

# Clouds in our current and in a warming climate

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## SYLLABUS

Remark: Some of the topics indicated with a \* will not be discussed in detail in the lectures because of time limitation, but are kept in those notes for the interested reader.

<b>A</b>	<b>Dry dynamics of the atmosphere: large-scale dominant balances</b>	<b>3</b>
A.1	Hydrostatic balance . . . . .	3
A.1.1	Introduction: equations of motion . . . . .	3
A.1.2	Typical scales of interest . . . . .	4
A.1.3	Hydrostatic balance . . . . .	5
A.1.4	Geopotential . . . . .	5
A.1.5	Pressure coordinates . . . . .	5
A.2	Hypsometric balance . . . . .	6
A.2.1	Hypsometric equation . . . . .	6
A.2.2	Special case of an isothermal atmosphere . . . . .	6
A.2.3	Equivalence between pressure and geopotential . . . . .	6
A.2.4	Weather maps . . . . .	7
A.3	Geostrophic balance . . . . .	7
A.3.1	Geostrophic balance and Rossby number . . . . .	7
A.3.2	Circulation around low and high pressures . . . . .	8
A.3.3	Properties of the geostrophic wind . . . . .	9
A.4	Thermal wind . . . . .	9
A.4.1	Thermal wind balance . . . . .	9
A.4.2	Slope of a front* . . . . .	11
A.5	Vorticity and Rossby waves . . . . .	11
<b>B</b>	<b>Atmospheric moisture and moist convection</b>	<b>14</b>
B.1	Fundamental aspects of clouds . . . . .	14
B.2	Dry thermodynamics . . . . .	14
B.2.1	First law of thermodynamics . . . . .	14
B.2.2	Ideal gas law . . . . .	14
B.2.3	Mixture of gases : Dalton's law . . . . .	15
B.2.4	Joule's law, enthalpy . . . . .	15
B.2.5	Summary of equations in specific form: . . . . .	16
B.3	Dry convection: application to the atmosphere . . . . .	16
B.3.1	Potential temperature $\theta$ , dry static energy, dry adiabatic lapse rate . . . . .	16
B.3.2	Stability to dry convection, Brunt-Väisälä frequency . . . . .	17
B.3.3	Centrifugal convection* . . . . .	21
B.3.4	Symmetric instability and slantwise convection* . . . . .	23
B.4	Entropy, Second law of thermodynamics and implications for the atmosphere . . . . .	24
B.4.1	Definition and link with $\theta^*$ . . . . .	24
B.4.2	Consequences, entropy maximisation* . . . . .	25
B.5	Moist thermodynamics . . . . .	26
B.5.1	When does water vapor condense into liquid water? The Clausius-Clapeyron equation . . . . .	26
B.5.2	Atmospheric thermodynamics: moist variables . . . . .	28
B.6	Moist convection . . . . .	30

B.6.1	Convection of unsaturated moist air - virtual temperature . . . . .	30
B.6.2	Equivalent potential temperature $\theta_e$ , moist static energy, moist adiabatic lapse rate . . . . .	31
B.6.3	How can we assess the stability of moist air? Skew-T diagrams . . . . .	32
B.6.4	Convective Available Potential Energy (CAPE), convective inhibition (CIN) . . . . .	33

## BOOKS

- "Atmospheric and Oceanic Fluid Dynamics", Vallis
- "Atmospheric Sciences", Wallace & Hobbes
- "Cloud Dynamics", Houze
- "Atmospheric Convection", Emanuel
- "Atmospheric Thermodynamics", Bohren & Albrecht
- "Physics of Climate", Peixoto & Oort

## INTRODUCTION

This class consists of two parts.

The first part (§ A) introduces some basic notions of geophysical fluid dynamics, i.e. the dynamics of the fluids on our planet, namely the oceans and the atmosphere. The dry dynamics of the atmosphere will be discussed in detail, introducing the equations of motion, and deriving the dominant balances at large (planetary and synoptic) scales.

The second part (§ B) introduces atmospheric moisture, which is present in the form of water vapor, as well as liquid and ice water condensates. Moist convection, cloud formation, and the life cycle of clouds will be discussed. We will end with a discussion of the role of clouds on the energetics of our planet, and how this role is expected to change in a warmer climate.

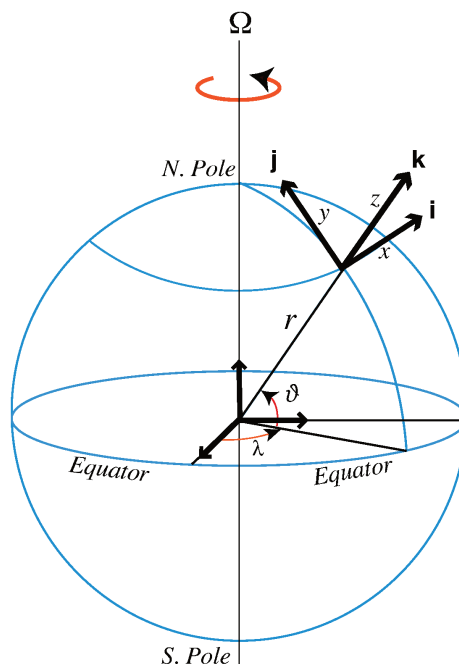
# A Dry dynamics of the atmosphere: large-scale dominant balances

## A.1 Hydrostatic balance

### A.1.1 Introduction: equations of motion

For many dynamical phenomena in the atmosphere and the oceans, the sphericity of the Earth is not central, and it is more convenient to work in a local Cartesian coordinate system. In the study of geophysical fluid dynamics, it is thus standard to work with a coordinate system in a plane tangent to the Earth (Fig. 1, adapted from the book of Vallis). The  $x$  direction corresponds to increasing longitude  $\lambda$ , the  $y$  direction corresponds to increasing latitude  $\vartheta$ , and  $z$  denotes the local vertical.

For small excursions on the plane around a reference latitude  $\vartheta_0$ ,  $(x, y, z) \approx (a\lambda \cos(\vartheta_0), a(\vartheta - \vartheta_0), z)$ , where  $a$  denotes the radius of the sphere. Consistently, the velocity  $(u, v, w)$  is the velocity in the tangent plane, in the west-east, south-north and vertical directions respectively.



**Fig. 2.3** The spherical coordinate system. The orthogonal unit vectors  $i, j$  and  $k$  point in the direction of increasing longitude  $\lambda$ , latitude  $\vartheta$ , and altitude  $z$ . Locally, one may apply a Cartesian system with variables  $x, y$  and  $z$  measuring distances along  $i, j$  and  $k$ .

Figure 1: Typical coordinate systems used in geophysical fluid dynamics, with  $x$  in the longitudinal direction (also called zonal direction),  $y$  in the latitudinal direction (also called meridional direction), and  $z$  is the local vertical direction.

We recall the equations of motion in this plane, under the effect of gravity  $g$ , including the Earth rotation (Coriolis parameter denoted  $f = 2\Omega \sin(\vartheta)$ , where  $\Omega$  denotes the Earth rotation frequency), and neglecting

viscous effects:

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho} \frac{\partial p}{\partial x} \quad (1)$$

$$\frac{Dv}{Dt} + fu = -\frac{1}{\rho} \frac{\partial p}{\partial y} \quad (2)$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho} \frac{\partial p}{\partial z} - g \quad (3)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \quad (4)$$

where  $p$  denotes pressure, and  $\rho$  density. Equations (1-2) are the horizontal momentum conservation equations, equation (3) the vertical momentum conservation equation, and equation (4) is the mass conservation equation (under the assumption that density variations are small compared to its reference value, see the book by Vallis for a more detailed discussion of this approximation known as the Boussinesq approximation, and for a derivation of these equations).

The first term on the left-hand side of equations (1-3) is the total Lagrangian acceleration

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z}. \quad (5)$$

It follows from the fact that Newton's law (mass  $\times$  acceleration = sum of forces) applies to atmospheric particles whose position  $x(t), y(t), z(t)$  depends on time, with  $dx/dt = u, dy/dt = v, dz/dt = w$ . Therefore the acceleration, for instance in the  $x$  direction  $D(u(t, x(t), y(t), z(t)))/Dt$ , is given by  $Du/Dt = \partial u/\partial t + u\partial u/\partial x + v\partial u/\partial y + w\partial u/\partial z$ , yielding equation (5).

The second term on the left-hand side of equations (1-2) is the Coriolis force, related to the Earth rotation. We will come back to this term and its physical interpretation when we will discuss the geostrophic balance.

The first term on the right-hand side of equations (1-3) is the pressure force. Finally, the  $-g$  term is the gravitational force, and the last equation holds for a non-divergent flow. (We note in passing that a thermodynamic equation completes the total set of equations for atmospheric motion, but we will not need it in this section).

Note that the Coriolis parameter  $f(y)$  depends on latitude as noted above, and thus is a function of  $y$ . If the region of study has small latitudinal extent around a latitude  $\vartheta_0$ , often this variation is neglected and  $f(y) \approx f_0 = 2\Omega \sin(\vartheta_0)$  is assumed constant, which is known as the  $f$ -plane approximation. Another common approximation is to include the variations of  $f$  but only to leading order keeping the linear term:  $f(y) \approx f_0 + \beta y$ , which is known as the  $\beta$ -plane approximation. In this case,  $\beta = df/dy = 2\Omega \cos(\vartheta_0)/a$  is the latitudinal variation of  $f$  with latitude.

This system of equations is not easily solved. One approach to gain insight into the dynamics, is to perform a scale analysis. This approach allows to highlight the leading order terms and dynamics. We will now use it to derive the hydrostatic balance.

### A.1.2 Typical scales of interest

For midlatitude synoptic systems, typical scales are:

$U \sim 10 \text{ m s}^{-1}$  horizontal velocity

$W \sim 1 \text{ cm s}^{-1}$  vertical velocity

$L \sim 10^6 \text{ m}$  horizontal scale (e.g., midlatitude cyclone or anticyclone)

$H \sim 10^4 \text{ m}$  vertical scale (depth of troposphere)

$\tau \sim L/U \sim 10^5 \text{ s}$  corresponding time scale ( $\sim 1$  day)

$f_0 \sim 10^{-4} \text{ s}^{-1}$  Coriolis frequency.

### A.1.3 Hydrostatic balance

The hydrostatic approximation states that:

$$\boxed{\frac{\partial p}{\partial z} = -\rho g.} \quad (6)$$

It is a very good approximation for the large-scale motions of the atmosphere and of the ocean.

It follows from a scale analysis of the vertical momentum equation (3). Equation (4) yields

$$\frac{U}{L} \sim \frac{W}{H}. \quad (7)$$

The horizontal momentum equations, which we rewrite in vector form with  $\mathbf{u} = (u, v, 0)$ ,  $\nabla_h$  horizontal gradient, and the unit vertical vector  $\mathbf{k} = (0, 0, 1)$ :

$$\frac{d\mathbf{u}}{dt} + f\mathbf{k} \wedge \mathbf{u} = -\frac{1}{\rho}\nabla_h p \Rightarrow \frac{U}{\tau} \sim \frac{P}{\rho L}, \quad (8)$$

where  $P$  denotes a typical pressure magnitude. Including these scalings into the vertical momentum equation yields:

$$\frac{dw}{dt} \sim \frac{W}{\tau} \quad (9)$$

$$\frac{1}{\rho}\frac{\partial p}{\partial z} \sim \frac{P}{\rho H} \sim \frac{LU}{H\tau} \text{ from equation (8)} \quad (10)$$

Thus the ratio of those two terms is

$$\frac{dw/dt}{1/\rho\partial p/\partial z} \sim \frac{WH}{LU} \sim \left(\frac{H}{L}\right)^2 \text{ from equation (7)}. \quad (11)$$

According to this scale analysis, for the large scales of interest here, we can neglect the vertical acceleration to very good approximation:  $(H/L)^2 \sim (10^{-2})^2 = 10^{-4} \ll 1$ .

For smaller scale phenomena, the vertical acceleration term can not be neglected. This is for instance the case for isolated clouds which have an aspect ratio order one. It follows from this scale analysis, and is consistent with the observation that vertical accelerations are important in cloud dynamics.

### A.1.4 Geopotential

In the atmosphere, the gravitational acceleration  $g$  varies in space. We introduce the geopotential  $d\Phi = gdz \Leftrightarrow \Phi = \int_0^z gdz$ . It is the work needed to raise a unit of mass from sea level to an altitude  $z$  (work against gravity). It has units of  $\text{J kg}^{-1}$ .

It is more standard to use the geopotential height, simply defined as  $Z = \Phi/g_0$ , where  $g_0 = 9.8 \text{ m s}^{-2}$  is a reference gravity constant.  $Z$  has a nice physical interpretation. Indeed, if we consider an atmospheric surface at pressure  $p_1$ , and ask the following question: what is the altitude of this isobaric surface? Then  $Z(p_1) = \int_0^{z(p_1)} gdz/g_0 \approx z(p_1)$  if  $g \approx g_0$ , which turns out to be the case in the atmosphere (variations of  $g$  are small). Thus the geopotential height  $Z(p_1)$  represents the altitude of the  $p_1$  isobar.

### A.1.5 Pressure coordinates

In meteorology, it is common to work in pressure coordinates instead of height. Let us derive the hydrostatic approximation in pressure coordinate. To that end, we will combine the hydrostatic approximation  $dp = -\rho g dz$  with the ideal gas law  $p = \rho RT$

$$\Rightarrow gdz = -\frac{RT}{p} dp \Rightarrow \boxed{\frac{\partial \Phi}{\partial p} = -\frac{RT}{p}}. \quad (12)$$

This last equation is the hydrostatic balance in pressure coordinates.

## A.2 Hypsometric balance

### A.2.1 Hypsometric equation

The hydrostatic approximation can be rewritten

$$\frac{\partial \Phi}{\partial \ln p} = -RT \quad (13)$$

which we integrate between  $p_0$  and  $p_1$ , yielding:

$$\Phi(p_1) - \Phi(p_0) = -R \int_{p_0}^{p_1} T d \ln p \quad (14)$$

$$= -R \int_{p_0}^{p_1} T d \ln p = -R\bar{T} \ln \left( \frac{p_1}{p_0} \right) \quad (15)$$

$$\Rightarrow \Phi(p_1) - \Phi(p_0) = R\bar{T} \ln \left( \frac{p_0}{p_1} \right), \quad (16)$$

$$\text{with } \bar{T} = \frac{\int_{p_0}^{p_1} T d \ln p}{\ln p_1 - \ln p_0} \quad (17)$$

Note that  $d \ln p \sim dz$  (exactly proportional in an isothermal hydrostatic atmosphere, see below equation (19)). So  $\bar{T}$  is representative of the mean temperature between isobars  $p_0$  and  $p_1$ .

$\delta\Phi = \Phi(p_1) - \Phi(p_0)$  represents the thickness between isobar  $p_0$  and isobar  $p_1$  (recall that  $\Phi$  is proportional to geopotential height). The hypsometric equation can be written

$$\boxed{\delta\Phi = R\bar{T} \ln \left( \frac{p_0}{p_1} \right)}. \quad (18)$$

It implies that the thickness between two isobars is proportional to the temperature between those two levels.

### A.2.2 Special case of an isothermal atmosphere

The hypsometric equation generalizes the relation between  $p$ ,  $z$  and  $T$  in the case of an isothermal atmosphere  $T = T_0$ . In that case,

$$d\Phi = g dz = -RT_0 d \ln p \Rightarrow d \ln p = -g / (RT_0) dz \Rightarrow \ln \left( \frac{p}{p_0} \right) = -\frac{g}{RT_0} (z - z_0) \quad (19)$$

Thus

$$p = p_0 \exp \left( -\frac{g(z - z_0)}{RT_0} \right) = p_0 \exp \left( -\frac{(z - z_0)}{H_p} \right) \quad (20)$$

pressure decreases exponentially with altitude. The typical decay length scale is  $H_p = RT_0/g \approx (287 \text{ J kg}^{-1} \text{ K}^{-1}) \times 288 \text{ K} / (9.8 \text{ m s}^{-2}) \approx 8400 \text{ m}$ . We see that for warmer temperatures, pressure decreases more slowly, and conversely for colder temperatures, pressure decreases more rapidly.

### A.2.3 Equivalence between pressure and geopotential

The geopotential can also be seen as a variable equivalent to pressure. Indeed, denoting  $\nabla$  the horizontal pressure gradient ( $\partial/\partial x, \partial/\partial y$ ),

$$\boxed{\nabla_{p=cstt} \Phi = \frac{1}{\rho} \nabla_{z=cstt} p.} \quad (21)$$

(This comes from the mathematical fact that

$$\nabla_{S=cstt} F = \nabla_{z=cstt} F + \frac{\partial F}{\partial z} \nabla_{S=cstt} z. \quad (22)$$

for any variable  $S$ ).

Therefore, the horizontal pressure gradient term  $\nabla p/\rho$  in the horizontal momentum equations (1-2) in altitude coordinates  $(x, y, z)$ , is replaced by  $\nabla \Phi$  in pressure coordinates  $(x, y, p)$ .

Thus, a low value of geopotential on an isobar is equivalent to a low value of pressure at constant altitude. In other words, a high pressure corresponds to a positive anomaly of geopotential height on an isobar, and conversely a low pressure corresponds to a negative geopotential anomaly.

These low and high pressure systems, which correspond to troughs and crests in the geopotential height field, have implications for atmospheric dynamics and weather. A high pressure, or geopotential crest, is associated with an anticyclonic circulation. Conversely a low pressure, or geopotential trough, is associated with a cyclonic circulation.

$\Rightarrow$  To summarize,  $\delta \Phi$  corresponds to  $\bar{T}$ , and  $\Phi$  corresponds to  $p$ .

### A.2.4 Weather maps

In meteorology, it is standard to analyze maps on different isobars:

Sea level, or 1000 hPa	Surface
850 hPa	1500 m    above the planetary boundary layer
500 hPa	5000 m    mid-troposphere
200 hPa	12 km    tropopause

The temperature in the atmosphere varies with height. The temperature is somewhat constant in the boundary layer, then decreases with height at a rate of about  $6^\circ \text{ km}^{-1}$  in the troposphere, until the tropopause. In the stratosphere above, temperature then increases with height.  $T$  at 850 hPa, i.e. above the boundary layer, is not directly influenced by the surface. For instance, it is not very sensitive to the diurnal cycle.

## A.3 Geostrophic balance

### A.3.1 Geostrophic balance and Rossby number

We perform a scale analysis of the horizontal momentum equation (8), which we rewrite in pressure coordinates:

$$\frac{d\mathbf{u}}{dt} + f\mathbf{k} \wedge \mathbf{u} = -\nabla_h \Phi \quad (23)$$

where recall that  $f = 2\Omega \sin(\text{lat})$ . A scale analysis of the left hand side terms yields:

$$\frac{d\mathbf{u}}{dt} \sim \frac{U^2}{L}, \quad \text{and} \quad f\mathbf{k} \wedge \mathbf{u} \sim fU. \quad (24)$$

The Rossby number is defined as the ratio of those terms:

$$Ro = \frac{\text{horizontal acceleration}}{\text{Coriolis force}} = \frac{U}{fL}. \quad (25)$$

If  $Ro \ll 1$ , the Coriolis force dominates the acceleration term, yielding

$$f\mathbf{k} \wedge \mathbf{u} = -\nabla_h \Phi \Leftrightarrow \begin{cases} -fv = -\frac{\partial \Phi}{\partial x} \\ fu = -\frac{\partial \Phi}{\partial y} \end{cases} \Leftrightarrow \begin{cases} u = -\frac{1}{f} \frac{\partial \Phi}{\partial y} \\ v = \frac{1}{f} \frac{\partial \Phi}{\partial x} \end{cases} \quad (26)$$

This is the geostrophic balance (in pressure coordinates), i.e. a balance between pressure and Coriolis forces.  $(u, v)$  given above is called the geostrophic wind.

For low pressure systems in mid-latitudes, lat  $\sim 45^\circ\text{N}$  so  $f \sim 10^{-4}\text{s}^{-1}$ ,  $L \sim 1000\text{ km}$  and  $U \sim 10\text{ m s}^{-1}$ , so that  $Ro \sim 0.1$ . The geostrophic balance holds to a good approximation.

For tropical cyclones, lat  $\sim 15^\circ\text{N}$  so  $f \sim 4 \times 10^{-5}\text{s}^{-1}$ ,  $L \sim 1000\text{ km}$  and  $U \sim 50\text{ m s}^{-1}$ , so that  $Ro \sim 1$ . A tropical cyclone is not in geostrophic balance.

For a bath-tub of kitchen sink vortex, lat  $\sim 45^\circ\text{N} \Rightarrow f \sim 10^{-4}\text{s}^{-1}$ ,  $L \sim 1\text{ m}$  and  $U \sim 1\text{ m s}^{-1}$ , so that  $Ro \sim 10^4$ ! These are NOT AT ALL in geostrophic balance, the Earth rotation does not impact these small-scale vortices.

Thus in  $p$  coordinates (with  $(u, v)$  along isobars), the geostrophic wind is given by

$$u_g = -\frac{1}{f_0} \frac{\partial \Phi}{\partial y}, \quad v_g = \frac{1}{f_0} \frac{\partial \Phi}{\partial x}.$$

It is worth noting the  $z$  coordinate expression of the geostrophic wind, assuming a constant density fluid  $\rho = \rho_0$ :

$$u_g = -\frac{1}{f_0 \rho_0} \frac{\partial p}{\partial y}, \quad v_g = \frac{1}{f_0 \rho_0} \frac{\partial p}{\partial x}. \quad (27)$$

### A.3.2 Circulation around low and high pressures

Figure 2 shows the cyclonic circulation around low pressure anomalies, and anticyclonic around high pressure anomalies, implied by geostrophic balance.

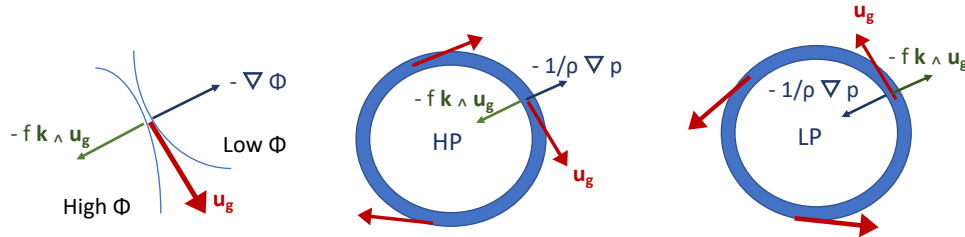


Figure 2: Geostrophic wind between low and high pressure anomalies in the Northern Hemisphere. Geostrophic balance, along with the fact that the Coriolis force is to the right of the velocity in the Northern Hemisphere, implies clockwise circulation around high pressure systems, and counter-clockwise circulation around low pressure systems.

#### - Effect of friction: hints for the homework

In the homework, you will investigate what happens when friction is added to the geostrophic balance. Friction can potentially be important near the surface, so for low-level flow. In that case, a simple friction term, proportional to the velocity and oriented opposite, will be added to the geostrophic balance (26):

$$f \mathbf{k} \wedge \mathbf{u} = -\nabla_h \Phi - K \mathbf{u} \Leftrightarrow \begin{cases} -fv = -\frac{\partial \Phi}{\partial x} - Ku \\ fu = -\frac{\partial \Phi}{\partial y} - Kv \end{cases} \quad (28)$$

As you will see in the homework, a consequence of the friction term is the deviation of the geostrophic winds towards low pressure anomalies. This implies a convergence of horizontal winds near the surface into



low pressure systems. Converging surface winds force upward motion, thereby advecting moist near-surface air to colder temperatures aloft. This can lead to cloud formation and precipitation.

Conversely, friction deviates geostrophic winds away from high pressure anomalies. This implies divergence of near-surface winds and subsidence of dry air from aloft, preventing cloud formation and favoring nice sunny weather.

### A.3.3 Properties of the geostrophic wind

Prop 1:  $(u_g, v_g)$  flows parallel to  $\text{iso}\Phi$  lines.

Indeed,  $(u_g, v_g) \cdot \nabla\Phi = 0$ .

Prop 2: The closer the  $\text{iso}\Phi$  lines are from each other, the stronger the geostrophic wind is.

Indeed,  $\|(u_g, v_g)\| \propto \|\nabla\Phi\|$ .

Prop 3: On the  $f$  plane (i.e. assuming  $f = f_0$  constant), the geostrophic wind is non divergent.

Indeed,  $\nabla \cdot (u_g, v_g) = \partial u_g / \partial x + \partial v_g / \partial y = 0$ . This implies that  $\partial w / \partial z = 0$  thus the geostrophic wind does not contribute directly to the vertical velocity.

Note that if  $f$  is not constant, then

$$\frac{\partial(fu_g)}{\partial x} + \frac{\partial(fv_g)}{\partial y} = 0 \Rightarrow f \left( \frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y} \right) + v_g \frac{\partial f}{\partial y} = 0 \Rightarrow f \frac{\partial w}{\partial z} = v_g \frac{\partial f}{\partial y} \quad (29)$$

The variations of  $f$  with latitude creates a vertical velocity (see Sverdrup balance in oceanography).

Prop 4: On the  $f$  plane (i.e. assuming  $f = f_0$  constant), the rotational of the geostrophic wind is proportional to the Laplacian of the geopotential.

Indeed

$$\xi_g = \nabla \wedge (u_g, v_g) = \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} = \frac{1}{f_0} \left( \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} \right) = \frac{1}{f_0} \Delta \Phi \quad (30)$$

Typically, vorticity is associated with two phenomena: changes of direction of the wind, and changes in wind magnitude (shear).

## A.4 Thermal wind

### A.4.1 Thermal wind balance

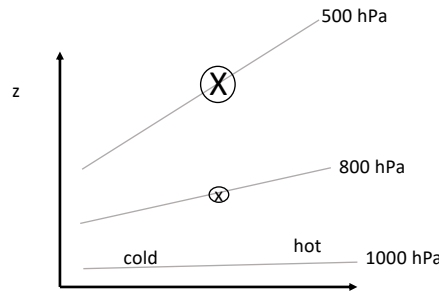


Figure 3: Thermal wind: in the presence of a horizontal temperature gradient, isobars are slanted (from hypsometry). The increasing horizontal gradient of geopotential with height, implies that the geostrophic wind increases with height.

To derive the thermal wind, we need two ingredients: geostrophy and hydrostasy.

As illustrated in figure 3, on the warm side from the hypsometric equation, isobars are further away from each other. If we assume that the surface approximately lies on an isobar, e.g. 1000 hPa isobar, then this implies that the geopotential height increases faster and faster with height in the warm sector. From geostrophic balance, a gradient of geopotential height on an isobar indicates a geostrophic wind. Furthermore, the gradient increases with height, thus the wind increases in height. This is the so-called thermal wind balance: a horizontal gradient of temperature yields a vertical gradient of horizontal wind.

Mathematically, in the case of the ocean, the density can be approximated by a constant  $\rho \sim 1000 \text{ kg m}^{-3}$  plus a small perturbation  $\rho'(x, y, z, t) < 10 \text{ kg m}^{-3}$ :

$$\rho(x, y, z, t) = \rho_0 + \rho'(x, y, z, t). \quad (31)$$

The pressure  $p_0(z) + p'(x, y, z, t)$  is assumed in hydrostatic balance with the density ( $\rho_0$  and  $\rho'(x, y, z, t)$  respectively). Combining the hydrostatic balance and geostrophy:

$$u_g = -\frac{1}{f_0 \rho_0} \frac{\partial p'}{\partial y} \quad (32)$$

and the hydrostatic balance

$$\frac{\partial p'}{\partial z} = -\rho' g, \quad (33)$$

we obtain

$$\frac{\partial u_g}{\partial z} = -\frac{1}{f_0 \rho_0} \frac{\partial^2 p'}{\partial y \partial z} \implies \boxed{\frac{\partial u_g}{\partial z} = \frac{g}{f_0 \rho_0} \frac{\partial \rho'}{\partial y}} \quad (34)$$

Similarly for  $v$ :

$$\boxed{\frac{\partial v_g}{\partial z} = -\frac{g}{f_0 \rho_0} \frac{\partial \rho'}{\partial x}} \quad (35)$$

Similarly in the atmosphere, in  $(x, y, p)$  coordinate, combining the geostrophic wind

$$f u = -\frac{\partial \Phi}{\partial y} \quad (36)$$

with hydrostatic balance

$$\frac{\partial \Phi}{\partial p} = -\frac{R T}{p} \quad (37)$$

yields

$$\frac{f \partial u}{\partial p} = -\frac{\partial^2 \Phi}{\partial y \partial p} \implies \boxed{\frac{\partial u}{\partial p} = \frac{R}{f p} \frac{\partial T}{\partial y}} \quad (38)$$

Similarly for  $v$

$$\boxed{\frac{\partial v}{\partial p} = -\frac{R}{f p} \frac{\partial T}{\partial x}} \quad (39)$$

(Note that in the last two equations, horizontal gradients are taken at  $p$  constant in the  $(x, y, p)$  coordinate system, so pressure can be taken out of the horizontal partial derivatives).

The vertical gradient of geostrophic wind is referred to as thermal wind. The thermal wind balance thus states that a horizontal gradient of temperature corresponds to a vertical gradient of horizontal wind. The thermal wind blows parallel to isotherms, with warm air on its right in the northern hemisphere. This is consistent with the jet stream in the atmosphere between the warm tropical air mass and the cold polar air mass, blowing eastward.

#### A.4.2 Slope of a front\*

A consequence of the thermal wind balance is that atmospheric fronts are slanted. Indeed, if we suppose that a front between a warm sector and a cold sector is vertical (figure 4a), then the iso-u lines are vertical, which contradicts the thermal wind balance. A vertical front is therefore not possible, since a horizontal gradient of temperature is associated with a vertical gradient of u.

Thus an atmospheric front is slanted, with a slope consistent with the thermal wind balance (figure 4b). In order to determine the slope, we perform a scale analysis of the thermal wind balance :

$$\frac{\partial u}{\partial \ln p} = \frac{R}{f} \frac{\partial T}{\partial y}. \quad (40)$$

From hydrostatic balance and the ideal gas law  $p = \rho RT$ ,

$$dp = -\rho g dz = -\frac{p}{RT} g dz \Rightarrow d(\ln p) = -\frac{g}{RT} dz. \quad (41)$$

Thus

$$\frac{\partial u}{\partial z} = -\frac{g}{RT} \frac{R}{f} \frac{\partial T}{\partial y} \Rightarrow \frac{\Delta z}{\Delta y} \sim -\frac{Tf}{g} \frac{\Delta u}{\Delta T} \quad (42)$$

where  $\Delta$  denotes change across the front. With typical values  $(T, f, g, \Delta u, \Delta T) \sim (280 \text{ K}, 10^{-4} \text{ s}^{-1}, 10 \text{ m s}^{-2}, 30 \text{ m s}^{-1}, 10 \text{ K})$ , we obtain a slope of

$$\frac{\Delta z}{\Delta y} \sim -\frac{1}{120}. \quad (43)$$

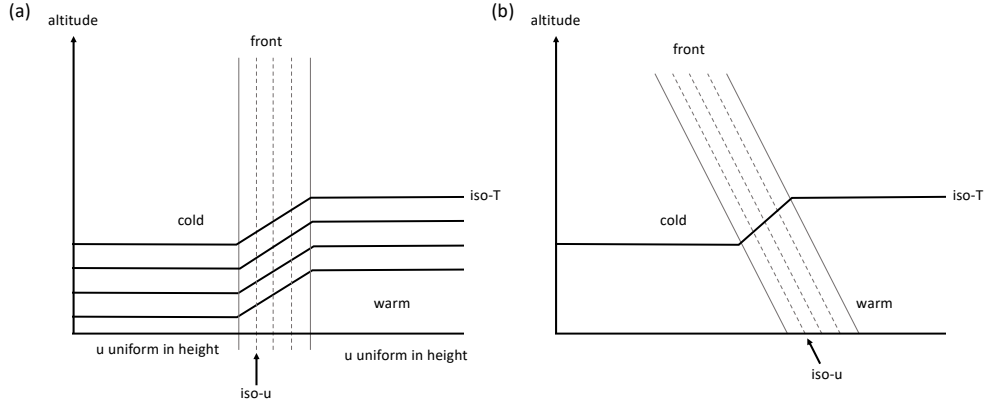


Figure 4: Slope of a front: A front can not be vertical (a) since the thermal wind implies that the horizontal gradient of temperature is associated with a vertical gradient of u. A front is thus necessarily slanted (b).

### A.5 Vorticity and Rossby waves

An important notion in geophysical fluids, and for fluids in general, is vorticity. It is a measure of the local rotation of the fluid. We will see that the conservation of absolute vorticity, defined as the sum of relative vorticity  $\xi_r = \partial v / \partial x - \partial u / \partial y$  and of planetary vorticity  $f$ , along with the variation of the Coriolis parameter  $f$  with latitude, gives rise to waves known as Rossby waves. One reference which may be useful for Rossby waves, is the MIT open courseware - see the link to the lecture notes on the course website

[http://www.lmd.ens.fr/muller/TEACHING\\_METEO/meteo.html](http://www.lmd.ens.fr/muller/TEACHING_METEO/meteo.html)

For simplicity, we place ourselves in the 2D case with horizontal motion only (i.e. there is no vertical motion:  $\partial / \partial z = w = 0$ ), and we assume a constant density fluid  $\rho = \rho_0$  constant. In that case, the equations

of motion (1-4) become:

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} \quad (44)$$

$$\frac{Dv}{Dt} + fu = -\frac{1}{\rho_0} \frac{\partial p}{\partial y} \quad (45)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \quad (46)$$

Recall that  $D/Dt$  denotes the Lagrangian derivative  $D/Dt = \partial/\partial t + u\partial/\partial x + v\partial/\partial y$ .

It is easily shown that this system of equations has an invariant, namely the absolute vorticity

$$\xi_a = \xi_r + f. \quad (47)$$

In other words,

$$\frac{D\xi_a}{Dt} = \frac{\partial\xi_r}{\partial t} + \mathbf{u} \cdot \nabla(\xi_r + f) = 0. \quad (48)$$

Physically, one can show that conservation of vorticity implies that an initial perturbation gives rise to a wavy pattern, with a phase propagating westward compared to the mean flow (will be discussed in more detail in class).

Mathematically, we can investigate the properties of those waves by linearizing the above equations around a reference state with mean zonal flow  $\bar{u}$ . We write the velocities and vorticity as the sum of this reference flow plus a perturbation (denoted with prime):  $u = \bar{u} + u'$ ,  $v = v'$ ,  $\xi_r = \xi_r'$ . The perturbation terms are supposed to be small compared to the mean state.

Furthermore, from equation (??), the perturbation flow is non divergent:

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0. \quad (49)$$

Therefore we can define a streamfunction  $\psi$  such that

$$u' = -\frac{\partial\psi}{\partial y} \text{ and } v' = \frac{\partial\psi}{\partial x}. \quad (50)$$

The system of equations can be transformed into one equation for this variable:

$$\left( \frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \Delta\psi + \beta \frac{\partial\psi}{\partial x} = 0, \quad (51)$$

where  $\beta = df/dy$  and  $\Delta$  denotes the Laplacian operator.

As is standard when studying waves, we look for a solution in the form of a propagating wave:

$$\psi(x, y, t) = A \exp(i(kx + ly - \omega t)).$$

Plugging into the above equation yields the following dispersion relationship (relationship between the frequency of the waves  $\omega$  and the horizontal wavenumbers  $(k, l)$ ):

$$\omega = \bar{u}k - \frac{\beta k}{k^2 + l^2}. \quad (52)$$

The phase speed for waves  $c_p = (c_{p,x}, c_{p,y})$ , which is the speed of wave crests and troughs, is given by

$$c_{p,x} = \frac{\omega}{k} = \bar{u} - \frac{\beta}{k^2 + l^2}; c_{p,y} = \frac{\omega}{l}. \quad (53)$$

We see that the phase of those waves propagates to the west compared to the mean flow. Also, the phase speed is a function of the wavenumbers  $k$  and  $l$ , so those waves are dispersive.

The group velocity for waves  $c_g = (c_{g,x}, c_{g,y})$ , which is the speed of wave energy, is given by

$$c_{g,x} = \frac{\partial \omega}{\partial k} = \bar{u} + \frac{\beta(k^2 - l^2)}{(k^2 + l^2)^2}; c_{g,y} = \frac{\partial \omega}{\partial l}. \quad (54)$$

Unlike the phase velocity, the group velocity can go eastward or westward, depending on the wavelengths.

For instance, for waves with zonal wavelengths  $k^{-1} \approx 6000$  km and  $l^{-1} \approx 3000$  km,  $c_{g,x} - \bar{u} \approx 6$  m s<sup>-1</sup>.

Note that for stationary waves,  $\omega = 0$  implies

$$\kappa = \sqrt{k^2 + l^2} = \sqrt{\frac{\beta}{\bar{u}}}$$

yielding a typical wavelength of  $2\pi/\kappa \approx 7000$  km. This is comparable to observed standing waves in the large-scale midlatitude dynamics (meanders of the midlatitude atmospheric jet).

## B Atmospheric moisture and moist convection

### B.1 Fundamental aspects of clouds

See online slides of introduction :

[http://www.lmd.ens.fr/muller/TEACHING\\_CLOUDS/Cours1-intros.pdf](http://www.lmd.ens.fr/muller/TEACHING_CLOUDS/Cours1-intros.pdf)

And slides on cloud types :

[http://www.lmd.ens.fr/muller/TEACHING\\_CLOUDS/Cours3-CloudTypes.pdf](http://www.lmd.ens.fr/muller/TEACHING_CLOUDS/Cours3-CloudTypes.pdf)

### B.2 Dry thermodynamics

#### B.2.1 First law of thermodynamics

See online thermodynamics notes :

[http://www.lmd.ens.fr/muller/TEACHING\\_CLOUDS/Cours3\\_thermo.pdf](http://www.lmd.ens.fr/muller/TEACHING_CLOUDS/Cours3_thermo.pdf)

$$\Rightarrow \boxed{dU = \delta W + \delta Q} \quad (55)$$

An example of work (crucial in atmospheric sciences): The work by pressure forces is given by  $\delta W = -pdV$  where  $V$  is the volume of the fluid. Note that this expression requires the change of volume to be done slowly enough (for instance by moving a piston slowly) that the pressure remains in equilibrium at the piston surface. In other words, the speed of the movement needed to be much slower than the speed of molecules, whose order of magnitude is the speed of sound  $\approx 300 \text{ m s}^{-1}$ . Such a transformation is called *quasi-static* (see Bohrn & Albrecht Chp1 for a more detailed discussion).

#### B.2.2 Ideal gas law

We consider an ideal gas of volume  $V$  with  $N$  molecules. Then

$$PV = NkT$$

where  $k$  is Boltzmann's constant ( $\approx 1.38 \times 10^{-23} \text{ J K}^{-1}$ ).

We rewrite in specific form, introducing the molecular mass  $m$  and the specific gas constant  $R = k/m$ :

$$p = \frac{Nm}{V} \frac{k}{m} T = \rho RT.$$

This is the ideal gas law in specific form :

$$\boxed{p = \rho RT \Leftrightarrow p\alpha = RT}, \quad (56)$$

where  $\alpha = 1/\rho$  denotes specific volume. For dry air,  $R \approx 286 \text{ J K}^{-1} \text{ kg}^{-1}$ .

Note that due to its dependence on the molecular mass of the gas, the specific gas constant  $R$  depends on the gas considered.

$\Rightarrow$  The pressure is related to the change of the momentum of molecules by collision. It is thus related to the molecular kinetic energy expressed in temperature.

Remark There is another formulation of the ideal gas law, in terms of moles:

$$pV = \frac{N}{N_A} (N_A k) T = \mathcal{N} R_* T$$

where  $\mathcal{N}$  is the number of moles,  $R_*$  the universal ideal gas constant ( $R_* \approx 8,3 \text{ J mol}^{-1} \text{ K}^{-1}$ ).

### B.2.3 Mixture of gases : Dalton's law

Dalton's law: The pressure of a mixture of ideal gases is the sum of the partial pressures of each ideal gas which would occupy the same volume  $V$ :  $p = p_1 + p_2 + \dots$ , with  $p_i V = N_i kT$  where  $N_i$  is the number of molecules of gas  $i$ .

The ideal gas law becomes  $p = \sum p_i = \sum N_i kT/V = NkT/V \Rightarrow pV = NkT$  where  $N = \sum N_i$  is the total number of molecules in the mixture of gases. In other words, the ideal gas law is unchanged for a mixture of ideal gases, with  $N$  the total number of molecules.

**Remark:** One implication of Dalton's law is that for a volume  $V$  at a given temperature  $T$  and pressure  $p$ , any ideal gas or mixture of ideal gases contains the same number of molecules  $N = pV/(kT)$ . Notably, if we compare a volume  $V$  at  $T, p$  of dry air the same volume of moist air (the latter containing water vapor molecules in addition to dry air molecules), the moist air is lighter than dry air. This is because the molecular mass of  $H_2O$  is smaller than either  $O_2$  or  $N_2$ . The number of molecules being the same by Dalton's law, some of the dry air molecules have been replaced by water vapor molecules, which are lighter. The fact that air containing water vapor is lighter than dry air is called the *virtual effect*.

**Specific form (per unit mass):** We introduce the density of gas  $i$  with molecular mass  $m_i$  and  $N_i$  molecules:  $\rho_i = N_i m_i/V$ . The total density is  $\rho = \sum_i \rho_i$ , the total number of molecules is  $N = \sum_i N_i$ . Then Dalton's law can be written

$$p = \frac{\sum_i N_i kT}{V} = \left( \frac{\sum_j N_j m_j}{V} \right) \frac{k}{\frac{\sum_k N_k m_k}{\sum_i N_i}} T$$

Thus for a mixture of gases:

$$\Rightarrow p = \sum_j \rho_j \frac{k}{\langle m \rangle} T \Rightarrow \boxed{p = \rho R T}, \quad (57)$$

where  $p = \sum_i p_i$ ,  $\rho = \sum_i \rho_i$ ,  $R = k/\langle m \rangle$ , and  $\langle m \rangle = \sum N_i m_i/N$  is the mean molecular mass.

### B.2.4 Joule's law, enthalpy

Note that internal energy is a state variable, hence a function of two variables  $U = U(T, V) = U(T, p)$ . Joule's law for an ideal gas states that

$$\frac{\partial U}{\partial V} \Big|_{T=const} = 0.$$

Thus  $\boxed{dU = C_v dT}$  for an ideal gas, where  $C_v$  is the heat capacity at constant volume.

- (1) At constant volume, adding or removing heat to/from an ideal gas yields a change of internal energy:  
Indeed from the first law,

$$\frac{dU}{dt} = C_v \frac{dT}{dt} = \dot{Q} + \dot{W} = \dot{Q} - p \frac{dV}{dt}$$

Thus at constant volume

$$\dot{Q} = \frac{dU}{dt} = C_v \frac{dT}{dt}$$

The larger  $C_v$ , the smaller the temperature response to heat added or removed.

- (2) At constant pressure, adding or removing heat to/from an ideal gas yields a change of enthalpy  
 $\boxed{H = U + pV = C_p T}$ ;

Indeed from the first law,

$$\frac{dU}{dt} = \dot{Q} + \dot{W} = \dot{Q} - p \frac{dV}{dt} \Rightarrow \frac{dU}{dt} + \frac{d(pV)}{dt} = \frac{dH}{dt} = \dot{Q} + V \frac{dp}{dt}$$

Note that  $H = U + pV = C_v T + NkT = C_p T$  where  $C_p = C_v + Nk$  is that heat capacity at constant pressure.

Thus at constant pressure

$$\dot{Q} = \frac{dH}{dt} = C_p \frac{dT}{dt}.$$

### B.2.5 Summary of equations in specific form:

In atmospheric science, we express thermodynamic quantities in specific form, i.e. per unit mass  $M = N \times m$  (as usual,  $N$  is the number of molecules and  $m$  the molecular mass).

Recall also that  $\alpha$  denotes specific volume  $1/\rho$  and we introduce

$$c_v = \frac{C_v}{M}, c_p = \frac{C_p}{M}, u = \frac{U}{M}, h = \frac{H}{M} \text{ and } \delta q = \frac{\delta Q}{M}.$$

Then the equations become:

$$p = \rho RT \Leftrightarrow p\alpha = RT : \text{Ideal gas law,} \quad (58)$$

$$\delta q = du + p d\alpha = dh - \alpha dp, \text{ with } h = u + p\alpha : \text{First law of thermodynamics,} \quad (59)$$

$$du = c_v dT \text{ and } dh = c_p dT, \text{ with } c_p = c_v + R : \text{Joule's law,} \quad (60)$$

Note that those specific variables are NOT extensive variables. One needs to multiply by the mass to recover extensive variables. In other words for a mixture of gases with masses  $M_1$  and  $M_2$ , and specific internal energies  $u_1$  and  $u_2$ , the total internal energy of the mixture is  $U = M_1 u_1 + M_2 u_2$  and the total specific internal energy is  $u = (M_1 u_1 + M_2 u_2)/(M_1 + M_2)$ .

## B.3 Dry convection: application to the atmosphere

Cloud formation is closely related to the convective movement of air. Thus a key question is what makes the air move. Note that although temperature decreases with height, the cold air aloft is not heavier and thus does not “fall” to the ground, as its density is not only a function of temperature, but also of pressure  $\rho(T, p)$ . The three are related through the ideal gas law

$$p = \rho RT$$

where  $R$  denotes the specific constant of the gas.

To determine the stability of air, we thus need to account for changes in  $T$  and  $p$  with height. We use the so-called “parcel method” to assess whether a parcel of air is unstable to upward motion. We consider a hypothetical parcel of air near the surface displaced vertically adiabatically, and ask the following question: if this parcel of air is displaced upwards, will it return to its original position, or will it keep rising? If the displaced parcel has lower density than the environment, it is lighter and will keep rising: the atmosphere is unstable to dry convection. If instead its density is larger, it is heavier and will accelerate back down: the atmosphere is stable to dry convection. The displacement is supposed to be slow enough that the pressure of the parcel is always in equilibrium with the pressure of the environment (“quasi-static” displacement; see § II.1).

### B.3.1 Potential temperature $\theta$ , dry static energy, dry adiabatic lapse rate

#### - Conservation of potential temperature

We now show that during the quasi-static adiabatic parcel displacement, there is an invariant called *potential temperature*

$$\boxed{\theta \stackrel{\text{def}}{=} T(p/p_0)^{-R/c_p}}, \quad (61)$$

where  $T$  denotes temperature,  $p$  pressure,  $p_0$  a reference pressure (typically 1000 hPa), and  $c_p$  heat capacity at constant pressure ( $c_p \approx 1006 \text{ J kg}^{-1} \text{ K}^{-1}$  for dry air). Then we will use this invariant to determine the condition under which the atmosphere is unstable to dry convection.



Applying the first law of thermodynamics to an infinitesimal displacement, the change in internal energy  $c_v dT$  is equal to the heat added, which is zero in this adiabatic displacement, plus the work done on the parcel, in that case due to pressure forces  $\delta W = -pd(1/\rho)$ :

$$c_v dT = -pd \left( \frac{1}{\rho} \right). \quad (62)$$

Using the ideal gas law  $p = \rho RT$  and recalling that  $c_v + R = c_p$ , we obtain

$$c_v dT = -p d \left( \frac{RT}{p} \right) = -RdT + RT \frac{dp}{p} \Leftrightarrow c_p dT - RT \frac{dp}{p} = 0 \quad (63)$$

implying

$$\frac{dT}{T} - \frac{R}{c_p} \frac{dp}{p} = d \ln(Tp^{-R/c_p}) = 0. \quad (64)$$

This shows that  $Tp^{-R/c_p}$  is constant, hence  $\theta = T(p/p_0)^{-R/c_p}$  is conserved during the displacement.

### - Conservation of dry static energy and dry adiabatic lapse rate

Before assessing the stability of the parcel, we note that in the special case of a hydrostatic atmosphere, i.e. assuming  $dp = -\rho g dz$ , the variable

$$h_{dry} \stackrel{\text{def}}{=} c_p T + gz \quad (65)$$

is conserved. It is called the *dry static energy*. Indeed if we make this hydrostatic approximation, equation (64) becomes

$$c_p dT + RT \rho g \frac{dz}{p} = d(c_p T + gz) = dh_{dry} = 0, \quad (66)$$

where we have again used the ideal gas law  $p = \rho RT$ . Thus the dry static energy  $c_p T + gz$  is conserved in an adiabatic quasi-static displacement under the hydrostatic approximation.

Furthermore in that case, it is readily seen from equation (66) that the dry adiabatic lapse rate  $\Gamma_d$ , defined as the decrease of temperature with height, is given by

$$\Gamma_d \stackrel{\text{def}}{=} -\frac{dT}{dz} = \frac{g}{c_p} \approx 10^\circ \text{ K / km}. \quad (67)$$

$\Rightarrow$  A dry adiabat loses 1 degree every 100m.

### B.3.2 Stability to dry convection, Brunt-Väisälä frequency

#### - How can we assess the stability of dry air?

We now return to our original question, namely is the displaced parcel lighter or heavier than its environment. Recall that the pressure of the parcel is equal to that of the environment, so that during the displacement its pressure changes following the environmental pressure, while its temperature changes such that the potential temperature is conserved. To fix ideas, let's raise a parcel upwards (a similar argument can be made for downward displacements). Figure 5 schematically describes the three possible cases, depending on the environmental potential temperature profile  $\bar{\theta}$ , shown in red as a function of height.

The displaced parcel starts with the near-surface environmental value  $\theta_p = \bar{\theta}$ , and conserves its potential temperature  $\theta_p$  during its adiabatic ascent (blue). Once displaced, comparing  $\theta_p$  with  $\bar{\theta}$  at the displaced pressure level determines whether the parcel is heavier or lighter than the environment. If  $\bar{\theta}$  increases with height (left panel), the displaced parcel has colder potential temperature than the environment  $\theta_p < \bar{\theta}$ , and thus is heavier and accelerates back down: the atmosphere is stable to dry convection. On the other hand, if  $\bar{\theta}$  decreases with height (right panel), the displaced parcel has warmer potential temperature than the

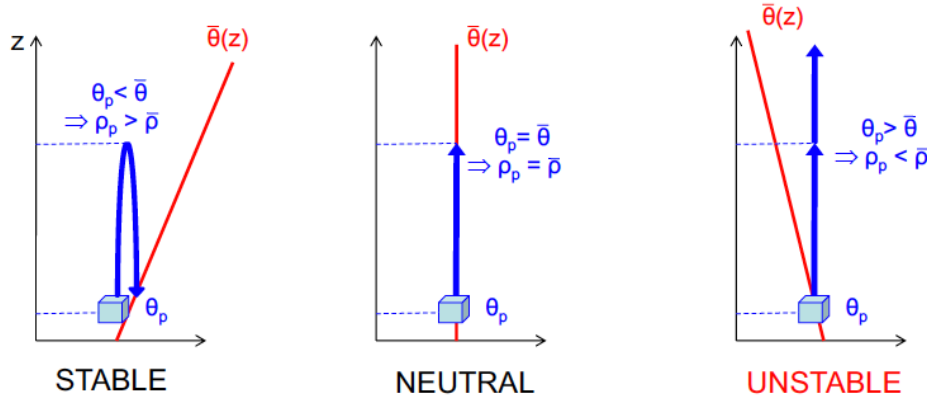


Figure 5: Schematic representation of the parcel method.

environment  $\theta_p > \bar{\theta}$ , and thus is lighter and keeps rising: the atmosphere is unstable to dry convection. If  $\bar{\theta}$  is constant with height (middle panel), the displaced parcel is neither accelerated downwards nor upwards: the atmosphere is neutral to dry convection.

In the stable case, the displaced parcel will accelerate back towards and passed its original equilibrium altitude. It will then be lighter than the environment and accelerate back up towards its equilibrium altitude. Thus the displaced parcel oscillates around its equilibrium height. The frequency of oscillation is known as the buoyancy frequency, or Brunt Väisälä frequency, which we will show is given by:

$$N = \sqrt{\frac{g}{\bar{\theta}} \frac{\partial \bar{\theta}}{\partial z}}.$$

In the unstable case, convection is very efficient at removing the instability. Indeed, in the planetary subcloud layer (first kilometer of the atmosphere or so), the convective adjustment time scale is very fast (minutes for dry convection) compared to destabilizing factors (surface warming, atmospheric radiative cooling...). Thus the observed state is very close to convective neutrality  $\bar{\theta} = \text{constant}$ , see for instance the review [Stevens “Atmospheric Moist Convection” 2005] and references therein.

But above this thin boundary layer, this observation does not hold anymore and  $\bar{\theta}$  is observed to increase with height. This is because above the subcloud layer, atmospheric convection involves phase change of water vapor. We need to revisit the above calculations to account for the significant latent heat released during the phase changes of water. This will be done in §V.

## - Hydrostatic case

### Exercise.

In this exercise, we want to derive conditions that determine whether an atmosphere is stable to dry convection. We will use the parcel method.

Consider the quasi-static adiabatic displacement of a parcel of dry air in an environment with density  $\rho$  and temperature  $T$ .

- The density and temperature of the parcel are denoted  $\rho'$  and  $T'$ . Give the vertical acceleration  $d^2 z' / dt^2$  of the parcel as a function of  $T$  and  $T'$ .
- Now suppose that the hydrostatic approximation holds. We saw in class that, in that case, the parcel follows the dry adiabatic lapse rate  $dT' / dz = -\Gamma_d$  during its displacement, where  $\Gamma_d = g / c_p$ . We displace the parcel from its equilibrium position  $z_0$  with temperature  $T = T' = T_0$  to a new altitude  $z_0 + z'$ . We suppose the displacement small (see figure below).

Show that the vertical acceleration of the parcel solves a differential equation of the form:

$$\frac{d^2 z'}{dt^2} + N^2 z' = 0,$$

where  $N^2$  will be expressed in terms of  $\Gamma_d$ ,  $T_0$ , and of the environmental lapse rate  $\Gamma = -dT/dz$  assumed constant (which is a reasonable approximation for a small displacement).

- (c) We now solve the particular case where  $N^2 > 0$ . We denote the initial displacement  $z'(t = 0) = z'_0$  and suppose zero initial velocity  $dz'/dt(t = 0) = 0$ . Solve for  $z'(t)$ .
- (d) More generally, discuss qualitatively the temporal evolution of the displacement  $z'(t)$  of the parcel as a function of  $N^2$ .

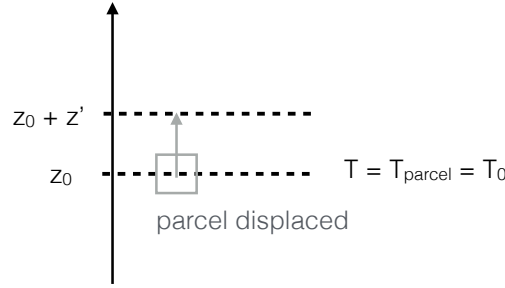


Figure 6: Schematic of parcel method

- (a) The environment has temperature  $T$ , density  $\rho$ , pressure  $p$ .  
 The parcel has temperature  $T'$  and density  $\rho'$ . By definition, a quasi-static displacement is such that the parcel pressure is in equilibrium with the environment  $p = p'$ .  
 By Archimedes' principle, the acceleration of the parcel is

$$\rho' \frac{d^2 z'}{dt^2} = (\rho - \rho')g,$$

with  $p = \rho RT = \rho' RT'$  (ideal gas law).

Thus

$$\frac{d^2 z'}{dt^2} = \frac{\rho - \rho'}{\rho'} g = \frac{p/RT - p/RT'}{p/RT'} g = \frac{T' - T}{T} g$$

- (b) We suppose hydrostatic balance and displace the parcel from  $z_0$  ( $T=T'=T_0$ ) to  $z_0 + z'$  (small displacement). The displaced parcel has vertical acceleration from (a) given by

$$\frac{d^2 z'}{dt^2} = \frac{T'(z_0 + z') - T(z_0 + z')}{T(z_0 + z')} g.$$

Now the parcel follows a dry adiabatic lapse rate  $\Gamma_d = g/c_p$  during its displacement, thus

$$T'(z_0 + z') = T'(z_0) - \Gamma_d z' = T_0 - \Gamma_d z'.$$

Similarly, if we denote  $\Gamma = -dT/dz$  the environmental lapse rate, then the temperature of the environment at  $z_0 + z'$  is

$$T(z_0 + z') = T(z_0) - \Gamma z' = T_0 - \Gamma z'.$$

Thus

$$\frac{d^2 z'}{dt^2} = \frac{(T_0 - \Gamma_d z') - (T_0 - \Gamma z')}{T_0 - \Gamma z'} g = \frac{\Gamma - \Gamma_d}{T_0 - \Gamma z'} g z' \approx \frac{\Gamma - \Gamma_d}{T_0} g z',$$

where in the last approximation, we have used the fact that the displacement is small, thus  $\Gamma z' \ll T_0$  is neglected.

$$\Rightarrow \boxed{\frac{d^2 z'}{dt^2} + N^2 z' = 0, \text{ where } N^2 = g \frac{\Gamma_d - \Gamma}{T_0}}$$

(c) In the particular case where  $N^2 > 0$ , the solutions of this equation are given by  $z'(t) = A \cos(Nt) + B \sin(Nt)$ , where  $A$  and  $B$  are constants. From initial conditions  $z'(t=0) = z'_0 \Rightarrow A = z'_0$  and  $dz'/dt = -AN \sin(Nt) + BN \cos(Nt) \Rightarrow B = 0$ . Thus  $z'(t) = z'_0 \cos(Nt)$ . The displacement oscillates with frequency  $N$  and amplitude  $z'_0$ .

(d) More generally, the temporal evolution of  $z'(t)$  depends on the sign of  $N^2$ . In other words, the stability of a temperature profile  $T(z)$  to dry convection under the hydrostatic approximation depends on the sign of  $N^2 = g \frac{\Gamma_d - \Gamma}{T_0}$ .  $\Gamma$  is the environmental lapse rate ( $= -dT/dz$ ) and  $\Gamma_d$  is the dry adiabatic lapse rate  $g/c_p$ .  $N$  is called the Brunt-Väisälä frequency.

If  $N^2 > 0$ , i.e. if  $\Gamma_d > \Gamma$ , then  $N$  is real and  $z'(t) \propto \cos(Nt), \sin(Nt)$ . The parcel oscillates with frequency  $N$ , Brunt-Väisälä frequency (sometimes also called buoyancy frequency). The atmosphere is stable to dry convection.

If  $N^2 < 0$ , i.e. if  $\Gamma_d < \Gamma$ , then  $N$  is imaginary and  $z'(t) \propto \exp(|N|t)$ . There is an exponentially growing solution  $z'(t) \propto \exp(|N|t)$  and the atmosphere is unstable to dry convection.

## - General case

We now consider the general case, i.e. we do not assume hydrostasy anymore. We will show that we can still derive a stability criterion. We know from the first law of thermodynamics that the parcel conserves its potential temperature  $\theta_{parcel} = T_{parcel}(p_{parcel}/p_0)^{-R/c_p}$  during the displacement. Thus  $\theta_{parcel}(z_0) = \theta_{parcel}(z' + z_0)$ . We denote the environmental potential temperature  $\theta$  and density  $\rho$ .

From Archimedes principle, the vertical acceleration of the parcel displaced from its equilibrium position  $z_0$  to the new position  $z' + z_0$  is

$$\frac{d^2 z'}{dt^2} = g \frac{\rho - \rho_{parcel}}{\rho_{parcel}} = g \frac{p/T - p/T_{parcel}}{p/T_{parcel}} = g \frac{T_{parcel} - T}{T} = g \frac{\theta_{parcel} - \theta}{\theta}$$

where we have used the ideal gas law and the fact that the parcel pressure is equal to that of the environment, and where all the quantities are evaluated at  $(z_0 + z')$ .

Now  $\theta_{parcel}(z_0 + z') = \theta_{parcel}(z_0) = \theta(z_0)$ , and  $\theta(z_0 + z') \approx \theta(z_0) + (\partial\theta/\partial z)z'$ . Thus

$$\frac{d^2 z'}{dt^2} = g \frac{\theta(z_0) - (\theta(z_0) + (\partial\theta/\partial z)z')}{\theta(z_0) + (\partial\theta/\partial z)z'} \approx g \frac{-(\partial\theta/\partial z)z'}{\theta(z_0)},$$

where in the denominator we have neglected the  $z'$  contribution since we are assuming a small displacement, thus this term is small.

Therefore

$$\frac{d^2 z'}{dt^2} + \frac{g}{\theta} \frac{\partial\theta}{\partial z} z' = \boxed{\frac{d^2 z'}{dt^2} + N^2 z' = 0},$$

with

$$\boxed{N^2 = \frac{g}{\theta} \frac{\partial\theta}{\partial z}}.$$

Similar to the previous, hydrostatic case, the atmosphere is stable if  $N^2 > 0$  i.e.  $\theta$  increases with  $z$ . Conversely, the atmosphere is unstable to dry convection if  $N^2 < 0$  i.e.  $\theta$  decreases with  $z$ .

## - Link between the 2

Note that we can recover the hydrostatic case from

$$N^2 = \frac{g}{\theta} \frac{\partial \theta}{\partial z} :$$

Indeed, assuming hydrostasy,

$$\frac{1}{\theta} \frac{\partial \theta}{\partial z} = \frac{1}{T} \frac{\partial T}{\partial z} - \frac{R}{c_p p} \frac{\partial p}{\partial z} = \frac{1}{T} \left( \frac{\partial T}{\partial z} - \frac{RT}{c_p p} (-\rho g) \right) = \frac{1}{T} \left( \frac{\partial T}{\partial z} + \frac{g}{c_p} \right) = \frac{1}{T} (\Gamma_d - \Gamma)$$

where we have used the ideal gas law  $p = \rho RT$ . Multiplyig by  $g$ , we indeed recover the hydrostatic Brunt-Väisälä frequency.

## - Remark

As we saw last time, the typical time scale for dry convection in the atmosphere  $\tau_N = 2\pi/N$  is of the order of 10 minutes.

To give some orders of magnitude, if for example we consider wind blowing on mountains with typical horizontal spacing of 10 km, a temperature around 20°C, and a lapse rate of 5° km<sup>-1</sup>, what value of wind speed yields a frequency that matches the frequency of gravity waves ?

⇒ With wind speed  $U$  over topography with length scale  $L$ , the time scale between two mountains is  $\tau_{topog} = L/U$ . The stratification time scale is  $\tau_N = 2\pi/N$ , with  $N = \sqrt{g(\Gamma_d - \Gamma)/T} \approx \sqrt{10 * (10 - 5) * 10^{-3}/293}$  s<sup>-1</sup>  $\approx 1.3 * 10^{-2}$  s<sup>-1</sup>. Thus  $\tau_N = 2\pi/N \approx 480$  s  $\approx 8$  min.

Both times scales will resonate when

$$\frac{L}{U} = \frac{2\pi}{N} \Rightarrow U = \frac{L}{2\pi} N = \frac{L}{\tau_N} \approx \frac{10^4}{480} \text{ m s}^{-1} \approx 20 \text{ m s}^{-1}.$$

### B.3.3 Centrifugal convection\*

We now investigate the stability of a rotating fluid to *horizontal* displacements. As we will discuss in more details later, the instability to vertical displacements discussed in the previous section is relevant to the tropics where the Coriolis parameter is small. But in midlatitudes where the planetary rotation is important, slantwise convection (with a displacement having both vertical and horizontal components) is relevant as well.

#### - Centrifugal convection: case of a rotating fluid in cylinder

We first focus on the simple case of a fluid in rotation around the vertical axis  $z$  and investigate the stability to horizontal displacements. We will apply the parcel method to a ring of fluid (in dark blue figure 7).

The Navier-Stokes equations in cylindrical coordinates, neglecting viscous effects, assuming no variation along the angle  $\theta$  :  $\partial/\partial\theta = 0$ , and assuming constant density  $\alpha = 1/\rho = \text{constant}$ , yield:

$$\frac{du}{dt} = -\alpha \frac{\partial p}{\partial r} + \frac{v^2}{r} = -\alpha \frac{\partial p}{\partial r} + \frac{M^2}{r^3} \quad (68)$$

$$\frac{dM}{dt} = -\alpha \frac{\partial p}{\partial \theta} = 0, \text{ where } M = rv. \quad (69)$$

From the second equation, we see that for an inviscid, symmetric motion ( $\partial/\partial\theta = 0$  i.e. symmetry around axis of rotation),  $M$  is conserved.

We assume that the fluid is in equilibrium :

$$\alpha \frac{\partial p}{\partial r} = \frac{M^2}{r^3}.$$

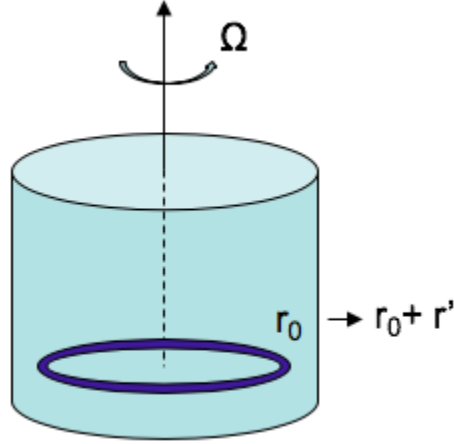


Figure 7: The ring of fluid in dark blue is displaced horizontally from  $r_0$  to  $r_0 + r'$ .

In other words, there is an equilibrium between pressure forces and centrifugal forces. We then consider the fluid ring of figure 7 in equilibrium with the environment at  $r_0$ , and displace it horizontally from  $r_0$  to  $r_0 + r'$ . As before, we use the parcel method and thus suppose that the displacement does not impact the environmental pressure field, and that the parcel pressure is equal to that of the environment  $p_{parcel} = p$ .

Then the equations of motion for the parcel at  $r = r_0 + r'$  are:

$$\frac{du_{parcel}}{dt} = \frac{d^2 r'}{dt^2} = -\alpha \frac{\partial p_{parcel}}{\partial r}(r) + \frac{M_{parcel}^2(r)}{r^3}.$$

But  $\alpha \partial p_{parcel} / \partial r = \alpha \partial p / \partial r = M^2(r) / r^3$  and  $M$  is conserved thus  $M_{parcel}(r) = M_{parcel}(r_0) = M(r_0)$ . Thus the equations of motion for the parcel at  $r = r_0 + r'$  become:

$$\frac{d^2 r'}{dt^2} = -\frac{M^2(r)}{r^3} + \frac{M^2(r_0)}{r^3} = \frac{M^2(r_0) - M^2(r)}{r^3} = \frac{(M(r_0) - M(r))(M(r_0) + M(r))}{r^3} \approx (M(r_0) - M(r)) \frac{2M(r_0)}{r_0^3},$$

where we have used the fact that the displacement is small  $r' \ll r_0$ . We further develop the first term on the right hand side:

$$M(r_0) - M(r) = M(r_0) - M(r_0 + r') \approx -\frac{dM}{dr} r'.$$

Thus

$$\boxed{\frac{d^2 r'}{dt^2} + \frac{2M(r_0)}{r_0^3} \frac{dM}{dr} r' = 0}.$$

As in the previous section for vertical displacement  $z'$ , we see that if  $dM/dr > 0$ , the fluid is stable to horizontal displacements. If  $dM/dr < 0$  on the other hand, the fluid is unstable to horizontal displacements.

Note that physically, this comes from the fact that the displaced fluid conserves its angular momentum  $M_{parcel}$ . If  $dM/dr < 0$ , the new conditions that the displaced parcel encounters correspond to a lower environmental  $M$  and thus by equilibrium to a lower pressure force. Thus its centrifugal force wins and the displaced parcel keeps advancing.

### - Centrifugal convection: atmospheric case

This is the case relevant to oscillations in a fluid in geostrophic equilibrium (equilibrium between pressure force and Coriolis force). We consider the displacement of a tube of fluid parallel to the  $x$ -axis (figure 8) in a fluid in geostrophic equilibrium. We suppose invariance along the  $y$ -axis  $\partial/\partial y = 0$ , and Coriolis parameter  $f = \text{constant}$ . We also suppose constant density thus  $\alpha = \text{constant}$ .

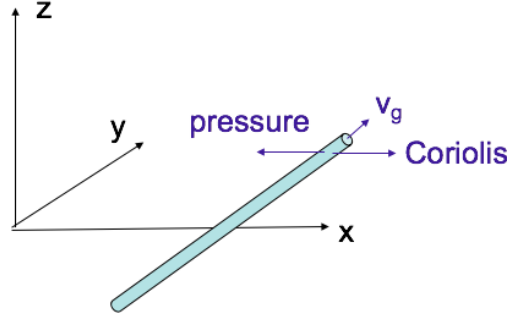


Figure 8: The tube of fluid in blue is displaced horizontally from  $x_0$  to  $x_0 + x'$ .  $v_g$  denotes the geostrophic wind.

The equations of motion for the displaced fluid are

$$\frac{du}{dt} - fv = -\alpha \frac{\partial p}{\partial x} = -fv_g \quad (70)$$

$$\frac{dv}{dt} + fu = -\alpha \frac{\partial p}{\partial y} = 0 \Rightarrow v + fx = \text{constant} = M, \quad (71)$$

where we have used the geostrophic velocity  $v_g$  in equilibrium with the pressure field, and we have denoted  $M$  the conserved quantity  $M = v + fx$  absolute momentum.

We apply the parcel method to the tube of fluid that we displace. As usual, we suppose that the movement does not perturb the pressure field. The equations of motion of the parcel thus yield at  $x = x_0 + x'$ :

$$\frac{du}{dt} - fv = -fv_g \Rightarrow \frac{du}{dt} = \frac{d^2 x'}{dt^2} = f(v - v_g) = f[(v + fx) - (v_g + fx)] = f(M_{\text{parcel}}(x_0 + x') - M_g(x_0 + x')).$$

Thus

$$\frac{d^2 x'}{dt^2} = f(M_g(x_0) - M_g(x_0 + x')) \approx -f \frac{dM_g}{dx} x' \Rightarrow \boxed{\frac{d^2 x'}{dt^2} + f \frac{dM_g}{dx} x' = 0}$$

By analogy with earlier sections, we see that if  $dM_g/dx < 0$ , the fluid is unstable to horizontal displacements. Conversely if  $dM_g/dx > 0$ , the fluid is stable. This instability is sometimes called inertial instability (in the stable case, the oscillations are called inertial oscillations).

#### B.3.4 Symmetric instability and slantwise convection\*

READING ASSIGNMENT: See Emanuel Chapter 12.

Overview: We suppose the reference state in hydrostatic and geostrophic balance. Then

$$\frac{Dw}{Dt} = B \approx g \frac{T - \bar{T}}{\bar{T}} \quad (72)$$

$$\frac{Du}{Dt} = f(M - \bar{M}), \quad (73)$$

where overbars denote the reference state temperature and absolute momentum. The atmosphere can be stable to horizontal and vertical displacements separately, but unstable to slantwise convection. This instability is known as symmetric instability. It occurs when

$$\frac{\partial \bar{\theta}}{\partial z} \Big|_{\bar{M}} < 0 \text{ or } \frac{\partial \bar{M}}{\partial x} \Big|_{\bar{\theta}} < 0$$

## B.4 Entropy, Second law of thermodynamics and implications for the atmosphere

### B.4.1 Definition and link with $\theta^*$

#### - Definition

Recall that during a reversible displacement of a parcel of air, the first law of thermodynamics yields (using same notations as before):

$$du = \delta Q - pd\alpha = \delta Q - \frac{RT}{\alpha} d\alpha, \quad \text{and from Joule's law } du = c_v dT \quad (74)$$

$$\Rightarrow \frac{\delta Q}{T} = c_v \left( \frac{dT}{T} + \frac{R}{c_v} \frac{d\alpha}{\alpha} \right) = c_v d \left( \ln T \alpha^{R/c_v} \right) \quad (75)$$

$$\Rightarrow \frac{\delta Q}{T} = c_v d \left( \ln T \alpha^{R/c_v} \right) \quad (76)$$

*Remark (notations):* This is also sometimes written  $c_v d \left( \ln T \alpha^{\gamma-1} \right)$ , where  $\gamma = c_p/c_v$  (recall that  $R = c_p - c_v$ ).

In terms of  $T$  and  $p$ :

$$dh = \delta Q + \alpha dp = \delta Q + \frac{RT}{p} dp, \quad \text{and } dh = c_p dT \quad (77)$$

$$\Rightarrow \frac{\delta Q}{T} = c_p \left( \frac{dT}{T} - \frac{R}{c_p} \frac{dp}{p} \right) = c_p d \left( \ln T p^{-R/c_p} \right) \quad (78)$$

$$\Rightarrow \frac{\delta Q}{T} = c_p d \left( \ln T p^{-R/c_p} \right) \quad (79)$$

Remark (notations): Using  $\kappa = R/c_p$ , this can also be written  $c_p d \left( \ln T p^{-\kappa} \right)$ .

Definition of entropy: We see that an invariant appears for reversible adiabatic displacements, which we denote  $s$ :

$$ds = \boxed{c_p d \ln \left( T p^{-R/c_p} \right) = c_v d \ln \left( T \alpha^{R/c_v} \right)} \quad (80)$$

$$\text{or equivalently } s = \boxed{s_0 + c_p \ln \left( \frac{T p^{-R/c_p}}{T_0 p_0^{-R/c_p}} \right) = s_0 + c_v \ln \left( \frac{T \alpha^{R/c_v}}{T_0 \alpha_0^{R/c_v}} \right)}, \quad (81)$$

where  $T_0, p_0$  and  $\alpha_0$  are constant reference temperature, pressure and density respectively.

This variable  $s$  is called entropy. It is a state variable, i.e. it only depends on  $T, p$  or  $T, \alpha$ . Thus during a transformation, the change of entropy  $\Delta s$  only depends on the initial and final states (not on the path between the 2 states).



We just showed that during a reversible transformation (for instance one for which  $\delta W = -pd\alpha$ ), the change of entropy is

$$ds = \frac{\delta Q_{rev}}{T}.$$

In particular, during a reversible, adiabatic displacement (thus  $Q_{rev} = 0$ ),  $s$  is conserved.

### - Atmospheric case: link with $\theta$

We see that

$$ds = c_p \frac{d\theta}{\theta}$$

thus entropy and potential temperature are closely related. This can also be written

$$s = c_p \ln \theta \quad (\text{up to constants}).$$

Adiabats (trajectories with constant  $\theta$ ) are also sometimes called isentropes ( $s$  constant).

### B.4.2 Consequences, entropy maximisation\*

The second law of thermodynamics states that for an isolated system, the change of entropy during a transformation is either 0 if this transformation is reversible, or is strictly positive if this transformation is irreversible.

This law gives us a stability criterion for an isolated system. Indeed, such a system with entropy  $s$  maximum will thus be stable, as it will not spontaneously change state (else its entropy would have to decrease which would contradict the second law).

For more on this, and the atmospheric implications of entropy maximisation, please read chapter 4 of Bohren and Albrecht (reference on the course syllabus), notably section 4.4:

READING ASSIGNMENT: Bohren Albrecht section 4.4.

Remark: For reference, we note the total entropy (not per unit mass)

$$dS = C_v d(\ln TV^{(C_p - C_v)/C_v}) = C_p d(\ln Tp^{-(C_p - C_v)/C_p}).$$

$U$  and  $S$  are extensive, i.e. for a system composed of 2 subsystems of internal energies  $U_1, U_2$  and entropies  $S_1, S_2$ , the total internal energy and entropy are  $U = U_1 + U_2$ ,  $S = S_1 + S_2$ . This is not true of the specific energy and entropy.

Example 1: Instantaneous adiabatic expansion:

If a gaz in a volume  $V_1$  is instantaneously expanded to a volume  $V_2$ , then the work done by pressure forces is  $W = 0$ . If the transformation is adiabatic, we thus have  $\Delta U = W + Q = 0 \Rightarrow \Delta T = 0$ .

Then the entropy change is  $\Delta S = C_v \ln(V_2/V_1)^{(C_p - C_v)/C_v} = Nk \ln(V_2/V_1)$ . The second law implies  $\Delta S \geq 0 \Rightarrow V_2 > V_1$ . The gaz spontaneously occupies a larger volume.

Example 2: Reversible adiabatic expansion:

In that case, the gaz is brought from a container of volume  $V_1$  to volume  $V_2$  by slowly moving a wall of the container. The work of pressure forces is  $\delta W = -pdV \Rightarrow ds = \delta Q/T = 0$ . The entropy is constant in this reversible adiabatic transformation.

Example 3: Entropic derivation of the dry adiabatic lapse rate:

Let's compute the dry adiabatic lapse rate  $\Gamma_d$  with the parcel method. The parcel displacement is supposed adiabatic and reversible, thus isentropic. We have

$$ds = c_p d \ln T - R d \ln p \Rightarrow \frac{ds}{dz} = 0 \Rightarrow \frac{c_p}{T} \frac{dT}{dz} - \frac{R}{p} \frac{dp}{dz} = 0.$$

If we further make the hydrostatic approximation,  $dp/dz = -\rho g$ , and use the ideal gas law  $p = \rho RT$ , we obtain

$$-\frac{dT}{dz} = \Gamma_d = \frac{g}{c_p}.$$

We thus recover the dry adiabatic lapse rate computed earlier.

Example 4: Application of entropy maximisation :  $\theta = \text{constant}$ .

The second law of thermodynamics provides a criterion for stability of an isolated system. Indeed, an isolated system with maximum entropy is stable, i.e. will not change state spontaneously nor under a small perturbation.

One implication is that the potential temperature  $\theta$  is constant with height in the atmospheric boundary layer (from the surface to cloud base, i.e. below the level of condensation of water vapor and concomitant latent heat release).

For more on this, the interested reader is referred to the book by Bohren and Albrecht page 164. We will not cover this in detail, but the main steps are :

- Consider an atmospheric layer between two isobars, of pressure  $p_1$  at height  $z_1$  and pressure  $p_2$  at height  $z_2$ . The atmosphere is supposed in hydrostatic equilibrium.
- First note that its mass  $M$  is constant. Indeed  $M = \int_{z_1}^{z_2} \rho dz = \int_{p_2}^{p_1} \frac{dp}{g} = \frac{1}{g}(p_1 - p_2)$ .
- Second note that the conservation of energy (dry static energy = enthalpy + potential energy) yields  $\int (c_p T + gz)\rho dz = \text{constant}$ .
- Finally note that the entropy of the layer is  $S = \int \rho c_p \ln \theta dz = \frac{c_p}{g} \int_{p_2}^{p_1} \ln \theta dp$ , and investigate the profiles of  $T$  and  $p$  which maximize  $S$  (the optimum is  $\theta(z) = \text{constant}$ ).

## B.5 Moist thermodynamics

### B.5.1 When does water vapor condense into liquid water? The Clausius-Clapeyron equation

What is the effect of moisture on convection? Beyond the virtual effect which will be introduced below, i.e. water vapor making air lighter, an important impact of moisture on convection is the condensation and concomitant latent heat released (we will focus on vapor - liquid phase transition, though all the results below can be extended to the ice phase).

When does water vapor condense into liquid water? The water vapor contained in air will condense when its partial pressure  $e_v$  exceeds a certain value, called the saturation partial pressure  $e_{v,s}$ . The latter is governed by the Clausius-Clapeyron equation, which can be derived using the thermodynamic equilibrium between liquid water and water vapor:

$$\boxed{\frac{de_{v,s}}{dT} = \frac{L_v e_{v,s}}{R_v T^2}}, \quad (82)$$

where  $e_{v,s}$  is the saturation vapor pressure,  $T$  the absolute temperature,  $L_v$  the latent heat of vaporization of water vapor and  $R_v$  the water vapor gas constant. There is net condensation when  $e_v > e_{v,s}(T)$ .

This law predicts that the saturation water vapor pressure strongly increases with temperature. A physical interpretation of this increase can be obtained by considering liquid water with a flat interface, above which water vapor is found with partial pressure  $e_v$ . Saturation corresponds to an equilibrium between evaporation from the liquid water below and condensation of the water vapor above.

- $e_v < e_{v,s}$  means that there is more evaporation than condensation,
- $e_v > e_{v,s}$  means that there is more condensation than evaporation,
- $e_v = e_{v,s}$  means there is as much condensation as there is evaporation.

Molecularly,  $e_{v,s}$  increases with temperature because the evaporation from the liquid phase increases with temperature, i.e. with the mean square velocity of the molecules. Thus the amount of water vapor required to equilibrate the evaporation is larger at larger temperatures.

We note here in passing that this is often phrased “warm air can hold more water vapor than cold air”. This is a useful shortcut to remember that the maximum amount of water vapor  $e_{v,s}$  attainable by a volume of air before it starts to condense, is an increasing function of temperature. But it gives the wrong impression that air is a “sponge” with holes in it, with the number of holes increasing with temperature. The saturation and condensation have nothing to do with “holes” in air, it simply has to do with equilibrium between evaporation and condensation. For a more in-depth discussion of the “sponge theory”, see the book by Bohren and Albrecht (reference on the course website).

The relative humidity measures the distance to saturation, and is defined as

$$RH = \frac{e_v}{e_{v,s}}. \quad (83)$$

It is typically expressed in percent (100% relative humidity corresponds to saturated air).

Remark 1: We just saw that  $e_{v,s}(T)$  only depends on temperature, following the Clausius-Clapeyron equation (82). As noted above,  $L_v$  is the latent heat of vaporization. It is the heat that the outside must provide to transform a unit mass of liquid into vapor. Or equivalently, it is the latent heat released per unit mass of condensation of vapor into liquid. It is given by  $L_v = h_v - h_l$ , difference of the specific enthalpy of water vapor and that of liquid water.

What happens during a phase change then? Let’s consider the liquid to vapor change, with liquid mass  $M_l$  and vapor mass  $M_v$ . The phase change occurs at constant pressure  $p$ , thus the heat  $Q = \Delta H$ , where  $H = M_v h_v + M_l h_l$  is the total enthalpy. Here  $h_v$  and  $h_l$  denote the specific enthalpies of vapor and liquid respectively. The enthalpy change during the phase change is only due to molecular rearrangements. For a mass  $M$  transformed,  $\Delta H = Q_{water} = M(h_v - h_l) = ML_v$ , where  $L_v$  is the heat to be added per unit mass of liquid water to make it evaporate. Note that the surrounding air provides heat to the water, and thus cools  $Q_{air} = -ML_v$ .

Similarly, during condensation, the latent heat released is the enthalpy difference  $ML_v = Q_{air}$ .

Remark 2: Consequence of Clausius-Clapeyron

We will see that  $L_v(T)$  depends on  $T$ , but if we assume that it is constant (a reasonable approximation for reasonable ranges of temperatures), then the Clausius-Clapeyron equation (82) can be solved, yielding

$$e_{v,s}(T) = e_{v,s}(T_0) \exp\left(\frac{L_v}{R_v} \left(\frac{1}{T_0} - \frac{1}{T}\right)\right)$$

In the case of a warming  $T = T_0 + \delta T$ , with  $\delta T \ll T_0$  (as is the case for instance of a warming of a few degrees from a reference temperature  $T_0 \sim 300K$ ), then

$$\frac{1}{T} = \frac{1}{T_0(1 + \delta T/T_0)} \approx \frac{1}{T_0} \left(1 - \frac{\delta T}{T_0}\right)$$

and

$$e_{v,s}(T) = e_{v,s}(T_0) \exp\left(\frac{L_v \delta T}{R_v T_0^2}\right)$$

In other words, the water vapor amount (as measured by the partial pressure) increases quasi-exponentially with warming!

This has implications for the response of the hydrological cycle to climate change. Notably, we expect an intensification of precipitation extremes following Clausius-Clapeyron, at a rate of about 7-8 % K<sup>-1</sup>. The mean precipitation on the other hand is expected to increase at a slower rate, due to energy constraints (see for instance the seminal paper Held & Soden 2006).

**Remark 3:** Similarly, the latent heat of sublimation is given by  $L_s = h_v - h_i$  (ice evaporates), and the latent heat of fusion is given by  $L_f = h_l - h_i$  (ice melts). Note that  $L_f + L_v = L_s$ , i.e. the enthalpic cost (total heat needed) to transform ice into vapor is the same as the cost of this transformation going through the liquid phase.

**Remark 4:** Actually,  $L_v(T)$  depends on temperature. Indeed,

$$\frac{\partial L_v}{\partial T} = \frac{\partial h_v}{\partial T} - \frac{\partial h_l}{\partial T} = c_{p,v} - c_{p,l}.$$

(Note that  $L_v$  is the enthalpy difference between the vapor and liquid phase of water at same saturation pressure  $e_{v,s}$  which depends only on  $T$ , so  $L_v$  is only function of  $T$ :  $\partial L_v / \partial T = dL_v / dT$ ).

Both  $c_{p,v} = 1870 \pm 25 \text{ J kg}^{-1} \text{ K}^{-1}$  and  $c_{p,l} = 4200 \pm 20 \text{ J kg}^{-1} \text{ K}^{-1}$  depend on temperature but weakly :  $\Delta c_{p,v} / c_{p,v}, \Delta c_{p,l} / c_{p,l} \sim 1 - 2\%$ . So we can to a good approximate use constants.

Then

$$L_v = L_{v,0} + (c_{p,v} - c_{p,l})(T - T_0).$$

$L_{v,0} \sim 2,5 \times 10^6 \text{ J kg}^{-1}$  at  $0^\circ\text{C}$ ,  $(c_{p,v} - c_{p,l})(T - T_0) \sim (2 \times 10^3 \text{ J kg}^{-1} \text{ K}^{-1}) (50 \text{ K}) \sim \mathcal{O}(10^5) \text{ J kg}^{-1}$ . Thus  $\Delta L_v / L_v \sim 5\%$  which is small but not always quantitatively negligible.

## B.5.2 Atmospheric thermodynamics: moist variables

READING ASSIGNMENT: Book by Kerry Emanuel, Chp 4.

Commonly used moist variables are:

- Water vapor density (where  $M_v$  denotes the mass in kg of water vapor in the volume  $V$  in  $\text{m}^3$ ):

$$\rho_v \stackrel{\text{def}}{=} \frac{M_v}{V},$$

- Dry air density (where  $M_d$  denotes the mass in kg of dry air in the volume  $V$  in  $\text{m}^3$ ):

$$\rho_d \stackrel{\text{def}}{=} \frac{M_d}{V},$$

- Total air density:

$$\rho = \rho_v + \rho_d,$$

- Specific humidity  $q_v$  (sometimes denoted  $q$ ):

$$q_v \stackrel{\text{def}}{=} \frac{\rho_v}{\rho},$$

- Mixing ratio  $r_v$  (sometimes denoted  $r$ ):

$$r_v \stackrel{\text{def}}{=} \frac{\rho_v}{\rho_d},$$

- Partial pressure of water vapor  $e_v$ , satisfying the ideal gas law (where  $R_v$  denotes specific constant of water vapor, recall that its value depends on the gas considered; here  $R_v = k_B / m_v$  with  $m_v$  molecular mass of water):

$$e_v = \rho_v R_v T,$$

- Partial pressure of dry air  $p_d$ , satisfying the ideal gas law (where  $R_d = k_B / m_d$  denotes specific constant of dry air):

$$p_d = \rho_d R_d T,$$

- The total pressure is then given by Dalton's law:

$$p = p_d + e_v,$$

- Dew point temperature  $T_d$ : Temperature at which a parcel must be cooled at constant pressure to reach saturation ,
- Virtual temperature  $T_v$ : Temperature that dry air would have in order to have the same density as moist air at same pressure.
- Wet bulb temperature  $T_w$ : Temperature that an air parcel would have if liquid water was evaporated at constant pressure until saturation.  
 $T_w$  can be measured with a thermometer covered in water-soaked cloth (wet-bulb thermometer) over which air is passed to make the liquid water evaporate.

Remark Note that  $T_d < T_w < T$ . Indeed, the dew point is obtained by cooling temperature until saturation (i.e.  $e_{v,s}(T)$  is decreased by cooling until the water vapor pressure  $e_v$  is reached  $e_v = e_{v,s}(T_d)$ ). But the wet bulb temperature is obtained not only by cooling (through latent cooling of evaporation) but also by increasing the water vapor amount (through evaporation). Thus the cooling needed to obtain saturation is reduced by the concomitant increased humidity.

EXERCISE:

- Express  $r_v$  as a function of  $q_v$
  - Express  $q_v$  as a function of  $r_v$
- $\rightarrow r_v = q_v/(1 - q_v); q_v = r_v/(1 + r_v)$ .

Recall that relative humidity is defined as  $RH = e_v/e_{v,s}$ .

Also, the specific humidity at saturation is denoted  $q_{v,s}$  and mixing ratio at saturation  $r_{v,s}$ .

EXERCISE:

- Show that

$$\frac{r_v}{r_{v,s}} = \frac{e_v}{e_{v,s}} \frac{p - e_{v,s}}{p - e_v}$$

where  $r_{v,s}$  denotes mixing ratio at saturation.

- Deduce by taking typical atmospheric values of the various variables that

$$RH \approx \frac{r_v}{r_{v,s}}.$$

$\rightarrow$  Note that  $\rho_{d,s} \neq \rho_d$ , and likewise  $p_{d,s} \neq p_d$  (indeed  $p = p_d + e_v = p_{d,s} + e_{v,s}$ ).

$$\frac{r_v}{r_{v,s}} = \frac{\rho_v/\rho_d}{\rho_{v,s}/\rho_{d,s}} = \frac{[e_v/(R_v T)]/[p_d(R_d T)]}{[e_{v,s}/(R_v T)]/[p_{d,s}/(R_d T)]} = \frac{e_v/p_d}{e_{v,s}/p_{d,s}} = \frac{e_v/(p - e_v)}{e_{v,s}/(p - e_{v,s})} = RH \frac{p - e_{v,s}}{p - e_v}.$$

In the atmosphere,  $p \sim 1000\text{hPa}$ ,  $e_v \sim 80\% e_{v,s}$ , and  $e_{v,s} \sim 10\text{hPa}$ , thus  $\frac{r_v}{r_{v,s}} \approx RH$  to a good approximation.

Note that this is climate dependent, and if the atmosphere was more humid (as for instance would be the case in much warmer conditions where the atmospheric water vapor amount would be important), then  $e$  may no longer be negligible compared to  $p$ .

EXERCISE:

Show that the mixing ratio  $r_{v,s}$  is not just a function of temperature, but also depends on  $p$ :

$$r_{v,s} = r_{v,s}(T, p) = \epsilon \frac{e_{v,s}}{p - e_{v,s}} \tag{84}$$

$$\rightarrow r_{v,s} = \frac{\rho_{v,s}}{\rho_{d,s}} = \frac{e_{v,s}/(R_v T)}{(p - e_{v,s})/(R_d T)} = \frac{e_{v,s}}{p - e_{v,s}} \epsilon \approx \epsilon \frac{e_{v,s}(T)}{p},$$

where  $\epsilon = R_d/R_v$  (see V.3 below). Thus  $r_{v,s}$  is inversely proportional to pressure, and increases quasi exponentially with T.

## B.6 Moist convection

### B.6.1 Convection of unsaturated moist air - virtual temperature

The virtual temperature is the temperature that dry air would have, to have the same density  $\rho$  as moist air at same pressure  $p$ . Recall that moist air is lighter than dry air (this is due to the lighter molecular mass of water  $H_2O$  compared to other air molecules  $N_2, O_2$ ). Thus in the following derivation of virtual temperature, we thus expect the virtual temperature to be warmer than the actual temperature (dry air needs to be warmer to have the same light density as moist air).

Let's derive the formula for the virtual temperature  $T_v$ , as a function of  $T$ ,  $r$  and the ratio of molecular mass of water vapor to dry air  $\epsilon$ :

$$\epsilon \stackrel{\text{def}}{=} \frac{m_v}{m_d} = \frac{R_d}{R_v} \approx 0.622.$$

By definition,  $T_v$  satisfies

$$p = \rho R_d T_v.$$

On the other hand, Dalton's law for partial pressures yields

$$p = \rho_v R_v T + \rho_d R_d T.$$

Therefore

$$T_v = T \left( \frac{\rho_v}{\rho_v + \rho_d} \frac{R_v}{R_d} + \frac{\rho_d}{\rho_v + \rho_d} \right) = T \left( \frac{1 + r_v/\epsilon}{1 + r_v} \right).$$

Note that since  $\epsilon < 1$ ,  $T_v > T$ , i.e. the virtual temperature is warmer than the actual temperature. This is expected since moist air is lighter than dry air, as the molecular mass of water vapor is smaller than the molecular mass of dry air. Therefore in order to have the same lighter density as moist air, dry air needs to be warmer.

Note that when considering the stability or instability to convection of moist, but unsaturated, air (i.e. without phase change of water), everything that we saw for dry convection in section III still holds if we replace the potential temperature  $\theta$  by the virtual potential  $\theta_v = T_v(p/p_0)^{-R/c_p}$ . In particular, the stability criterion becomes  $d\theta_v/dz > 0$ . Similarly, the Brunt Väisälä frequency becomes:

$$N = \sqrt{\frac{g}{\theta_v} \frac{\partial \bar{\theta}_v}{\partial z}}.$$

**Remark 1:** One can similarly define the specific gas constant and heat capacity that accounts for water vapor:

$$R = R_d \frac{1 + r_v/\epsilon}{1 + r_v}, \quad c_p = c_{p,d} \frac{1 + r(c_{p,v}/c_{p,d})}{1 + r_v}$$

**Remark 2:** One can also define the so-called "density temperature", which accounts both for the virtual effect and the condensate loading (weight of liquid and ice condensates, with mixing ratios respectively  $r_l$  and  $r_i$ ):

$$T_p = T \frac{1 + r_v/\epsilon}{1 + r_T}, \quad \text{where } r_T = r_v + r_l + r_i$$

Beyond the virtual effect just discussed, i.e. water vapor making air lighter, an important impact of moisture on convection is the condensation and concomitant latent heat released (we will focus on vapor - liquid phase transition, though all the results below can be extended to the ice phase). In the next section, we will include water phase changes when investigating moist convection.

## B.6.2 Equivalent potential temperature $\theta_e$ , moist static energy, moist adiabatic lapse rate

### - Conservation of equivalent potential temperature

We saw that to determine the stability of dry air, it was important to derive a conserved quantity under adiabatic displacements, namely the potential temperature  $\theta$  for dry convection. Can we derive a similar conserved quantity for moist air? The answer is yes, as we will now show, though this quantity, called equivalent potential temperature, is *approximately* conserved under adiabatic displacements.

For simplicity, we neglect the temperature dependence of the latent heat of vaporization  $L_v$ , as well as the temperature and water vapor dependence of the specific constant of air  $R$  and of the heat capacity at constant pressure  $c_p$ , which in the following denote constants for dry air. We also neglect the virtual effect discussed above. These approximations are reasonable in our climate, where the amount of water vapor is small ( $r_v \ll 1 \text{ kg kg}^{-1}$ ). This would not be true in the presence of significant water vapor in air, as for instance would be the case in much warmer climates; for more details on how to include those dependencies, the interested reader is referred to Kerry Emanuel's book (reference on the course website).

As before, we apply the first law of thermodynamics to an infinitesimal displacement of a parcel of air. We first suppose that the parcel is saturated, i.e.  $r_v = r_{v,s}$  where  $r_v$  is the water vapor mixing ratio and  $r_{v,s}$  the mixing ratio at saturation. The only difference with our earlier dry case equation is that we need to take into account the latent heat released during the condensation of water vapor  $-L_v dr_v$ , where  $dr_v$  denotes the change in water vapor mixing ratio:

$$c_v dT = \delta W + \delta Q_{\text{cond}} = -p d\left(\frac{1}{\rho}\right) - L_v dr_v \quad (85)$$

$$\Leftrightarrow c_p dT - RT \frac{dp}{p} = -L_v dr_{v,s} \quad (86)$$

$$\Leftrightarrow d \ln \left( T p^{-R/c_p} \right) = -L_v \frac{dr_{v,s}}{c_p T} \approx d \left( \frac{L_v r_{v,s}}{c_p T} \right). \quad (87)$$

The latter approximation holds as long as

$$\frac{dr_{v,s}}{r_{v,s}} \gg \frac{dT}{T},$$

which is typically the case in the troposphere. We thus obtain

$$T p^{-R/c_p} \exp \left( \frac{L_v r_{v,s}}{c_p T} \right) = \text{constant},$$

leading to the introduction of a new variable approximately conserved for saturated adiabatic motion ( $r_v = r_{v,s}$ ), the equivalent potential temperature:

$$\theta_e \stackrel{\text{def}}{=} \theta \exp \left( \frac{L_v r_v}{c_p T} \right),$$

where  $\theta$  is the dry potential temperature. Now note that in the case where the parcel is not saturated, there is no condensation of water vapor and  $r_v$  is conserved, so that  $\theta_e$  is also conserved. Thus  $\theta_e$  is (approximately) conserved under adiabatic motion, saturated or not.

### - Conservation of moist static energy and moist adiabatic lapse rate

Before investigating implications for the stability of air to moist convection, we first note that under the hydrostatic approximation, we can define the moist equivalent of the dry static energy in equation (65). It is called the moist static energy:

$$h \stackrel{\text{def}}{=} c_p T + gz + L_v r_v.$$

With the hydrostatic approximation,  $h$  is conserved under adiabatic displacements. Indeed from equation (86) and the ideal gaz law,

$$dh = c_p dT + g dz + L_v dr_v \stackrel{\text{hydrostasy}}{=} c_p dT - \frac{dp}{\rho} + L_v dr_v \stackrel{(86)}{=} 0$$

From this equation, we can derive the moist adiabatic lapse rate  $\Gamma_s$ , defined as the decrease of temperature with height under a saturated reversible displacement. Recall that  $r_{v,s}$  is a function of temperature and pressure (84), and we further make the hydrostatic approximation. Thus:

$$\frac{dT}{dz} = -\frac{g}{c_p} - \frac{L_v}{c_p} \frac{dr_{v,s}}{dz} = -\frac{g}{c_p} - \frac{L_v}{c_p} \left( \frac{\partial r_{v,s}}{\partial p} \frac{\partial p}{\partial z} + \frac{r_{v,s}}{T} \frac{dT}{dz} \right) \quad (88)$$

$$\Leftrightarrow \frac{dT}{dz} = -\frac{g}{c_p} \left( \frac{1 - L_v \rho \partial r_{v,s} / \partial p}{1 + L_v / c_p \partial r_{v,s} / \partial T} \right) \quad (89)$$

$$\Leftrightarrow \Gamma_s = \Gamma_d \left( \frac{1 - L_v \rho \partial r_{v,s} / \partial p}{1 + L_v / c_p \partial r_{v,s} / \partial T} \right) \quad (90)$$

From (84),

$$\rho \frac{\partial r_{v,s}}{\partial p} = -\rho \frac{r_{v,s}}{p - e_{v,s}} = -r_{v,s} \frac{1 + r_{v,s}}{R_d T} \text{ and} \quad (91)$$

$$\frac{\partial r_{v,s}}{\partial T} = \frac{r_{v,s}}{T^2} \frac{L_v}{R_v} \frac{p}{p - e_{v,s}} = \frac{r_{v,s}}{T^2} \frac{L_v}{R_v} \left( 1 + \frac{r_{v,s}}{\epsilon} \right). \quad (92)$$

Thus

$$\Gamma_s = \Gamma_d \left( \frac{1 + L_v r_{v,s} \frac{1 + r_{v,s}}{R_d T}}{1 + \frac{L_v^2 r_{v,s}}{R_v c_p T^2} \left( 1 + \frac{r_{v,s}}{\epsilon} \right)} \right) \quad (93)$$

In Earth's atmosphere, the numerator is smaller than the denominator, so that the temperature decrease with altitude on a moist adiabat is smaller ( $\sim \mathcal{O}(5^\circ \text{ km}^{-1})$ ) than the temperature decrease following a dry adiabat. This is expected since latent heat released reduces the decrease of temperature with height with pressure decrease.

### B.6.3 How can we assess the stability of moist air? Skew-T diagrams

The question addressed here, is how can we assess the stability of moist air ? Traditionally, skew-T diagrams are used in meteorology (figure 9), which allow to easily compare a measured temperature profile (red curve) to theoretical dry and moist adiabatic profiles (constant  $\theta$  and  $\theta_e$  respectively). Such diagrams have iso-temperature lines slanted at  $45^\circ$  to the right (slanted thin brown lines on the figure, hence the name "skew-T"; for more details on those diagrams, see the online MetEd module on Skew-T Mastery (link on the course website, online courses from the COMET program MetEd by UCAR). Since temperature typically decreases with height, observed temperature profiles are largely vertical when reported on those diagrams. The green curve shows the observed dewpoint temperature, which is the temperature at which a parcel must be cooled at constant pressure to reach saturation. This dewpoint temperature curve depends on the environmental humidity (the more humid the air, the closer the dewpoint temperature is to the environmental temperature).

We use the parcel method to evaluate the stability of this environmental temperature profile, by lifting a hypothetical parcel of air from the ground. Comparing, at a given pressure, the temperature of the parcel (shown in black) with the environmental temperature gives us information on its upward (warmer) or downward (colder) acceleration, and thus of its stability to vertical displacement.

- At first, the near-surface parcel is unsaturated. It thus follows a dry adiabatic curve (line of constant  $\theta$ , thin green lines on the figure), until the lifted condensation level (LCL) is reached, where the parcel reaches saturation.



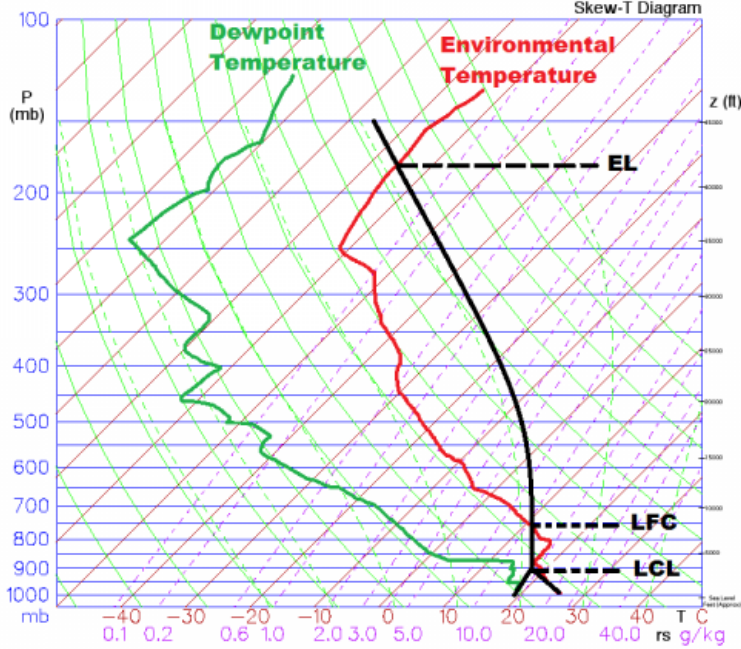


Figure 9: Skew-T diagram, showing observed temperature profile in red, observed dewpoint temperature in green. The parcel method consists of assessing the stability of a near-surface parcel to an upward displacement (parcel temperature shown in black). Adapted from the weathertogether.net blog.

- Above the lifted condensation level,  $\theta_e$  will be conserved for the parcel which is undergoing moist adiabatic ascent (line of constant  $\theta_e$ , thin dashed green lines). Note that its temperature decreases with height slower than the dry adiabatic curve, due to the latent heat released as water vapor condenses.
- At a certain height, called level of free convection (LFC), the parcel becomes warmer than the environment. As long as this moist adiabatic curve is warmer than the environmental temperature profile, as “warm air rises”, the parcel is convectively unstable and keeps ascending.
- This ends when the parcel reaches its equilibrium level (EL), where the parcel’s temperature is equal to the environmental temperature.

In order to find the lifted condensation level, the dew point temperature is used, as well as lines of constant saturation mixing ratio  $iso-r_{v,s}$  (thin dashed purple lines on the skew-T diagram). By definition of the dewpoint temperature, at the surface,  $r_{v,s}(T_{d,sfc}, p_{sfc}) = r_{v,sfc}$  where  $r_{v,sfc}$  is the water vapor mixing ratio of the parcel at the surface. The lifted condensation level is thus located where the  $iso-r_{v,s}$  line passing through the surface dewpoint temperature intersects the parcel temperature.

#### B.6.4 Convective Available Potential Energy (CAPE), convective inhibition (CIN)

During the ascent, we just saw that the upward acceleration is related to the difference between the parcel temperature and the environmental temperature. This can be quantified further, in fact the area between the parcel and the environmental temperature is directly related to the potential energy of convection.

This can be clarified by considering the vertical momentum equation

$$\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = -g \frac{\rho'}{\rho} - \frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \Delta w, \quad (94)$$

where  $\rho$  denotes the density of the environment,  $\rho'$  the density of the parcel minus that of the environment, and the last term is the viscous force. In strong vertical ascent, we can expect the leading order balance to be between the vertical advection and the buoyancy force

$$w \frac{\partial w}{\partial z} = -g \frac{\rho'}{\rho} \Leftrightarrow \frac{w^2(z)}{2} = \int -\frac{\rho'}{\rho} g dz. \quad (95)$$

The right-hand side has units of a specific energy.

If  $-\rho' < 0$ , i.e. if the parcel is lighter than the environment as is the case between LFC and EL (figure 9), the integral represents an upper bound for the kinetic energy of the rising parcel. It is called the Convective Available Potential Energy, or CAPE. It is an upper bound as it neglects forces opposing the motion including viscosity and pressure gradients, as well as turbulent entrainment of dry environmental air at the edge of the rising plume, leading to partial evaporation of condensates and latent cooling, thus opposing the rising motion. Conversely, if  $-\rho' > 0$ , i.e. if the parcel is heavier than the environment as is the case below the LFC (figure 9), it represents the potential energy barrier to upward convection, as the parcel acceleration is downward. It is called the convective inhibition, or CIN.

CAPE can be rewritten as a function of temperature using the ideal gas law  $p = \rho RT$  and assuming that pressure perturbations between the parcel and the environment are small:

$$\frac{\rho'}{\rho} = -\frac{T'}{T}.$$

If we further make the hydrostatic approximation,

$$\text{CAPE} = \int_{p_{EL}}^{p_{LFC}} \frac{T'}{T} \frac{dp}{\rho} = R \int_{p_{EL}}^{p_{LFC}} T' d \ln p.$$

This expression shows that this convective potential energy is proportional to the area between the parcel and the environmental temperature on the skew-T  $\ln p$  diagram shown in figure 9.

CAPE can be used to derive an upper bound for vertical velocities of buoyant parcels

$$w_{max} = 2\sqrt{\text{CAPE}}.$$

As mentioned earlier, for simplicity, we neglected virtual effects. We note though, that it is straightforward to include them in the above computation, yielding the more general formula

$$\text{CAPE} = R \int_{p_{EL}}^{p_{LFC}} T'_v d \ln p.$$

Similarly,

$$\text{CIN} = R \int_{p_{LFC}}^{p_{SFC}} T'_v d \ln p, \quad \text{where } p_{SFC} \text{ denotes surface pressure (note that } \text{CIN} < 0).$$

Thus during the ascent, the area between the parcel temperature and the environment temperature on the skew-T diagram is a measure of atmospheric instability. The larger the CAPE, the stronger the upward motion during the ascent. If enough atmospheric instability is present, cumulus clouds are capable of producing severe convection and storms. In later lectures, we will give a brief overview of the life cycle of such severe convective clouds. We will also use the fundamental knowledge gained in previous lectures to clarify the physical processes leading to cloud formation, for the different cloud types of the cloud classification.