METEOROLOGY
LECTURE NOTES - MASTER 1 & MASTER 2, ENS PARIS, PSL
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Introduction

This Meteorology class consists of two parts.

The first part (taught by C. Muller) aims at providing physical intuition of large-scale atmospheric weather patterns. We will focus on planetary and synoptic scales. A review of the main equilibria relevant to Earth's atmosphere, will be followed by a discussion of several important weather phenomena in mid-latitudes, including extratropical cyclones and fronts. If time permits, tropical meteorology will also be discussed.
The grade for this first part will be based on one homework, and one practical session in class.

The second part of the course (taught by Aurelien Podglajen) aims at introducing some key, simple models of the atmosphere. These simple models help shed light into the large-scale dynamics and energy transport of the atmosphere.

PART 1

A Large-scale equilibria of the atmosphere

A.1 Hydrostatic balance

Reminder : equations of motion

We recall the equations of motion of the atmosphere, under the effect of gravity g, including the Earth rotation (Coriolis parameter denoted $f = 2\Omega \sin(lat)$, where Ω denotes the Earth rotation frequency, and lat latitude), and neglecting viscous effects :

$$\frac{du}{dt} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} \quad (1)$$

$$\frac{dv}{dt} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} \quad (2)$$

$$\frac{dw}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \quad (3)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \quad (4)$$

The first term on the left-hand side of equations (1-3) is the total Lagrangian acceleration ($d/dt = \partial/\partial t + u\partial/\partial x + v\partial/\partial y + w\partial/\partial z$). The second term on the left-hand side of equations (1-2) is the Coriolis force, and the first term on the right-hand side of equations (1-3) is the pressure force. Finally, the $-g$ term is the gravitational force, and the last equation holds for a non-divergent flow. (We note in passing that a thermodynamic equation completes the total set of equations for atmospheric motion, but we will not need it in this section).

This system of equations is not easily solved. One approach to gain insight into the dynamics, is to perform a scale analysis. This approach allows to highlight the leading order terms and dynamics. We will use it to derive the hydrostatic approximation.

Typical scales of interest

For midlatitude synoptic systems, typical scales are :
$U \sim 10$ m s^{-1} horizontal velocity
$W \sim 1$ cm s^{-1} vertical velocity
$L \sim 10^6$ m horizontal scale (e.g., midlatitude cyclone or anticyclone)

$H \sim 10^4$ m vertical scale (depth of troposphere)
$\tau \sim L/U \sim 10^5$ s corresponding time scale ($\sim$ 1 day)
$f_0 \sim 10^{-4}$ s^{-1} Coriolis frequency.

Hydrostatic approximation

The hydrostatic approximation states that :

$$\boxed{\frac{\partial p}{\partial z} = -\rho g.} \tag{5}$$

It is a very good approximation for the large-scale motions of the atmosphere and of the ocean.

It follows from a scale analysis of the vertical momentum equation (3). Equation (4) yields

$$\frac{U}{L} \sim \frac{W}{H}. \tag{6}$$

The horizontal momentum equations, which we rewrite in vector form with $\mathbf{u} = (u, v, 0)$, ∇_h horizontal gradient, and the unit vertical vector $\mathbf{k} = (0, 0, 1)$:

$$\frac{d\mathbf{u}}{dt} + f\mathbf{k} \wedge \mathbf{u} = -\frac{1}{\rho}\nabla_h p \;\Rightarrow\; \frac{U}{\tau} \sim \frac{P}{\rho L}, \tag{7}$$

where P denotes a typical pressure magnitude. Including these scalings into the vertical momentum equation yields :

$$\frac{dw}{dt} \sim \frac{W}{\tau} \tag{8}$$

$$\frac{1}{\rho}\frac{\partial p}{\partial z} \sim \frac{P}{\rho H} \sim \frac{LU}{H\tau} \text{ from equation (7)} \tag{9}$$

Thus the ratio of those two terms is

$$\frac{dw/dt}{1/\rho \partial p/\partial z} \sim \frac{WH}{LU} \sim \left(\frac{H}{L}\right)^2 \text{ from equation (6).} \tag{10}$$

According to this scale analysis, for the large scales of interest here, we can neglect the vertical acceleration to very good approximation : $(H/L)^2 \sim (10^{-2})^2 = 10^{-4} \ll 1$.

For smaller scale phenomena, the vertical acceleration term can not be neglected. This is for instance the case for isolated clouds which have an aspect ratio order one. It follows from this scale analysis, and is consistent with the observation that vertical accelerations are important in cloud dynamics.

Geopotential

In the atmosphere, the gravitational acceleration g varies in space. We introduce the geopotential $d\Phi = g dz \Leftrightarrow \Phi = \int_0^z g dz$. It is the work needed to raise a unit of mass from sea level to an altitude z (work against gravity). It has units of J kg^{-1}.

It is more standard to use the geopotential height, simply defined as $Z = \Phi/g_0$, where $g_0 = 9.8$ m s^{-2} is a reference gravity constant. Z has a nice physical interpretation. Indeed, if we consider an atmospheric surface at pressure p_1, and ask the following question : what is the altitude of this isobaric surface ? Then $Z(p_1) = \int_0^{z(p_1)} g dz/g_0 \approx z(p1)$ if $g \approx g_0$., which turns out to be the case in the atmosphere (variations of g are small). Thus the geopotential height $Z(p_1)$ represents the altitude of the p_1 isobar.

Pressure coordinates

In meteorology, it is qommon to work in pressure coordinates instead of height. Let us derive the hydrostatic approximation in pressure coordinate. To that end, we will combine the hydrostatic approximation $dp = -\rho g dz$ with the ideal gaz law $p = \rho RT$

$$\Rightarrow g dz = -\frac{RT}{p} dp \Rightarrow \boxed{\frac{\partial \Phi}{\partial p} = -\frac{RT}{p}}. \tag{11}$$

This last equation is the hydrostatic balance in pressure coordinates.

A.2 Hypsometric balance

Hypsometric equation

The hydrostatic approximation can be rewritten

$$\frac{\partial \Phi}{\partial \ln p} = -RT \tag{12}$$

which we integrate between p_0 and p_1, yielding :

$$\begin{aligned}
\Phi(p_1) - \Phi(p_0) &= -R \int_{p_0}^{p_1} T \, d\ln p & (13)\\
&= -R \int_{p_0}^{p_1} T \, d\ln p = -R\overline{T} \ln\left(\frac{p_1}{p_0}\right) & (14)\\
\Rightarrow \Phi(p_1) - \Phi(p_0) &= R\overline{T} \ln\left(\frac{p_0}{p_1}\right), & (15)\\
\text{with} \quad \overline{T} &= \frac{\int_{p_0}^{p_1} T \, d\ln p}{\ln p_1 - \ln p_0} & (16)
\end{aligned}$$

Note that $d\ln p \sim dz$ (in an isothermal hydrostatic atmosphere, see below). So $\overline{T}$ is representative of the mean temperature between isobars p_0 and p_1.

$\delta\Phi = \Phi(p_1) - \Phi(p_0)$ represents the thickness between isobar p_0 and isobar p_1 (recall that Φ is proportional to geopotential height). The hypsometric equation can be written

$$\boxed{\delta\Phi = R\overline{T} \ln\left(\frac{p_0}{p_1}\right).} \tag{17}$$

It implies that the thickness between two isobars is proportional to the temperature between those two levels.

Special case of an isothermal atmosphere

The hypsometric equation generalizes the relation between p, z and T in the case of an isothermal atmosphere $T = T_0$. In that case,

$$d\Phi = gdz = -R\,T_0 \, d\ln p \;\Rightarrow\; d\ln p = -g\,/\,(RT_0)\,dz \;\Rightarrow\; \ln\left(\frac{p}{p_0}\right) = -\frac{g}{RT_0}(z - z_0) \tag{18}$$

Thus

$$p = p_0 \exp\left(-\frac{g(z-z_0)}{RT_0}\right) = p_0 \exp\left(-\frac{(z-z_0)}{H_p}\right) \tag{19}$$

pressure decreases exponentially with altitude. The typical decay length scale is $H_p = RT_0/g \approx$ (287 J kg^{-1} K^{-1}) $\times$ 288 K / (9.8 m s^{-2}) $\approx$ 8400 m. We see that for warmer temperatures, pressure decreases more slowly, and conversely for colder temperatures, pressure decreases more rapidly.

Equivalence between pressure and geopotential

The geopotential can also been seen as a variable equivalent to pressure. Indeed, denoting ∇ the horizontal pressure gradient $(\partial/\partial x, \partial/\partial y)$,

$$\boxed{\nabla_{p=cstt}\Phi = \frac{1}{\rho}\nabla_{z=cstt}p.} \tag{20}$$

(This comes from the mathematical fact that

$$\nabla_{S=cstt}F = \nabla_{z=cstt}F + \frac{\partial F}{\partial z}\nabla_{S=cstt}z. \tag{21}$$

for any variable S).

Therefore, the horizontal pressure gradient term $\nabla p/\rho$ in the horizontal momentum equations (1-2) in altitude coordinates (x, y, z), is replaced by $\nabla\Phi$ in pressure coordinates (x, y, p).

Thus, a low value of geopotential on an isobar is equivalent to a low value of pressure at constant altitude. In other words, a high pressure corresponds to a positive anomaly of geopotential height on an isobar, and conversely a low pressure corresponds to a negative geopotential anomaly.

These low and high pressure systems, which correspond to troughs and crests in the geopotential height field, have implications for atmospheric dynamics and weather. A high pressure, or geopotential crest, is associated with an anticyclonic circulation. Conversely a low pressure, or geopotential trough, is associated with a cyclonic circulation.

$\Rightarrow$ To summarize, $\delta\Phi$ corresponds to $\overline{T}$, and Φ corresponds to p.

Weather maps

In meteorology, it is standard to analyze maps on different isobars :

Sea level, or 1000 hPa	Surface	
850 hPa	1500 m	above the planetary boundary layer
500 hPa	5000 m	mid-troposphere
200 hPa	12 km	tropopause

The temperature in the atmosphere varies with height. The temperature is somewhat constant in the boundary layer, then decreases with height at a rate of about 6 ° km^{-1} in the troposphere, until the tropopause. In the stratosphere above, temperature then increases with height. T at 850 hPa, i.e. above the boundary layer, is not directly influenced by the surface. For instance, it is not very sensitive to the diurnal cycle.

A.3 Geostrophic balance

Geostrophic balance and Rossby number

We perform a scale analysis of the horizontal momentum equation (7), which we rewrite in pressure coordinates :

$$\frac{d\mathbf{u}}{dt} + f\mathbf{k} \wedge \mathbf{u} = -\nabla_h \Phi \tag{22}$$

where recall that $f = 2\Omega \sin(lat)$. A scale analysis of the left hand side terms yields :

$$\frac{d\mathbf{u}}{dt} \sim \frac{U^2}{L}, \quad \text{and} \quad f\mathbf{k} \wedge \mathbf{u} \sim fU. \tag{23}$$

The Rossby number is defined as the ratio of those terms :

$$\boxed{Ro = \frac{\text{horizontal acceleration}}{\text{Coriolis force}} = \frac{U}{fL}.} \tag{24}$$

If $Ro \ll 1$, the Coriolis force dominates the acceleration term, yielding

$$f\mathbf{k} \wedge \mathbf{u} = -\nabla_h \Phi \Leftrightarrow \begin{cases} -fv = & -\dfrac{\partial \Phi}{\partial x} \\ fu = & -\dfrac{\partial \Phi}{\partial y} \end{cases} \Leftrightarrow \begin{cases} u = & -\dfrac{1}{f}\dfrac{\partial \Phi}{\partial y} \\ v = & \dfrac{1}{f}\dfrac{\partial \Phi}{\partial x} \end{cases} \tag{25}$$

This is the geostrophic balance (in pressure coordinates), i.e. a balance between pressure and Coriolis forces. (u, v) given above is called the geostrophic wind.

For low pressure systems in mid-latitudes, lat $\sim 45°$N so $f \sim 10^{-4} s^{-1}$, $L \sim 1000$ km and $U \sim 10$ m s^{-1}, so that $Ro \sim 0.1$. The geostrophic balance holds to a good approximation.

For tropical cyclones, lat $\sim 15°$N so $f \sim 4 \times 10^{-5} s^{-1}$, $L \sim 1000$ km and $U \sim 50$ m s^{-1}, so that $Ro \sim 1$. A tropical cyclone is not in geostrophic balance.

For a bath-tub of kitchen sink vortex, lat $\sim 45°$N $\Rightarrow f \sim 10^{-4} s^{-1}$, $L \sim 1$ m and $U \sim 1$ m s^{-1}, so that $Ro \sim 10^4$! These are NOT AT ALL in geostrophic balance, the Earth rotation does not impact these small-scale vortices.

Thus in p coordinates (with (u, v) along isobars), the geostrophic wind is given by

$$\boxed{u_g = -\frac{1}{f_0}\frac{\partial \Phi}{\partial y}, \quad v_g = \frac{1}{f_0}\frac{\partial \Phi}{\partial x}.}$$

It is worth noting the z coordinate expression of the geostrophic wind, assuming a constant density fluid $\rho = \rho_0$:

$$\boxed{u_g = -\frac{1}{f_0\rho_0}\frac{\partial p}{\partial y}, \quad v_g = \frac{1}{f_0\rho_0}\frac{\partial p}{\partial x}.} \tag{26}$$

Circulation around low and high pressures

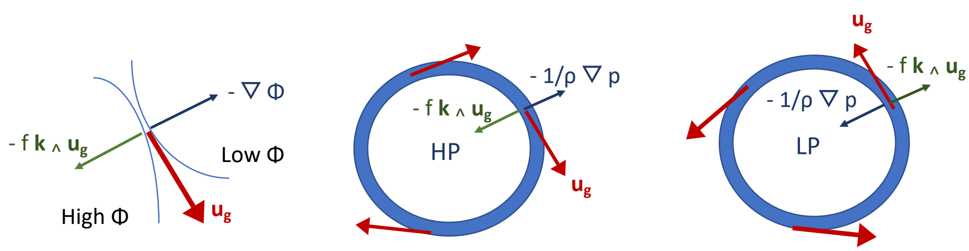

FIGURE 1 – Geostrophic wind between low and high pressure anomalies in the Northern Hemisphere. Geostrophic balance, along with the fact that the Coriolis force is to the right of the velocity in the Northern Hemisphere, implies clockwise circulation around high pressure systems, and counter-clockwise circulation around low pressure systems.

Effect of fricton : hints for the homework

In the homework, you will investigate what happens when friction is added to the geostrophic balance. Friction can potentially be important near the surface, so for low-level flow. In that case, a simple friction term, proportional to the velocity and oriented opposite, will be added to the geostrophic balance (25) :

$$f\mathbf{k} \wedge \mathbf{u} = -\nabla_h \Phi - K\mathbf{u} \Leftrightarrow \begin{cases} -fv = & -\dfrac{\partial \Phi}{\partial x} - Ku \\ fu = & -\dfrac{\partial \Phi}{\partial y} - Kv \end{cases} \tag{27}$$

As you will see in the homework, a consequence of the friction term is the deviation of the geostrophic winds towards low pressure anomalies. This implies a convergence of horizontal winds near the surface into low pressure systems. Converging surface winds force upward motion, thereby advecting moist near-surface air to colder temperatures aloft. This can lead to cloud formation and precipitation.

Conversely, friction deviates geostrophic winds away from high pressure anomalies. This implies divergence of near-surface winds and subsidence of dry air from aloft, preventing cloud formation and favoring nice sunny weather.

Properties of the geostrophic wind

Prop 1 : (u_g, v_g) flows parallel to isoΦ lines.
Indeed, $(u_g, v_g) \cdot \nabla\Phi = 0$.

Prop 2 : The closer the isoΦ lines are from each other, the stronger the geostrophic wind is.
Indeed, $||(u_g, v_g)|| \propto ||\nabla\Phi||$.

Prop 3 : On the f plane (i.e. assuming $f = f_0$ constant), the geostrophic wind is non divergent.
Indeed, $\nabla \cdot (u_g, v_g) = \partial u_g/\partial x + \partial v_g/\partial y = 0$. This implies that $\partial w/\partial z = 0$ thus the geostrophic wind does not contribute directly to the vertical velocity.

Note that if f is not constant, then

$$\frac{\partial(fu_g)}{\partial x} + \frac{\partial(fv_g)}{\partial y} = 0 \;\Rightarrow\; f\left(\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y}\right) + v_g\frac{\partial f}{\partial y} = 0 \;\Rightarrow\; f\frac{\partial w}{\partial z} = v_g\frac{\partial f}{\partial y} \tag{28}$$

The variations of f with latitude creates a vertical velocity (see Sverdrup balance in oceanography).

Prop 4 : On the f plane (i.e. assuming $f = f_0$ constant), the rotational of the geostrophic wind is proportional to the Laplacian of the geopotential.
Indeed

$$\xi_g = \nabla \wedge (u_g, v_g) = \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} = \frac{1}{f_0}\left(\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2}\right) = \frac{1}{f_0}\Delta\Phi \tag{29}$$

Typically, vorticity is associated with two phenomena : changes of direction of the wind, and changes in wind magnitude (shear).

A.4 Thermal wind

Thermal wind balance

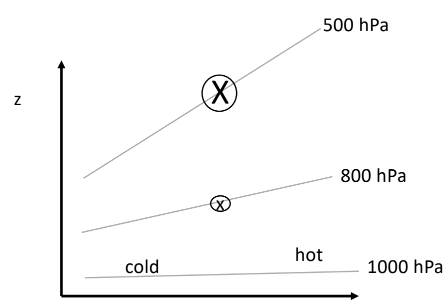

FIGURE 2 – Thermal wind : in the presence of a horizontal temperature gradient, isobars are slanted (from hypsometry). The incraeasing horizontal gradient of geopotential with height, implies that the geostrophic wind increases with height.

To derive the thermal wind, we need two ingredients : geostrophy and hydrostasy.

As illustrated in figure 2, on the warm side from the hypsometric equation, isobars are further away from each other. If we assume that the surface approximately lies on an isobar, e.g. 1000 hPa isobar, then this implies that the geopotential height increases faster and faster with height in the warm sector. From geostrophic balance, a gradient of geopotential height on an isobar indicates a geostrophic wind. Furthermore, the gradient increases with height, thus the wind increases in height. This is the so-called thermal wind balance : a horizontal gradient of temperature yields a vertical gradient of horizontal wind.

Mathematically, in the case of the ocean, the density can be approximated by a constant $\rho \sim 1000$ kg m^{-3} plus a small perturbation $\rho'(x, y, z, t) < 10$ kg m^{-3} :

$$\rho(x, y, z, t) = \rho_0 + \rho'(x, y, z, t). \tag{30}$$

The pressure $p_0(z)+p'(x, y, z, t)$ is assumed in hydrostatic balance with the density (ρ_0 and $\rho'(x, y, z, t)$ respectively). Combining the hydrostatic balance and geostrophy :

$$u_g == -\frac{1}{f_0\rho_0}\frac{\partial p'}{\partial y} \tag{31}$$

and the hydrostatic balance

$$\frac{\partial p'}{\partial z} = -\rho' g, \tag{32}$$

we obtain

$$\frac{\partial u_g}{\partial z} = -\frac{1}{f_0\rho_0}\frac{\partial^2 p'}{\partial y \partial z} \implies \boxed{\frac{\partial u_g}{\partial z} = \frac{g}{f_0\rho_0}\frac{\partial \rho'}{\partial y}} \quad (33)$$

Similarly for v :

$$\boxed{\frac{\partial v_g}{\partial z} = -\frac{g}{f_0\rho_0}\frac{\partial \rho'}{\partial x}} \quad (34)$$

Similarly in the atmosphere, in (x, y, p) coordinate, combining the geostrophic wind

$$fu = -\frac{\partial \Phi}{\partial y} \quad (35)$$

with hydrostatic balance

$$\frac{\partial \Phi}{\partial p} = -\frac{RT}{p} \quad (36)$$

yields

$$\frac{f\partial u}{\partial p} = -\frac{\partial^2 \Phi}{\partial y \partial p} \implies \boxed{\frac{\partial u}{\partial p} = \frac{R}{fp}\frac{\partial T}{\partial y}} \quad (37)$$

Similarly for v

$$\boxed{\frac{\partial v}{\partial p} = -\frac{R}{fp}\frac{\partial T}{\partial x}} \quad (38)$$

(Note that in the last two equations, horizontal gradients are taken at p constant in the (x, y, p) coordinate system, so pressure can be taken out of the horizontal partial derivatives).

The vertical gradient of geostrophic wind is referred to as thermal wind. The thermal wind balance thus states that a horizontal gradient of temperature corresponds to a vertical gradient of horizontal wind. The thermal wind blows parallel to isotherms, with warm air on its right in the northern hemisphere. This is consistent with the jet stream in the atmosphere between the warm tropical air mass and the cold polar air mass, blowing eastward.

Slope of a front

A consequence of the thermal wind balance is that atmospheric fronts are slanted. Indeed, if we suppose that a front between a warm sector and a cold sector is vertical (figure 3a), then the iso-u lines are vertical, which contradicts the thermal wind balance. A vertical front is therefore not possible, since a horizontal gradient of temperature is associated with a vertical gradient of u.

Thus an atmospheric front is slanted, with a slope consistent with the thermal wind balance (figure 3b). In order to determine the slope, we perform a scale analysis of the thermal wind balance :

$$\frac{\partial u}{\partial \ln p} = \frac{R}{f}\frac{\partial T}{\partial y}. \quad (39)$$

From hydrostatic balance and the ideal gas law $p = \rho RT$,

$$dp = -\rho g dz = -\frac{p}{RT} g dz \Rightarrow d(\ln p) = -\frac{g}{RT} dz. \quad (40)$$

Thus

$$\frac{\partial u}{\partial z} = -\frac{g}{RT}\frac{R}{f}\frac{\partial T}{\partial y} \Rightarrow \frac{\Delta z}{\Delta y} \sim -\frac{Tf}{g}\frac{\Delta u}{\Delta T} \quad (41)$$

where Δ denotes change accross the front. With typical values $(T, f, g, \Delta u, \Delta T) \sim$ (280 K, 10^{-4} s^{-1}, 10 m s^{-2}, 30 m s^{-1}, 10 K), we obtain a slope of

$$\frac{\Delta z}{\Delta y} \sim -\frac{1}{120}. \quad (42)$$

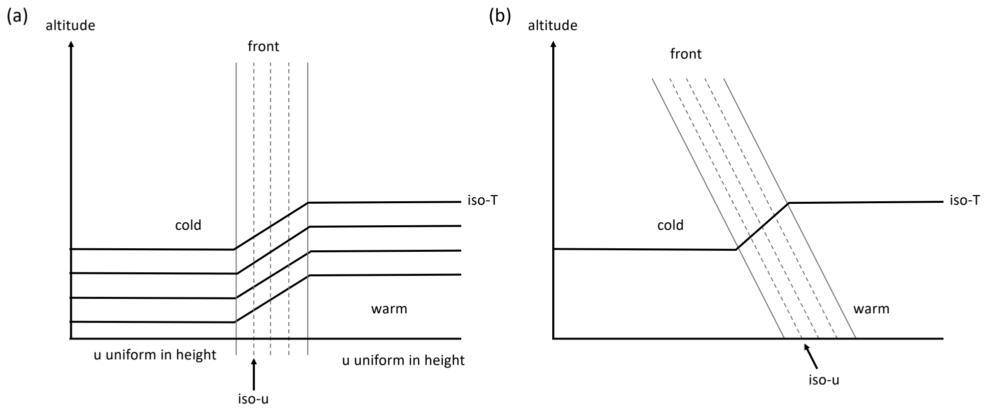

FIGURE 3 – Slope of a front : A front can not be vertical (a) since the thermal wind implies that the horizontal gradient of temperature is associated with a vertical gradient of u. A front is thus necessarily slanted (b).

B Mid-latitude meteorology

One reference which may be useful is the MIT open courseware - see the link to the lecture notes on the course website `http://www.lmd.ens.fr/muller/TEACHING_METEO/meteo.html`

B.1 Rossby waves

B.2 Frontal systems

B.3 Extratropical cyclones/anticyclones