
$$k_6 = -(k_1 + k_2) - \sqrt{k_1^2 + k_1k_2 + k_2^2}. \tag{D.17}$$

A necessary and sufficient condition for k_5 and k_6 to be integer is

$$k_1^2 + k_1k_2 + k_2^2 = z^2, \tag{D.18}$$

where z is an integer. Setting

$$x = k_1 + k_2, \quad y = k_1 - k_2, \tag{D.19}$$

we obtain, from (D.18), the quadratic diophantine equation

$$3x^2 + y^2 = 4z^2, \tag{D.20}$$

which is a variant of the famous Pythagoras equation $(x^2 + y^2 = z^2)$ and which may be solved by similar techniques. We shall not dwell on these matters.

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