MEC 654
Polytechnique-UPMC-Caltech
Year 2014-2015

Turbulence

chapter 7

turbulence and vortices :
an introduction

7. turbulence and vortices : an introduction

- **dissipation function** $\epsilon(\underline{x},t) = {}^t\underline{\underline{\tau}} : \nabla \underline{u} > 0$

- **incompressible and homogeneous newtonian fluid with constant viscosity**

$$\epsilon(\underline{x},t) = 2\eta\, \underline{\underline{d}} : \underline{\underline{d}}$$

- **other form** (chap. 3, §3.4)

$$\epsilon(\underline{x},t) = \eta \overbrace{\boxed{\underline{\omega}^2}}^{\varepsilon_2 \text{ pseudo-dissipation}} + 2\eta\, div(\nabla \underline{u}.\underline{u})$$

squared vorticity (enstrophy)

- **averaging in a volume**

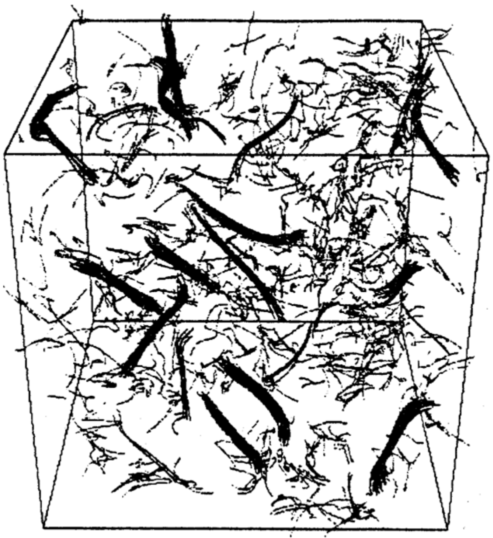
$$\langle \epsilon \rangle_V \equiv \frac{1}{V}\iiint_V \epsilon(\underline{x},t)\, dV = \frac{1}{V}\iiint_V \eta\, \underline{\omega}^2 dV + \underbrace{2\eta \iint_S (\nabla \underline{u}.\underline{u}).\underline{n}\, dS}_{\substack{=0 \\ \text{on a conveniently} \\ \text{chosen boundary}}}$$

$\Rightarrow$ turbulence : a « vortex factory »

7. turbulence and vortices : an introduction (…)

✓ a numerical simulation of the 3D incompressible Navier-Stokes equations in a periodic box starting from a random fluctuation velocity field (isotropic turbulence)

iso-vorticity $|\underline{\omega}|$



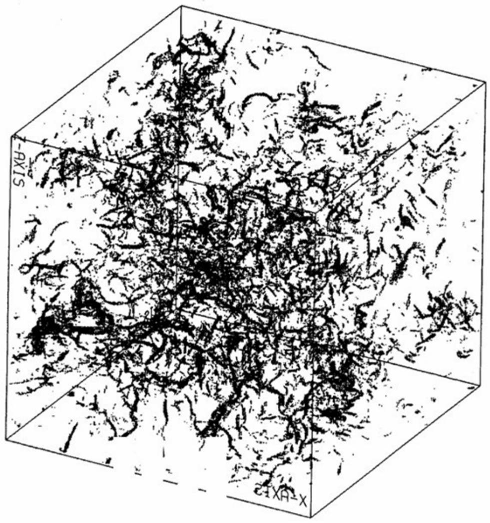
She et al. 1991

7. turbulence and vortices : an introduction (…)

✓ same, but with a larger energy injection

⇨ thinner vortices $\eta = \left(\frac{\nu^3}{\varepsilon_0} \right)^{1/4} \downarrow$

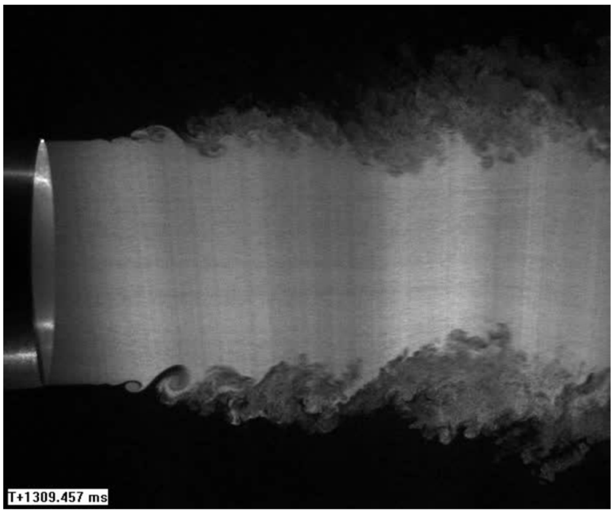
iso-vorticité
$|\underline{\omega}|$



Vincent & Meneguzzi (1991)

7. turbulence and vortices : an introduction (…)

- **a jet**



Davoust 2011

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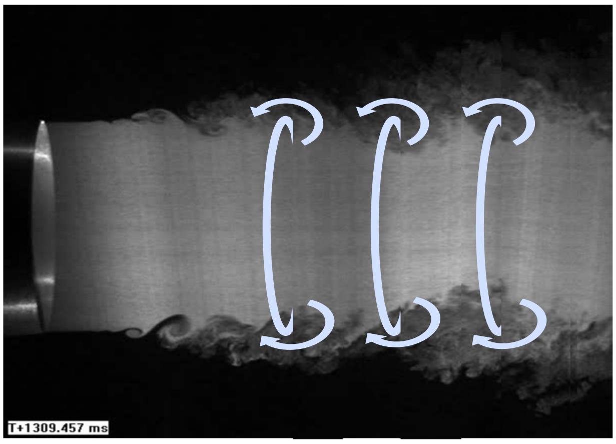
7. turbulence and vortices : an introduction (…)

- **a jet (…)**



Davoust (2011)

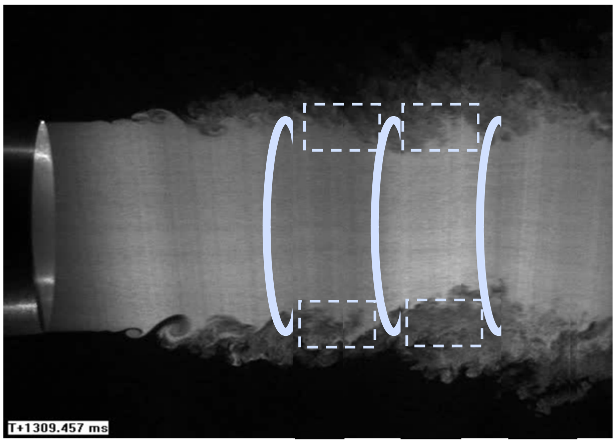
azimuthal vorticity rings
(Crow & Champagne 1971)

Kelvin – Helmholtz
instability mechanism
(Batchelor & Gill 1962)

acoustic source
(Tinney et. al 2008)

7. turbulence and vortices : an introduction (…)

- **a jet (…)**



Davoust (2011)

« braid region »
unsteady strain field
(we will see soon)

promotes
streamwise vortices
(Lin & Corcos 1984)

crucial for mixing
and growth
(Liepmann & Gharib 1992)

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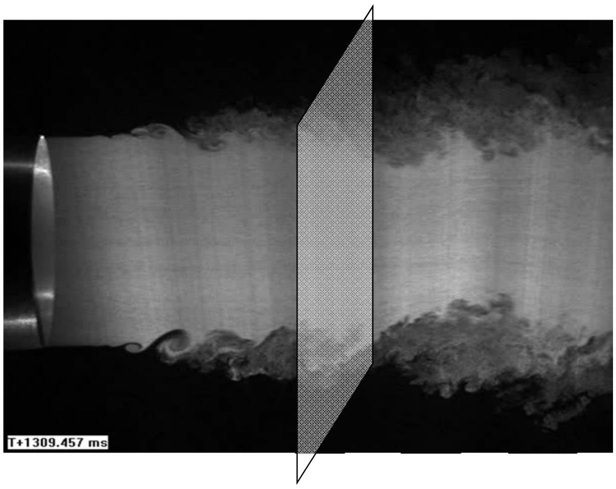
7. turbulence and vortices : an introduction (…)

- **a jet (…)**

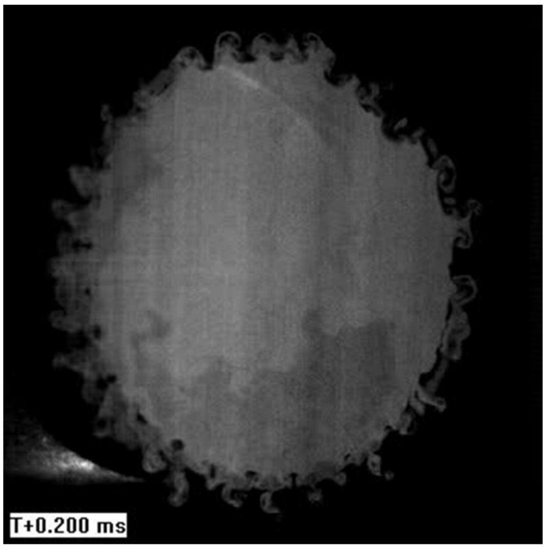


a cross section
($x\,/D = 2$)

9

7. turbulence and vortices : an introduction (…)

- **a jet (…)**

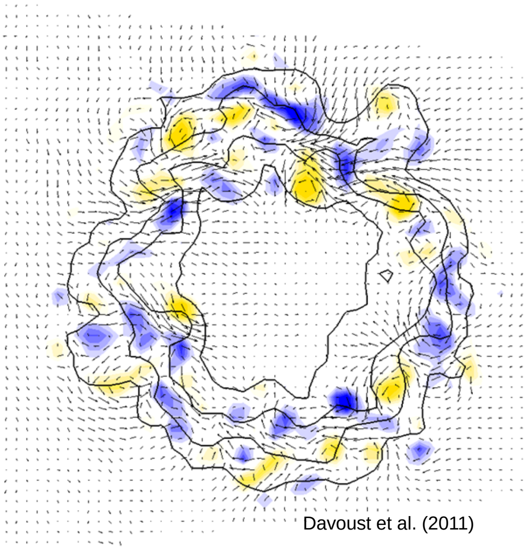


a cross section
($x\,/D = 2$)

Davoust (2011)

7. turbulence and vortices : an introduction (…)

• **a jet (…)**



Davoust et al. (2011)

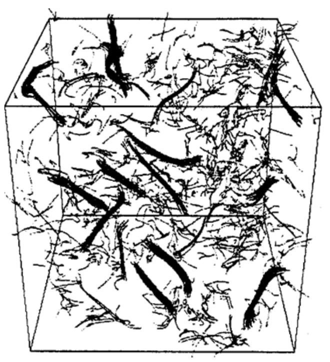
a cross section
($x / D = 2$)

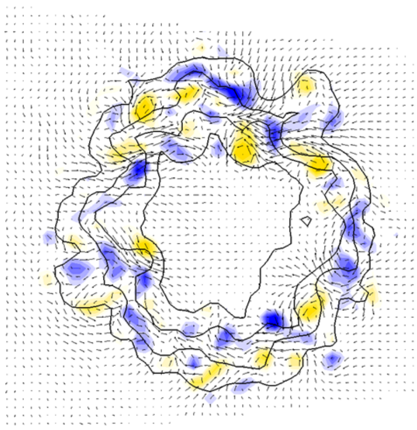
time resolved PIV :
component of the vorticity
$\underline{\omega}$ aligned with the jet axis

7. turbulence and vortices : an introduction (…)

• **observation**

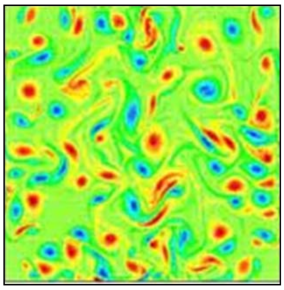
✓ in 3D flows turbulence looks like a cascade of vortices, the smaller ones being those where dissipation takes place





7. turbulence and vortices : an introduction (…)

✓ numerical simulation of **2D** incompressible Navier-Stokes equations in a periodic box starting from a random fluctuation velocity field (**2D turbulence**)

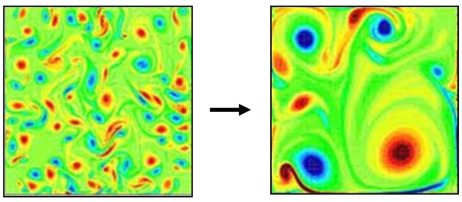


Turbulence_in_periodic_box_long_simulation.mp4

7. turbulence and vortices : an introduction (…)

• **observation**

⇨ in 2D flows turbulence looks like an "inverse cascade" of vortices where small scale vortices form larger ones



⇨ this is in contradiction with the dynamics of 3D flows where the large scales break down into smaller ones, in accordance with Richardson-Kolmogorov' cascade model

⇨ vorticity thus reveals that 3D and 2D turbulent flows behave differently

⇨ for understanding why, we need to consider vortex dynamics

chapter 8

kinematics of vorticity

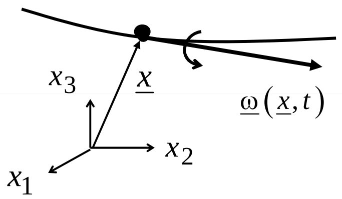
8.1 some definitions

- **vorticity** $\boxed{\underline{\omega} = \underline{rot}\ \underline{u}}$

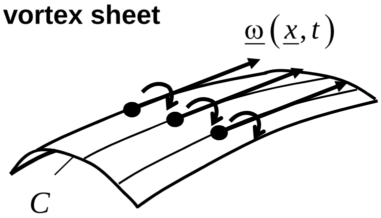
- **vortex line**

✓ equation $d\,\underline{x} \wedge \underline{\omega}\left(\underline{x}, t\right) = 0$

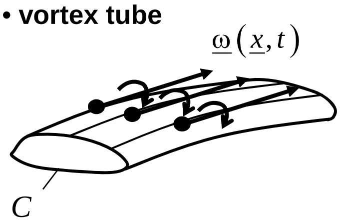
$$\Rightarrow \quad \frac{dx_1}{\omega_1\left(\underline{x}, t\right)} = \frac{dx_2}{\omega_2\left(\underline{x}, t\right)} = \frac{dx_3}{\omega_3\left(\underline{x}, t\right)}$$



- **vortex sheet**



- **vortex tube**



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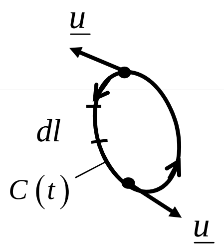
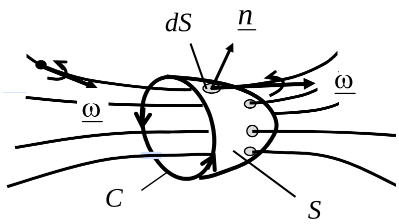
8.1 some definitions (…)

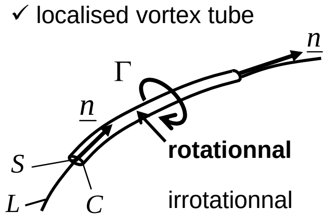
- **circulation**

$$\Gamma = \oint_{C(t)} \underline{u} \cdot \underline{dl} \xrightarrow{\text{Stokes formulae}} \Gamma = \iint_{S(t)} \underline{\omega} \cdot \underline{n}\, dS$$

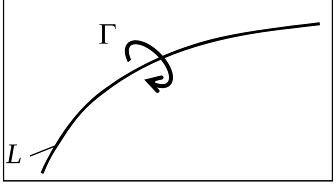




- **vortex**



$$\xrightarrow[\Gamma = const.]{S \to 0}$$



8.2 Hemholtz's laws

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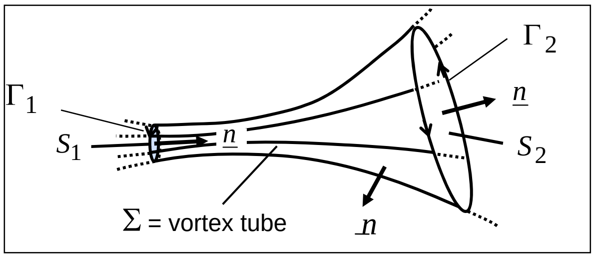
- **vorticity is solenoidal**

$$\boxed{\begin{array}{c}\forall\, \underline{x},\ \forall\, t \\ div\ \underline{\omega} = 0\end{array}} \Rightarrow \forall V,\ \forall t \quad \iiint_{V(t)} div\ \underline{\omega}\ dV = 0 \overset{\text{Green-Ostrogradski}}{=} \oiint_{S(t)} \underline{\omega} \cdot \underline{n}\ dS$$

$\Rightarrow$ the flux of vorticity across any closed surface is nill

✓ lets consider a volume bounded by a vortex tube + 2 cross sections S_1 and S_2



$$\Rightarrow \quad \Gamma_1 = \oint_{C_1} \underline{u} \cdot \underline{dl} = \oiint_{S_1} \underline{\omega} \cdot \underline{n}\ dS = \oiint_{S_2} \underline{\omega} \cdot \underline{n}\ dS = \oint_{C_2} \underline{u} \cdot \underline{dl} = \Gamma_2$$

$\Rightarrow$ the circulation of a vortex tube ~~vorticity~~ is an intrinsic quantity attached to the tube (it is identical in all sections).

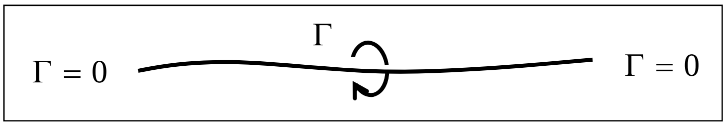
8.2 Hemholtz's laws (…)

• **first Helmholtz' law**

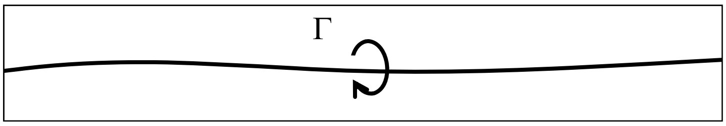
first Helmholtz' law - the circulation of a vortex tube is a constant, independent of the tube section

• **consequences**

✓ impossible

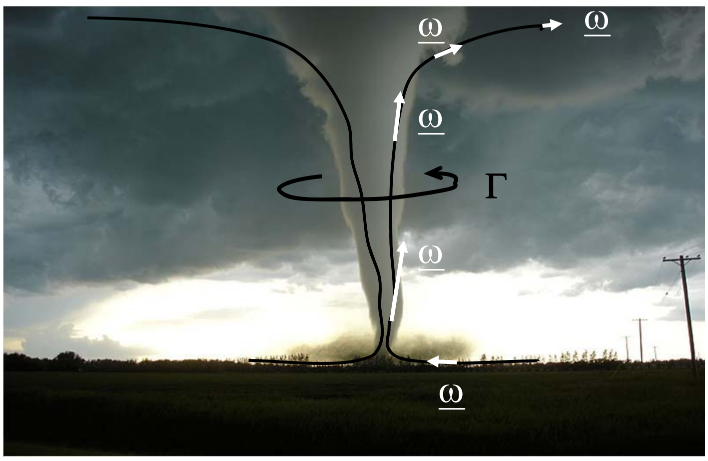


✓ possible



✓ possible

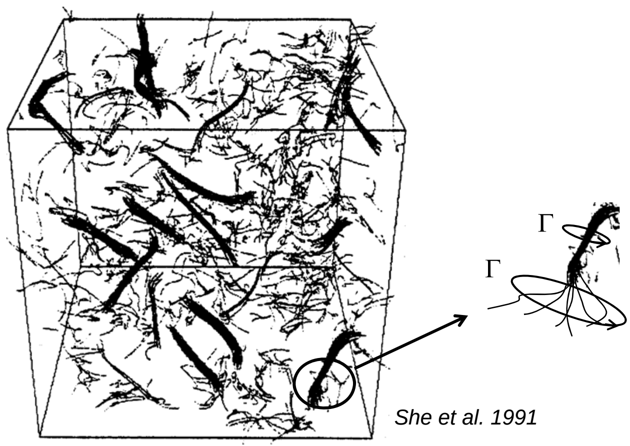
8.2 Hemholtz's laws (…)



8.2 Hemholtz's laws (…)

• **exemple : vorticity in 3D isotropic turbulence**

iso-vorticité $\| \underline{\omega} \|$



She et al. 1991

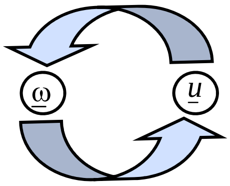
✓ this does not violate Helmholtz laws : $\| \underline{\omega} \|$ is thresholded so we only see the thinner sections of the tubes

chapter 9
vorticity dynamics

9.1 vorticity / velocity

vorticity equation (Helmholtz)

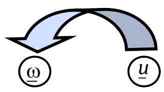
convection
distorsion



création : $\underline{u} = \underline{rot}^{-1}(\underline{\omega})$

Biot-Savart relation

9.2 equation of vorticity (Helmholtz equation)



- **law of dynamics**

$$\frac{d\underline{u}}{dt} = \frac{\partial \underline{u}}{\partial t} + \nabla \underline{u}\,.\,\underline{u} = \underline{f} - \frac{1}{\rho}\underline{grad}\ p + \frac{1}{\rho} div\ \underline{\underline{\tau}} \qquad \textbf{(1)}$$

- **take the curl of (1)**

$$\underline{rot}\left[\frac{d\underline{u}}{dt} = \underbrace{\frac{\partial \underline{u}}{\partial t} + \nabla \underline{u}\,.\,\underline{u}}_{\text{material derivative (acceleration)}} = \underbrace{\underline{f} - \frac{1}{\rho}\underline{grad}\ p + \frac{1}{\rho} div\ \underline{\underline{\tau}}}_{\text{sources}}\right]$$

- **sources : pressure term**

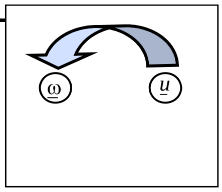
✓ **identities** $\underline{rot}\ \underline{grad}\,(\,.\,) = 0$, $\underline{rot}(a\ \underline{A}) = a\ \underline{rot}\ \underline{A} + \underline{grad}\ (a) \wedge \underline{A}$

$$\left.\begin{array}{l} a = -1/\rho \\ \underline{A} = \underline{grad}\ p \end{array}\right\} \Rightarrow \underline{rot}\left(-\frac{1}{\rho}\underline{grad}\ p\right) = \underbrace{\frac{1}{\rho^2}\underline{grad}\,\rho \wedge \underline{grad}\ p}_{\text{baroclinic production}}$$

- **sources : total**

$$\boxed{\underbrace{\underline{rot}\ \underline{f}}_{\text{production due to volumic forces}} + \underbrace{\frac{1}{\rho^2}\underline{grad}\,\rho \wedge \underline{grad}\ p}_{\text{baroclinic production}} + \underbrace{\underline{rot}\left(\frac{1}{\rho} div\,\underline{\underline{\tau}}\right)}_{\text{viscous diffusion}}}$$

9.2 equation of vorticity (…)



- **material derivative :** $\underline{rot}\,\dfrac{d\underline{u}}{dt} = \underline{rot}\left[\dfrac{\partial \underline{u}}{\partial t} + \nabla \underline{u}\,.\,\underline{u}\right]$

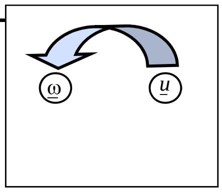
✓ vectorial identities :

$$\left\{\begin{array}{l}
\underline{rot}\,\dfrac{\partial \underline{u}}{\partial t} = \dfrac{\partial}{\partial t}\underline{rot}\,\underline{u} = \dfrac{\partial \underline{\omega}}{\partial t} \\
\nabla \underline{u}\,.\,\underline{u} = \overbrace{\underline{rot}\,\underline{u}}^{\underline{\omega}} \wedge \underline{u} + \underline{grad}\left(\dfrac{1}{2}\underline{u}^2\right) \quad \text{Lamb decomposition} \\
\underline{rot}\,\underline{grad}\left(\dfrac{1}{2}\underline{u}^2\right) = 0 \\
\underline{rot}\left(\underline{\omega} \wedge \underline{u}\right) = \nabla \underline{\omega}.\underline{u} + \underline{\omega}\,div\,\underline{u} - \nabla \underline{u}.\underline{\omega} - \underline{u}\,\cancel{div\,\underline{\omega}}
\end{array}\right.$$

$$\Rightarrow \quad \underline{rot}\,\frac{d\underline{u}}{dt} = \underbrace{\frac{\partial \underline{\omega}}{\partial t} + \nabla \underline{\omega}.\underline{u}}_{d\underline{\omega}/dt} + \underline{\omega}\,div\,\underline{u} - \nabla \underline{u}.\underline{\omega}$$

$$\Rightarrow \quad \boxed{\underline{rot}\,\frac{d\underline{u}}{dt} = \underbrace{\frac{d\underline{\omega}}{dt}}_{\text{advection}} + \underbrace{\underline{\omega}\,div\,\underline{u} - \nabla \underline{u}.\underline{\omega}}_{\text{distorsion : effect of } \underline{u} \text{ on } \underline{\omega}}}$$

9.2 equation of vorticity (…)



- **total**

Helmholtz's equation

$$\boxed{\begin{array}{l}
\overbrace{\dfrac{\partial \underline{\omega}}{\partial t} + \nabla \underline{\omega}.\underline{u}}^{d\underline{\omega}/dt} + \overbrace{\underline{\omega}\,div\,\underline{u} - \nabla \underline{u}.\underline{\omega}}^{\text{distorsion}} \\
= \underbrace{\underline{rot}\,\underline{f}}_{\text{production due to volumic forces}} + \underbrace{\dfrac{1}{\rho^2}\underline{grad}\,\rho \wedge \underline{grad}\,p}_{\text{baroclinic production}} + \underbrace{\underline{rot}\left(\dfrac{1}{\rho}div\,\underline{\underline{\tau}}\right)}_{\text{viscous diffusion}}
\end{array}}$$

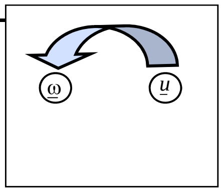
- **compared to Navier-Stokes :**

✓ no pressure

✓ two new mechanisms :

- ➢ baroclinity
- ➢ distorsion

9.3 baroclinity



• **static**
$\underline{u} = 0$

Helmholtz's equation

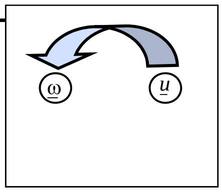
$$\overbrace{\frac{\partial \underline{\omega}}{\partial t} + \cancel{\nabla \underline{\omega}.\underline{u}}}^{d\underline{\omega}/dt} + \overbrace{\cancel{\underline{\omega}\, div\, \underline{u}} - \cancel{\nabla \underline{u}.\underline{\omega}}}^{\text{distorsion}} = \underbrace{\underline{rot}\ \underline{f}}_{\text{production due to volumic forces}} + \underbrace{\frac{1}{\rho^2}\underline{grad}\,\rho \wedge \underline{grad}\, p}_{\text{baroclinic production}} + \underbrace{\underline{rot}\left(\frac{1}{\rho}\cancel{div\,\underline{\underline{\tau}}}\right)}_{\text{viscous diffusion}}$$

• **conservative forces :** $\underline{f} = -\underline{grad}\,\phi$ with, e.g., $\phi = gz$ (gravity)

$\underline{rot}\ \underline{grad}\,\phi = 0$ $\Rightarrow$ conservative forces can not produce a torque

$$\Rightarrow \boxed{\frac{\partial \underline{\omega}}{\partial t} = \underbrace{\frac{1}{\rho^2}\underline{grad}\,\rho \wedge \underline{grad}\, p}_{\text{baroclinic production}}}$$

9.3 baroclinity (…)

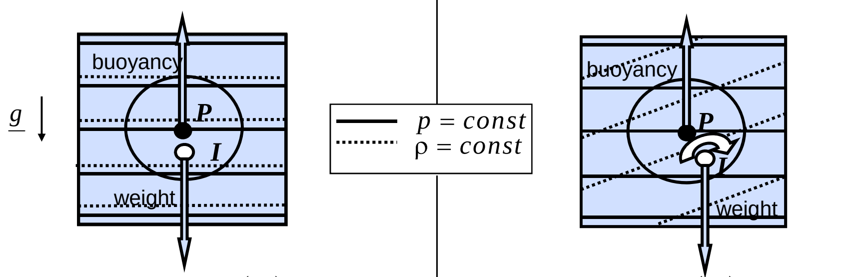


• **static (…)**
$\underline{u} = 0$

$$\boxed{\frac{\partial \underline{\omega}}{\partial t} = \underbrace{\frac{1}{\rho^2}\underline{grad}\,\rho \wedge \underline{grad}\, p}_{\text{baroclinic production}}}$$

$\underline{grad}\,\rho \wedge \underline{grad}\, p = 0 \;\Rightarrow\; \rho = \rho(p)$

$\underline{grad}\,\rho \wedge \underline{grad}\, p \neq 0 \;\Rightarrow\; \rho \neq \rho(p)$



barotropy : $\rho = \rho(p) \Rightarrow \underline{\omega} = 0$

baroclinity : $\rho \neq \rho(p) \Rightarrow \underline{\omega} \neq 0$

$\Rightarrow$ only a barotropic fluid can remain static

9.4 Kelvin's and Lagrange's theorems

- **circulation** $\Gamma = \oint_{C(t)} \underline{u} \cdot \underline{dl}$

✓ hypotheses :
H1 – a surface S (t) based on a closed contour $\mathcal{C}$ (t) which crosses no discontinuity
H2 – ideal fluid (non viscous-non conducting)
H3 – conservative volumic forces

✓ from Helmholtz equation one gets (see annex) : $\frac{d\Gamma}{dt} = \iint_{\mathcal{S}(t)} \big(\overbrace{\cancel{\frac{1}{\rho^2} \underline{grad}\, \rho \wedge \underline{grad}\, p}}^{\text{baroclinic production}} \big) \cdot \underline{n}\, dS$
H4

✓ hypothesis : **H4 –** barotropic fluid : $\rho = \rho\,(p)$

- **Kelvin's theorem**

in in a flow of barotropic ideal fluid subjected to conservative external forces, circulation along a closed material contour which does not cross any discontinuity is constant :

$$\Gamma = \oint_{C(t)} \underline{u} \cdot dl = \iint_{S(t)} \underline{\omega} \cdot \underline{n}\, dS = const.$$

- **Lagrange's theorem (corollary)**

if a flow of barotropic ideal fluid subjected to conservative external forces is irrotational at instant t, it remains irrotationnal

annex – Kelvin's theorem : demonstration

$$\frac{d\Gamma}{dt} = \frac{d}{dt} \oint_{C(t)} (\underline{u} \cdot \underline{dl}) = \oint_{C(t)} \Big[\frac{d\underline{u}}{dt} \cdot \underline{dl} + \underline{u} \cdot \overbrace{\frac{d\,(\underline{dl})}{dt}}^{\underline{u}} \Big]$$

$$= \oint_{C(t)} \Big[\frac{d\underline{u}}{dt} \cdot \underline{dl} + \underline{u} \cdot d\underline{u} \Big] = \oint_{C(t)} \frac{d\underline{u}}{dt} \cdot \underline{dl} + \cancel{\oint_{C(t)} d\left(\frac{1}{2} \underline{u}^2\right)}$$

H1 - no discontinuity

$$\Rightarrow \frac{d\Gamma}{dt} = \oint_{C(t)} \frac{d\underline{u}}{dt} \cdot \underline{dl} = \oint_{C(t)} \Big[\underline{f} - \frac{1}{\rho} \underline{grad}\, p \Big] \cdot \underline{dl}$$

H2 - ideal fluid

Stokes' formulae

$$\Rightarrow \frac{d\Gamma}{dt} = \oint_{C(t)} \Big[\underline{f} - \frac{1}{\rho} \underline{grad}\, p \Big] \cdot \underline{dl} = \iint_{S(t)} \underline{rot} \Big(\cancel{\underline{f}} - \frac{1}{\rho} \underline{grad}\, p \Big) \cdot \underline{n}\, dS$$

H3 - $\underline{f} = -\underline{grad}\, \phi$

✓ identity $\underline{rot}(a\, \underline{A}) = a\, \underline{rot}\, \underline{A} + \underline{grad}\,(a) \wedge \underline{A}$

H4 - barotropy

$$\begin{cases} a = 1/\rho \\ \underline{A} = \underline{grad}\, p \end{cases} \Rightarrow \frac{d\Gamma}{dt} = \iint_{\mathcal{S}(t)} \Big(\frac{1}{\rho^2} \underline{grad}\, \rho \wedge \underline{grad}\, p \Big) \cdot \underline{n}\, dS = 0$$

9.4 Kelvin's and Lagrange's theorems (…)

- **note : first Helmholtz' law ≠ Kelvin's theorem**

first Helmholtz' law ≠ Kelvin's theorem

⇩ circulation of a vorticity tube is the same for the whole tube

⇩ in in a flow of barotropic ideal fluid subjected to conservative external forces, circulation of a vortex tube is constant

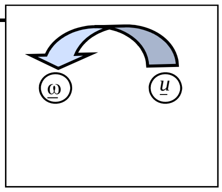
⇩ if either of these assumptions is not satisfied, circulation changes…

… but it remains the same for a given tube

9.5 vorticity distorsion

- **Helmholtz's equation**

$$\overbrace{\frac{\partial \underline{\omega}}{\partial t} + \nabla \underline{\omega} . \underline{u}}^{d\underline{\omega}/dt} + \overbrace{\underline{\omega}\, div\, \underline{u} - \nabla \underline{u} . \underline{\omega}}^{\text{distorsion}}$$
$$= \underbrace{\cancel{\underline{rot}\, \underline{f}}}_{\text{production by volumic forces}} + \underbrace{\frac{1}{\rho^2}\, \cancel{\underline{grad}\, \rho \wedge \underline{grad}\, p}}_{\text{baroclinic production}} + \underbrace{\cancel{\underline{rot}\left(\frac{1}{\rho}\, \underline{div}\, \underline{\underline{\tau}}\right)}}_{\text{viscous diffusion}}$$



- **flow of a barotropic ideal fluid with conservative forces**

$$\boxed{\frac{d\underline{\omega}}{dt} + \overbrace{\underbrace{\underline{\omega}\, div\, \underline{u}}_{\text{compressibility}} - \underbrace{\nabla \underline{u} . \underline{\omega}}_{\text{stretching}}}^{\text{distorsion of } \underline{\omega}} = 0} \quad (1)$$

⇨ if $\underline{\omega}(t) = 0$, $\underline{\omega}(t' > t) = 0$ Lagrange's theroem

- **compressibility :** $div\,\underline{u} = -\frac{1}{\rho}\frac{d\rho}{dt}$ ⇨ $\frac{\underline{\omega}}{\rho}\, div\,\underline{u} = -\frac{\underline{\omega}}{\rho^2}\frac{d\rho}{dt} = \underline{\omega}\,\frac{d}{dt}\left(\frac{1}{\rho}\right)$

$$\frac{(1)}{\rho} \Rightarrow \frac{1}{\rho}\frac{d\underline{\omega}}{dt} + \frac{\underline{\omega}}{\rho}\, div\, \underline{u} = \frac{1}{\rho}\frac{d\underline{\omega}}{dt} + \underline{\omega}\,\frac{d}{dt}\left(\frac{1}{\rho}\right) = \boxed{\frac{d}{dt}\left(\frac{\underline{\omega}}{\rho}\right) = \underbrace{\nabla \underline{u} . \frac{\underline{\omega}}{\rho}}_{\text{distorsion}}}$$

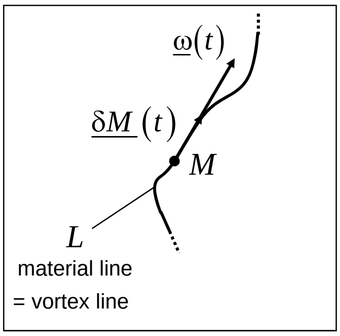
9.5 vorticity distorsion (…)

• **consequence : the material nature of vorticity**

✓ vorticity distorsion (1) $\frac{d}{dt}\left(\frac{\underline{\omega}}{\rho}\right) = \nabla \underline{u} \, . \frac{\underline{\omega}}{\rho}$

✓ kinematics : distorsion of material elements (2) $\frac{d}{dt}\underline{\delta M} = \nabla \underline{u} \, . \, \underline{\delta M}$

(1) - (2) ⇨ $\boxed{\frac{\underline{\omega}}{\rho} = const. \times \underline{\delta M}}$



⇨ in a flow of barotropic ideal fluid with conservative forces, vorticity weighted by fluid density behaves as fluid elements

✓ incompressible and homogeneous fluid flows

in a flow of incompressible homogeneous fluid ($\rho = const.$) this is vorticity that has this property

✓ note : an incompressible homogeneous fluid ($\rho = const.$) is a particular barotropic fluid no baroclinity : $\underline{grad}\,\rho \wedge \underline{grad}\,p = 0$)

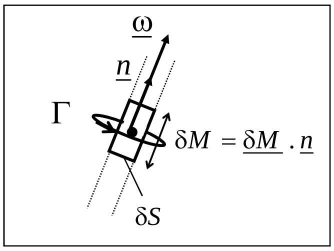
9.5 vorticity distorsion (…)

• **interpretation : conservation of angular momentum**

$\frac{\underline{\omega}}{\rho} = const. \times \underline{\delta M}$ ⇨ $\underline{\omega} = const. \times \rho \times \underline{\delta M}$

✓ let consider an elementary vortex tube

$$\underbrace{\underline{\omega} \, . \, \underline{n} \, \delta S}_{\Gamma \text{ circulation}} = const. \times \underbrace{\rho \times \underline{\delta M} \, . \, \underline{n} \, \delta S}_{\delta m \text{ masse = const.}}$$

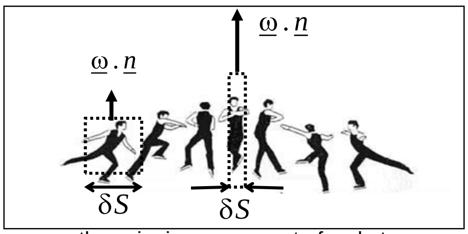


⇨ $\boxed{\Gamma = \underline{\omega} \, . \, \underline{n} \, \delta S = const.}$ Kelvin's theorem

✓ conservation of angular momentum

$$\boxed{I \, \dot{\theta} = const.}$$

δS moment of inertia — $\underline{\omega} \, . \, \underline{n}$ rotation rate — Γ angular momentum



the spinning movement of a skater

9.6 viscous diffusion of vorticity

- **Helmoltz's equation : general**

$$\boxed{\frac{d\underline{\omega}}{dt} + \underline{\omega}\,div\,\underline{u} - \nabla\underline{u}\,.\,\underline{\omega} = \underline{rot}\;\underline{f} + \frac{1}{\rho^2}\underline{grad}\,\rho \wedge \underline{grad}\;p + \underline{rot}\left[\,\frac{1}{\rho}div\,\underline{\underline{\tau}}\,\right]}$$

- **newtonian fluids**

✓ newtonian :

$$\underline{\underline{\tau}} = \kappa\,div\,\underline{u}\;\underline{\underline{1}} + 2\,\eta\,\left[\,\underline{\underline{d}} - \frac{1}{3}div(\underline{u})\underline{\underline{1}}\,\right]$$

✓ viscous term (see annex… awful)

$$\underline{rot}\left[\,\frac{1}{\rho}div\,\underline{\underline{\tau}}\,\right] = \nu\;\Delta\underline{\omega} + \frac{2}{\rho}\cancel{\underline{grad}}\,\eta \wedge \left(\,\Delta\underline{u} + \underline{grad}\;div\,\underline{u}\,\right) +$$

$$+\cancel{\underline{grad}\left(\frac{1}{\rho}\right)} \wedge \left[\,\eta\left(\,\Delta\underline{u} + \underline{grad}\;div\,\underline{u}\,\right) + 2\underline{\underline{d}}\,.\,\underline{grad}\;\eta + \underline{grad}\,(\lambda\,div\,\underline{u})\,\right]$$

- **simplifications**
  - incompressibilty : $div\,\underline{u} = 0$
  - homogeneity : $\rho = const.$
  - constant viscosity : $\eta = const.$
  - conservative forces : $\underline{f} = -\underline{grad}\;\phi$

$$\Rightarrow\quad \boxed{\underline{rot}\left[\,\frac{1}{\rho}div\,\underline{\underline{\tau}}\,\right] = \nu\;\Delta\underline{\omega}}$$

annex – viscous term : demonstration

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- general $\underline{rot}\left(\,\frac{1}{\rho}div\,\underline{\underline{\tau}}\,\right)$
- identity : $\underline{rot}\left(a\,\underline{A}\right) = a\,\underline{rot}\;\underline{A} + \underline{grad}\;a \wedge \underline{A}$

$$\Rightarrow\quad \underline{rot}\left(\,\frac{1}{\rho}div\,\underline{\underline{\tau}}\,\right) = \frac{1}{\rho}\,\underline{rot}\;div\,\underline{\underline{\tau}} + \underline{grad}\left(\,\frac{1}{\rho}\,\right)\wedge div\,\underline{\underline{\tau}}$$

- newtonian fluid $\underline{\underline{\tau}} = \kappa\,div\,\underline{u}\;\underline{\underline{1}} + 2\,\eta\,\left[\,\underline{\underline{d}} - \frac{1}{3}div(\underline{u})\underline{\underline{1}}\,\right]$
- other form $\underline{\underline{\tau}} = \lambda\,div\,\underline{u}\;\underline{\underline{1}} + 2\,\eta\,\underline{\underline{d}}$ with $\lambda = \kappa - \frac{2}{3}\eta$
- identity : $div\left(a\,\underline{\underline{T}}\right) = a\,div\,\underline{\underline{T}} + \underline{\underline{T}}\,.\,\underline{grad}\;a$

$$\Rightarrow\quad \begin{cases} div\left(\,\lambda\,div\,\underline{u}\;\underline{\underline{1}}\,\right) = \lambda\,div\,\underline{u}\;\cancel{div\,\underline{\underline{1}}} + \underline{\underline{1}}.\underline{grad}\left(\,\lambda\,div\,\underline{u}\,\right) = \underline{grad}\left(\,\lambda\,div\,\underline{u}\,\right) \\ div\left(\,2\,\eta\,\underline{\underline{d}}\,\right) = 2\,\eta\,div\,\underline{\underline{d}} + 2\underline{\underline{d}}\,.\,\underline{grad}\;\eta \end{cases}$$

$$\Rightarrow\quad div\,\underline{\underline{\tau}} = 2\,\eta\,div\,\underline{\underline{d}} + 2\underline{\underline{d}}\,.\,\underline{grad}\;\eta + \underline{grad}\,(\lambda\,div\,\underline{u})$$

- identity : $\underline{rot}\left(a\,\underline{A}\right) = a\,\underline{rot}\,\underline{A} + \underline{grad}\;a \wedge \underline{A}$

$$\Rightarrow\quad \underline{rot}\left[\,\frac{1}{\rho}div\,\underline{\underline{\tau}}\,\right] = \frac{1}{\rho}\,\underline{rot}\;div\,\underline{\underline{\tau}} + \underline{grad}\left(\,\frac{1}{\rho}\,\right)\wedge div\,\underline{\underline{\tau}}$$

annex – viscous term : demonstration (…)

• total

$$\begin{cases} \underline{rot}\left[\dfrac{1}{\rho} div\,\underline{\underline{\tau}}\right] = \dfrac{1}{\rho}\underline{rot}\; div\,\underline{\underline{\tau}} + \underline{grad}\left(\dfrac{1}{\rho}\right)\wedge div\,\underline{\underline{\tau}} \\ div\,\underline{\underline{\tau}} = 2\,\eta\, div\,\underline{\underline{d}} + 2\underline{\underline{d}}\,.\,\underline{grad}\;\eta + \underline{grad}\left(\lambda\, div\,\underline{u}\right) \end{cases}$$

$$\Rightarrow \quad \underline{rot}\left[\dfrac{1}{\rho} div\,\underline{\underline{\tau}}\right] = \dfrac{1}{\rho}\underline{rot}\left[\,2\,\eta\, div\,\underline{\underline{d}} + 2\underline{\underline{d}}\,.\,\underline{grad}\;\eta + \underline{grad}\left(\lambda\, div\,\underline{u}\right)\right]$$
$$+\underline{grad}\left(\dfrac{1}{\rho}\right)\wedge\left[\,2\,\eta\, div\,\underline{\underline{d}} + 2\underline{\underline{d}}\,.\,\underline{grad}\;\eta + \underline{grad}\left(\lambda\, div\,\underline{u}\right)\right]$$

• identities :

$$\begin{cases} \underline{rot}\;\underline{grad}\; a = 0 \\ \underline{rot}\left(a\,\underline{A}\right) = a\;\underline{rot}\,\underline{A} + \underline{grad}\; a\wedge\underline{A} \end{cases}$$

$$\Rightarrow \quad \underline{rot}\left[\dfrac{1}{\rho} div\,\underline{\underline{\tau}}\right] = \dfrac{2\eta}{\rho}\underline{rot}\; div\,\underline{\underline{d}} + \dfrac{2}{\rho}\underline{grad}\;\eta\wedge div\,\underline{\underline{d}} +$$
$$+\underline{grad}\left(\dfrac{1}{\rho}\right)\wedge\left[\,2\,\eta\, div\,\underline{\underline{d}} + 2\underline{\underline{d}}\,.\,\underline{grad}\;\eta + \underline{grad}\left(\lambda\, div\,\underline{u}\right)\right]$$

• identities :

$$\begin{cases} div\,\nabla\underline{u} = \Delta\underline{u} \\ div\,{}^t\nabla\underline{u} = \underline{grad}\; div\,\underline{u} \end{cases}$$

$$\Rightarrow \quad 2div\,\underline{\underline{d}} = div\left(\nabla\underline{u} + {}^t\nabla\underline{u}\right) = \Delta\underline{u} + \underline{grad}\; div\,\underline{u}$$

annex – viscous term : demonstration (…)

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• total

$$\begin{cases} \underline{rot}\left[\dfrac{1}{\rho} div\,\underline{\underline{\tau}}\right] = \dfrac{2\eta}{\rho}\underline{rot}\; div\,\underline{\underline{d}} + \dfrac{2}{\rho}\underline{grad}\;\eta\wedge div\,\underline{\underline{d}} + \\ \qquad +\underline{grad}\left(\dfrac{1}{\rho}\right)\wedge\left[\,2\,\eta\, div\,\underline{\underline{d}} + 2\underline{\underline{d}}\,.\,\underline{grad}\;\eta + \underline{grad}\left(\lambda\, div\,\underline{u}\right)\right] \\ 2div\,\underline{\underline{d}} = div\left(\nabla\underline{u} + {}^t\nabla\underline{u}\right) = \Delta\underline{u} + \underline{grad}\; div\,\underline{u} \end{cases}$$

$$\Rightarrow \quad \underline{rot}\left[\dfrac{1}{\rho} div\,\underline{\underline{\tau}}\right] = \nu\;\underline{rot}\;\Delta\underline{u} + \dfrac{2}{\rho}\underline{grad}\;\eta\wedge\left(\,\Delta\underline{u} + \underline{grad}\; div\,\underline{u}\,\right) +$$
$$+\underline{grad}\left(\dfrac{1}{\rho}\right)\wedge\left[\,\eta\left(\,\Delta\underline{u} + \underline{grad}\; div\,\underline{u}\,\right) + 2\underline{\underline{d}}\,.\,\underline{grad}\;\eta + \underline{grad}\left(\lambda\, div\,\underline{u}\right)\right]$$

• identity : $\underline{rot}\left(\Delta\underline{A}\right) = \Delta\left(\underline{rot}\;\underline{A}\right)$

$$\Rightarrow \quad \boxed{\begin{aligned} &\underline{rot}\left[\dfrac{1}{\rho} div\,\underline{\underline{\tau}}\right] = \nu\;\;\Delta\underline{\omega} + \dfrac{2}{\rho}\underline{grad}\;\eta\wedge\left(\,\Delta\underline{u} + \underline{grad}\; div\,\underline{u}\,\right) + \\ &\qquad +\underline{grad}\left(\dfrac{1}{\rho}\right)\wedge\left[\,\eta\left(\,\Delta\underline{u} + \underline{grad}\; div\,\underline{u}\,\right) + 2\underline{\underline{d}}\,.\,\underline{grad}\;\eta + \underline{grad}\left(\lambda\, div\,\underline{u}\right)\right] \end{aligned}}$$

9.7 the equation of vorticity for a flow of a newtonian incompressible homogeneous fluid of constant viscosity with conservative forces

• **Helmoltz's equation : general**

$$\boxed{\frac{d\underline{\omega}}{dt} + \underline{\omega}\, div\, \underline{u} - \nabla \underline{u}\,.\,\underline{\omega} = \underline{rot}\ \underline{f} + \frac{1}{\rho^2}\underline{grad}\ \rho \wedge \underline{grad}\ p + \underline{rot}\left[\frac{1}{\rho} div\, \underline{\underline{\tau}}\right]}$$

• **simplifications**

- incompressibilty $div(\underline{u}) = 0$
- homogeneity $\rho = const.$
- constant viscosity $\eta = const.$
- conservative forces $\underline{f} = -\underline{grad}\ \phi$

$\Rightarrow$ $$\boxed{\frac{d\underline{\omega}}{dt} = \frac{\partial \underline{\omega}}{\partial t} + \underbrace{\nabla \underline{\omega}\,.\,\underline{u}}_{\text{convection}} = \underbrace{\nabla \underline{u}\,.\,\underline{\omega}}_{\text{distorsion}} + \underbrace{\nu\ \Delta \underline{\omega}}_{\text{diffusion}}}$$

$\Rightarrow$ from now we will only consider this type of flows

chapter 10

Biot-Savart's induction law

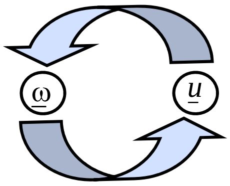
10.1 vorticity / velocity

10.2 Biot-Savart relation

10.1 vorticity / velocity

vorticity equation (Helmholtz)

convection
distorsion



création

Biot-Savart relation $\underline{u} = \underline{rot}^{-1}(\underline{\omega})$

10.2 Biot-Savart relation

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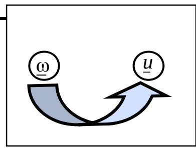


• **solution** (see annex)

particular solution of $\begin{cases} div\,\underline{u} = 0 \\ \underline{rot}\,\underline{u} = \underline{\omega} \end{cases}$

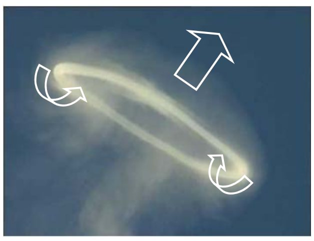
$$\boxed{\underline{u}(\underline{x}) = -\frac{1}{4\pi}\iiint_\Omega \frac{\underline{x}-\underline{x}'}{\|\underline{x}-\underline{x}'\|^3} \wedge \underline{\omega}(\underline{x}')\,d\Omega \quad + \underline{grad}\,\varphi}$$

solution of the homogeneous problem

$\begin{cases} div\,\underline{u} = 0 \\ \underline{rot}\,\underline{u} = 0 \end{cases} \Rightarrow \Delta\varphi = 0$

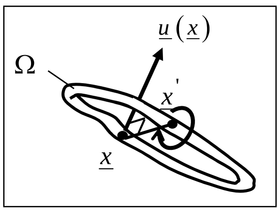
this is zero in an unbounded fluid

• **exemple**



self propelled vortex ring

=



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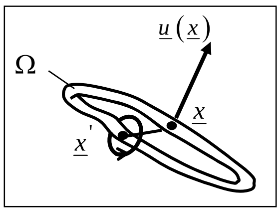
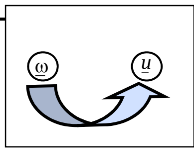


annex – Biot-Savart : demonstration



- **hypotheses**
  - ✓ rotationnel flow in a bounded domain Ω
  - ✓ incompressibility : $div\,\underline{u} = 0$
- **method**
  - ✓ identity : $div\left(\underline{rot}\,\underline{A}\right) = 0 \Longrightarrow$ introducing a potential vector $\underline{A}$: $\boxed{\underline{u} = \underline{rot}\,\underline{A}}$ (1)
  - ✓ unicity : $div\,\underline{A} = 0$
  - ✓ équation : $\underline{\omega} = \underline{rot}\,\underline{u} = \underline{rot}\left(\underline{rot}\,\underline{A}\right)$
  - ✓ identity : $\Delta\underline{A} = \underline{grad}\left(\cancel{div\,\underline{A}}\right) - \underline{rot}\left(\underline{rot}\,\underline{A}\right) = -\underline{rot}\,\underline{u} = -\underline{\omega}$

$\Longrightarrow$ $\boxed{\Delta\underline{A} = -\underline{\omega}}$ (2)

Poisson equation

$\Longrightarrow$ given $\underline{\omega}$: solve (2) and use (1) to get $\underline{u}$

annex – Biot-Savart : demonstration

- solution (principle)

$$\Delta\underline{A} = -\underline{\omega} \Longleftarrow \begin{cases} \Delta G\left(\underline{x}\right) = \delta\left(\underline{x}\right) & \text{fondamental equation} \\ \Longrightarrow G\left(\underline{x}\right) = -\dfrac{1}{4\pi\|\underline{x}\|} & \text{Green's kernel} \end{cases}$$

$$\Downarrow$$

$$\underline{A}\left(\underline{x}\right) = -\underline{\omega} * G = \frac{1}{4\pi}\iiint_\Omega \frac{\underline{\omega}\left(\underline{x}'\right)}{\|\underline{x}-\underline{x}'\|}\,d\Omega$$

$$\Longrightarrow \underline{u}\left(\underline{x}\right) = \underline{rot}_{\underline{x}}\,\underline{A} = \frac{1}{4\pi}\iiint_\Omega \underline{rot}_{\underline{x}}\frac{\underline{\omega}\left(\underline{x}'\right)}{\|\underline{x}-\underline{x}'\|}\,d\Omega$$

- identity : $\underline{rot}_{\underline{x}}\left(a\,\underline{A}\right) = a\,\underline{rot}_{\underline{x}}\,\underline{A} + \underline{grad}_{\underline{x}}\,a \wedge \underline{A}$

$$\begin{cases} a = 1/\|\underline{x}-\underline{x}'\| \\ \underline{A} = \underline{\omega}\left(\underline{x}'\right) \end{cases} \Longrightarrow \underline{rot}_{\underline{x}}\frac{\underline{\omega}\left(\underline{x}'\right)}{\|\underline{x}-\underline{x}'\|} = \underline{grad}_{\underline{x}}\frac{1}{\|\underline{x}-\underline{x}'\|} \wedge \underline{\omega}\left(\underline{x}'\right)$$

- one has : $\underline{grad}_{\underline{x}}\dfrac{1}{\|\underline{x}-\underline{x}'\|} = -\dfrac{\underline{grad}_{\underline{x}}\|\underline{x}-\underline{x}'\|}{\|\underline{x}-\underline{x}'\|^2} = -\dfrac{\underline{x}-\underline{x}'}{\|\underline{x}-\underline{x}'\|^3}$
- solution $\underline{u}\left(\underline{x}\right) = -\dfrac{1}{4\pi}\iint_{\Omega'}\dfrac{\underline{x}-\underline{x}'}{\|\underline{x}-\underline{x}'\|^3} \wedge \underline{\omega}\left(\underline{x}'\right)d\Omega$

this is a particular solution of $\left(div\,\underline{u} = 0,\ \underline{rot}\,\underline{u} = \underline{\omega}\right)$ to which one must had the solution $\underline{u}\left(\underline{x}\right) = \underline{grad}\,\varphi$ of the homogeneous problem $\left(div\,\underline{u} = 0,\ \underline{rot}\,\underline{u} = 0\right)$

chapter 11

vorticity in 2D flows

11.1 2D flows

11.2 2D Biot-Savart relation

11.3 vortex patches

11.4 cascade of enstrophy – conservation of energy

11.5 an inverse cascade of energy

11.6 3D → 2D dynamics : a first inventory

11.7 chaos

10.8 2D dynamics : conclusion

10.9 2D-3D dynamics

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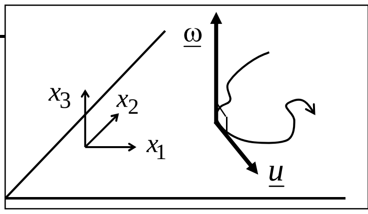
11.1 2D flows



• **the case of 2D flows**

✓ in a 2D flow, velocity gradients are limited to a plane and vorticity has only one component, normal to the plane

$\underline{u} = (u_1, u_2, 0)(x_1, x_2)$ 2 components

$$\nabla \underline{u} = \begin{pmatrix} \partial u_1/\partial x_1 & \partial u_1/\partial x_2 & 0 \\ \partial u_2/\partial x_1 & \partial u_2/\partial x_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ 4 components

$$\underline{\omega} = \left(0, 0, \omega = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right)$$ 1 component

$\Rightarrow$ $\boxed{\nabla \underline{u} \, . \, \underline{\omega} = 0}$ no vorticity distorsion

• **Helmholtz** $\boxed{\dfrac{d\,\underline{\omega}}{dt} = \cancel{\nabla \underline{u} . \underline{\omega}} + \nu \, \Delta \underline{\omega}}$ $\Rightarrow$ $\boxed{\dfrac{d\,\underline{\omega}}{dt} = \dfrac{\partial \underline{\omega}}{\partial t} + \nabla \underline{\omega} . \underline{u} = \nu \, \Delta \underline{\omega}}$

convection - diffusion equation

remember : valid for a flow of newtonian, incompressible, homogeneous fluid with constant viscosity and conservative forces

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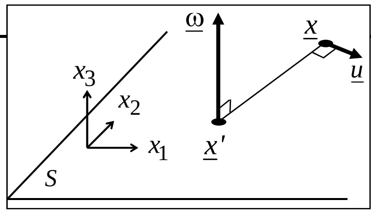
11.2 2D Biot-Savart relation



- **2D flows :** $\dfrac{\partial}{\partial x_3} = 0\,,\quad \underline{u} = \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix}$

✓ one can show that the 3D Biot-Savart relation

$$\underline{u}\left(\underline{x}\right) = -\frac{1}{4\pi}\iiint_{\Omega} \frac{\underline{x}-\underline{x}'}{\left\|\underline{x}-\underline{x}'\right\|^3} \wedge \underline{\omega}\left(\underline{x}'\right) d\Omega$$

reads (see annex) : $$\underline{u}\left(\underline{x}\right) = -\frac{1}{2\pi}\iint_{S} \frac{\underline{x}-\underline{x}'}{\left\|\underline{x}-\underline{x}'\right\|^2} \wedge \omega\left(\underline{x}'\right) \underline{e}_z dS$$

annex – Biot-Savart 2D : demonstration

✓ 2D flow **:** $\dfrac{\partial}{\partial x_3} = 0\,,\quad \underline{u} = \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix}$

✓ streamfunction $\underline{u} = \nabla\psi \wedge \underline{e}_z \Longrightarrow u_1 = \dfrac{\partial\psi}{\partial x_2}\,, u_2 = -\dfrac{\partial\psi}{\partial x_1}$ (1)

✓ vorticity $\underline{\omega} = \underline{rot}\ \underline{u} = \omega\, \underline{e}_3 = -\Delta\psi \Longrightarrow \Delta\psi = -\omega$ (2)

- solution of (2) :

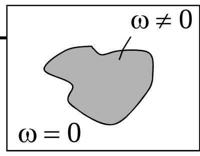
Green $\Delta G = \delta\left(\underline{x}\right) \;\Rightarrow\; G\left(\underline{x}\right) = \dfrac{Ln\left\|\underline{x}\right\|}{2\pi}$

so : $\psi\left(\underline{x}\right) = -\underset{\substack{\uparrow \\ \text{convolution}}}{\omega * G} = -\dfrac{1}{2\pi}\iint \omega\left(\underline{x}'\right) Ln\left\|\underline{x}-\underline{x}'\right\| dS$ (3)

✓ substituing (3) in (1), with : $\nabla_{\underline{x}} Ln\left\|\underline{x}-\underline{x}'\right\| = \frac{1}{2}\nabla_{\underline{x}}\left[\left(x_1 - x_1'^2\right)^2 + \left(x_2 - x_2'\right)^2\right] = \dfrac{\underline{x}-\underline{x}'}{\left\|\underline{x}-\underline{x}'\right\|^2}$

gives the Biot-Savart formulae $\underline{u}\left(\underline{x}\right) = \dfrac{1}{2\pi}\iint_S \dfrac{\omega\left(\underline{x}'\right)\underline{e}_z \wedge \left(\underline{x}-\underline{x}'\right)}{\left\|\underline{x}-\underline{x}'\right\|^2} dS$

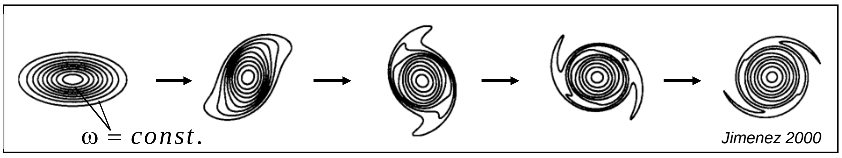
11.3 vortex patches



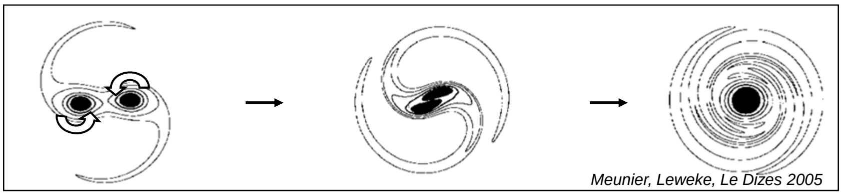
- **vortex patch = 2D bounded vorticity regions**

✓ bounded vorticity = exponential decay of ω away from the patch

✓ an elliptic patch

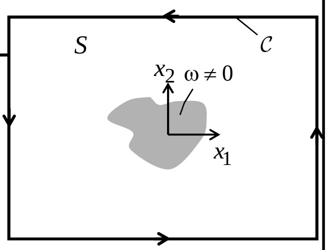


✓ two co-rotating vortices



- **how explain these 2D flow vorticity dynamics ?**
- **2 key notions**
  - ✓ movement invariants
  - ✓ enstrophy cascade

11.3 vortex patches (…)



- **vortex patch**

✓ circulation $\quad \Gamma = \iint_S \omega \, dS \overset{\text{Stokes formulae}}{=} \oint_{\mathcal{C}} \underline{u} \cdot \underline{dl}$

Kelvin theorem : $\quad d\Gamma/dt = 0 \quad$ (see also annex)

- **invariants of the movement**

✓ linear momentum $\quad \boxed{\underline{I} = \iint_S \underline{x} \wedge \underline{\omega} \, dS} \quad \underline{x} \wedge \underline{\omega} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} = \begin{pmatrix} x_2\omega \\ -x_1\omega \\ 0 \end{pmatrix} \Rightarrow \boxed{\begin{cases} I_1 = \iint_S x_2\omega \, dS \\ I_2 = -\iint_S x_1\omega \, dS \end{cases}}$

⇨ this is the flow momentum per unit mass

one can show that : $\quad \boxed{d\underline{I}/dt = 0} \quad$ momentum conservation (see annex)

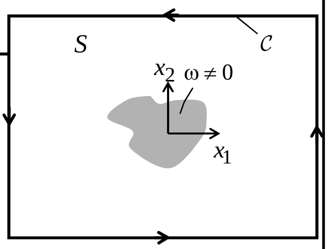
✓ angular momentum $\quad \boxed{\underline{M} = \iint_S \underline{x} \wedge (\underline{x} \wedge \underline{\omega}) dS} \Rightarrow \boxed{\begin{cases} \underline{M} = -M \underline{e}_3 \\ M = \iint_S (\, x_1^2 + x_2^2 \,) \omega \, dS \end{cases}}$

one can show that : $\quad \boxed{dM/dt = -2\nu\Gamma} \quad$ (see annex : hard)

so $\quad \boxed{\lim_{\mathrm{Re} \to \infty} dM/dt = 0}$

- **Reynolds number** $\quad \mathrm{Re} = \oint_{\mathcal{C}} \underline{u} \cdot \underline{dl} \big/ \nu = \Gamma/\nu$

11.3 vortex patches (…)



- **summary**

given vortex patches of circulation $\Gamma = const.$, invariants of the movement are :

✓ linear momentum : $\begin{cases} I_1 = \iint_S x_2 \omega \; dS = const. \\ I_2 = -\iint_S x_1 \omega \; dS = const. \end{cases}$

✓ angular momentum : $M = \iint_S (x_1^2 + x_2^2) \omega \, dS$

$$\lim_{\mathrm{Re}=\Gamma/\nu >> 1} dM/dt = 0$$

✓ to which one must add energy, as defined later

11.3 vortex patches (…)

⇨ so, one may characterize the spatial evolution of any 2D vortex distribution by considering the two following quantities

- **centroid**

✓ linear moment $\begin{cases} I_1 = \iint_S x_2 \, \omega \; dS = const. \\ I_2 = -\iint_S x_1 \, \omega \; dS = const. \end{cases}$ ⇨ $\begin{cases} x_{1c} = \iint_S x_1 \, \omega \, dS \Big/ \iint_S \omega \, dS = -I_2/\Gamma = const. \\ x_{2c} = \iint_S x_2 \, \omega \, dS \Big/ \iint_S \omega \, dS = I_1/\Gamma = const. \end{cases}$

⇨ $\boxed{\underline{x}_c \left(x_{1c}, x_{2c} \right) = \underline{x}_c \left(\frac{-I_2}{\Gamma}, \frac{I_1}{\Gamma} \right)}$

centroid

the centroid of the vorticity field is conserved

⇨ take it as the origin

- **dispersion radius**

✓ angular momentum $M = \iint_S \left(x_1^2 + x_2^2 \right) \omega \, dS = \iint_S \left\| \underline{x} - \underline{x}_c \right\|^2 \omega \, dS$

⇨ $\boxed{a^2 = \dfrac{\iint_S \left\| \underline{x} - \underline{x}_c \right\|^2 \omega \, dS}{\iint_S \omega \, dS} = \dfrac{M}{\Gamma}}$

a = dispersion radius

the dispersion of vorticity around the centroid is characterized by radius *a*

annex – 2D invariants : demonstrations

- circulation $\Gamma = \iint_S \omega \, dS$

H1 - incompressibility
H2 - homogeneity (barotropy)
H3 - constant viscosity
H4 - conservative forces
H5 - 2D

✓ vorticity $\dfrac{\partial \underline{\omega}}{\partial t} = -\nabla\, \underline{\omega}.\underline{u} + \nu \Delta \underline{\omega}$

✓ identities
$$\begin{cases} \underline{rot}\left(\underline{\omega} \wedge \underline{u}\right) = \nabla\, \underline{\omega}.\underline{u} + \underline{u}\ \cancel{div\,\underline{\omega}} - \underline{\omega}\ \overset{H1}{\cancel{div\,\underline{u}}} - \nabla\, \overset{H5}{\cancel{\underline{u}}}.\underline{\omega} \quad = \nabla\, \underline{\omega}.\underline{u} \\ \underline{rot}\,\underline{rot}\,\underline{\omega} = \underline{grad}\ \cancel{div\,\underline{\omega}} - \Delta \underline{\omega} \end{cases}$$

$$\Rightarrow \quad \frac{\partial \underline{\omega}}{\partial t} = \underline{rot}\left(\underline{u} \wedge \underline{\omega} - \nu\, \underline{rot}\, \underline{\omega}\right) \qquad \textbf{(1)}$$

$$\textbf{(1)} \quad \Rightarrow \quad \frac{d}{dt}\Gamma = \iint_S \frac{\partial \omega}{\partial t} dS = \iint_S \underline{rot}\left(\underline{u} \wedge \underline{\omega} - \nu\, \underline{rot}\, \underline{\omega}\right) dS = \oint_{\mathcal{C}} \left(\underline{u} \wedge \underline{\omega} - \nu\, \underline{rot}\, \underline{\omega}\right) \wedge \underline{n}\, dl = 0$$

$\boxed{\underline{\omega} = 0 \text{ on } \mathcal{C}}$

Note - The flow being that of a Newtonian, incompressible, homogenous (barotropic) fluid of constant viscosity with conservative forces + ideal fluid on the contour $\mathcal{C}$. All conditions for application of Kelvin's theorem (valid also for 3D flows) are thus fulfilled. The above demonstration is thus an alternative to that of Kelvin's theorem for the case of 2D flows

annex - 2D invariants : demonstrations (…)

- linear momentum $\underline{I} = \iint_S \underline{x} \wedge \underline{\omega}\, dS$

$$\begin{cases} \underline{\omega} = \omega\, \underline{e}_z \\ \underline{x} = x\, \underline{e}_x + y\, \underline{e}_y \end{cases} \Rightarrow \quad \underline{I} = \iint_S \omega\left(\underline{x} \wedge \underline{e}_z\right) dS = \iint_S \omega\, x\left(\underline{e}_x \wedge \underline{e}_z\right) dS + \iint_S \omega\, y\left(\underline{e}_y \wedge \underline{e}_z\right) dS$$
$$= -\underline{e}_y \underbrace{\iint_S \omega\, x\, dS}_{I_y} + \underline{e}_x \underbrace{\iint_S \omega\, y\, dS}_{I_x}$$

- angular momentum $\underline{M} = \iint_S \underline{x} \wedge \left(\underline{x} \wedge \underline{\omega}\right) dS$

$$\begin{cases} \underline{\omega} = \omega\, \underline{e}_z \\ \underline{x} = x\, \underline{e}_x + y\, \underline{e}_y \end{cases} \Rightarrow \quad \underline{M} = \iint_S \omega\, \underline{x} \wedge \left(\underline{x} \wedge \underline{e}_z\right) dS = \iint_S \left[-\omega\left(\underline{x} \wedge \underline{e}_y\right) + \omega\left(\underline{x} \wedge \underline{e}_x\right) \right] dS$$
$$= -\iint_S \omega\left(x^2 + y^2\right) dS = -M\, \underline{e}_z$$

$$\Rightarrow \quad M = \iint_S \omega\left(x^2 + y^2\right) dS$$

annex – 2D invariants : demonstrations (…)

- **linear moment** $\boxed{\underline{I} = \iint_S \underline{x} \wedge \underline{\omega}\, dS}$

✓ vorticity (1) $\Rightarrow$ $\frac{d}{dt}\underline{I} = \iint_S \underline{x} \wedge \frac{\partial \underline{\omega}}{\partial t} dS = \iint_S \underline{x} \wedge \underline{rot}\left(\underline{u} \wedge \underline{\omega} - \nu\, \underline{rot}\, \underline{\omega}\right) dS$

✓ identity $\underline{B} = \iint_S \underline{x} \wedge \underline{rot}\, \underline{A}\, dS = \iint_S \underline{A}\, dS - \oint_{\mathcal{C}} \underline{x} \wedge \left(\underline{n} \wedge \underline{A}\right) dl$ see demo below

✓ on $\mathcal{C}$: $\underline{\omega} = 0 \Rightarrow \underline{A} = \underline{u} \wedge \underline{\omega} - \nu\, \underline{rot}\, \underline{\omega} = 0 \Rightarrow \frac{d}{dt}\underline{I} = \iint_S \left(\underline{u} \wedge \underline{\omega} - \nu\, \underline{rot}\, \underline{\omega}\right) dS$

✓ identity $\begin{cases} \underline{u} \wedge \underline{\omega} = \underline{grad}\left(u^2/2 \right) - \nabla \underline{u}.\underline{u} \\ \nabla \underline{u}.\underline{u} = div\left(\underline{u} \otimes \underline{u}\right) - \underline{u}\ \cancel{div\, \underline{u}} \end{cases}$ (see Lamb decomposition)

$$\Rightarrow \frac{d\underline{I}}{dt} = \iint_S \left[div\left(\tfrac{1}{2}\underline{u}^2 \underline{\underline{1}} - \underline{u} \otimes \underline{u} \right) - \nu\, \underline{rot}\, \underline{\omega} \right] dS = \oint_{\mathcal{C}} \left[\tfrac{1}{2}\underline{u}^2\, \underline{n} - \underline{u}\left(\underline{u}.\underline{n}\right) - \nu\, \underline{\omega} \wedge \underline{n} \right] dl$$

✓ $\underline{u} \sim 1/r$ (Biot-Savart) and $\underline{\omega} = 0$ on $\mathcal{C}$ $\Rightarrow \lim_{\mathcal{C}\to\infty} \frac{d\underline{I}}{dt} = 0$

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annex - invariants : demonstrations (…)

- **identity (2D)** $\boxed{\underline{B} = \iint_S \underline{x} \wedge \underline{rot}\, \underline{A}\, dS = \iint_S \underline{A}\, dS - \oint_{\mathcal{C}} \underline{x} \wedge \left(\underline{n} \wedge \underline{A}\right) dl}$

✓ $\underline{x} \wedge \underline{rot}\, \underline{A} = \varepsilon_{ijk} x_j\, \varepsilon_{kpq} \frac{\partial A_q}{\partial x_p} = \left(\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}\right) x_j \frac{\partial A_q}{\partial x_p} = x_j \frac{\partial A_j}{\partial x_i} - x_j \frac{\partial A_i}{\partial x_j}$

where $\varepsilon_{ijk}\varepsilon_{kpq} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}$

$$\Rightarrow \begin{cases} B_x = \iint_S y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) dxdy = \iint_S \left(\frac{\partial y A_y}{\partial x} - \frac{\partial y A_x}{\partial y} \right) dxdy + \iint_S A_x\, dxdy \\ B_y = \iint_S x\left(\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x} \right) dxdy = -\iint_S \left(\frac{\partial x A_y}{\partial x} - \frac{\partial x A_x}{\partial y} \right) dxdy + \iint_S A_y\, dxdy \end{cases}$$

✓ Green's theorem $\iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = \oint_{\mathcal{C}} \left(P dx + Q dy \right)$

$$\Rightarrow \begin{cases} B_x = \oint_{\mathcal{C}} \left(y A_x dx + y A_y dy \right) + \iint A_x dxdy \\ B_y = -\oint_{\mathcal{C}} \left(x A_x dx + x A_y dy \right) + \iint A_y dxdy \end{cases}$$

✓ check that the line integrals correspond to $-\underline{x} \wedge \left(\underline{n} \wedge \underline{A}\right) dl$ with $\underline{n}\, dl = \left(-dy, dx, 0\right)$

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annex - invariants : demonstrations (…)

- **angular momentum (2D)** $\boxed{\underline{M} = -M\,\underline{e}_z \;,\; M = \frac{1}{2}\iint_S \left(x^2+y^2\right)\omega\,dxdy}$

✓ nota : vectorial form : $\underline{M} = \iint \underline{x} \wedge \left(\underline{x} \wedge \underline{\omega}\right) dxdy$

✓ vorticity : use the 2D scalar equation $\dfrac{\partial\omega}{\partial t} = -\underline{u}.\underline{grad}\,\omega + \nu\,\Delta\omega$

✓ identity $\underline{u}.\underline{grad}\,\omega = div\left(\omega\underline{u}\right) - \omega\,\cancel{div\underline{u}} = div\left(\omega\underline{u}\right) \;\Rightarrow\; \dfrac{\partial\omega}{\partial t} = -div\left(\omega\,\underline{u}\right) + \nu\,\Delta\omega$

$$\Rightarrow \frac{dM}{dt} = \frac{1}{2}\iint_S \left(x^2+y^2\right)\frac{\partial\omega}{\partial t}\,dxdy = -\frac{1}{2}\iint_S \left(x^2+y^2\right)\left[div\left(\omega\,\underline{u}\right) - \nu\,\Delta\underline{\omega}\right]dxdy$$

$$\Rightarrow \frac{dM}{dt} = -\frac{1}{2}\iint_S \left[\left(x^2+y^2\right)div\left(\omega\,\underline{u}\right)\right]dxdy - \frac{1}{2}\iint_S \nu\left[\left(x^2+y^2\right)\Delta\omega\right]dxdy$$
$$= -\frac{1}{2}\iint_S div\left[\left(x^2+y^2\right)\omega\,\underline{u}\right]dxdy + \frac{1}{2}\iint_S \left[\omega\,\underline{u}.\underline{grad}\left(x^2+y^2\right)\right]dxdy - \frac{1}{2}\iint_S \nu\left[\left(x^2+y^2\right)\Delta\omega\right]dxdy$$

✓ $\iint_S div\left[\left(x^2+y^2\right)\omega\,\underline{u}\right]dxdy = \oint_C \left(x^2+y^2\right)\omega\,\underline{u}.\underline{n}\,dl$

✓ $\underline{u} \sim 1/r$ (Biot-Savart) so as $\left(x^2+y^2\right)\omega\,\underline{u}.\underline{n}\,dl \sim r^2\omega\left(r\right)$ .This vanishes if ω is exponentially decaying

✓ $\underline{u}.\underline{grad}\left(x^2+y^2\right) = \left(u_x\underline{e}_x + u_y\underline{e}_y\right).\left(2x\underline{e}_x + 2y\underline{e}_y\right) = 2xu_x + 2yu_y = 2\underline{x}.\underline{u}$

annex - invariants : demonstrations (…)

$$\Rightarrow \frac{dM}{dt} = \iint_S \omega\left(xu+yv\right)dxdy + \frac{1}{2}\iint_S \nu\left[\left(x^2+y^2\right)\Delta\omega\right]dxdy$$

✓ Biot-Savard (2D) $\underline{u}\left(\underline{x}\right) = \dfrac{1}{2\pi}\oiint_S \dfrac{\omega\left(\underline{x}'\right)\underline{e}_z \wedge \left(\underline{x}-\underline{x}'\right)}{\left\|\underline{x}-\underline{x}'\right\|^2}\,d\underline{x}'$

$$\Rightarrow \begin{cases} u\left(x,y\right) = -\dfrac{1}{2\pi}\iint_S \dfrac{\left(y-y'\right)\omega\left(x',y'\right)}{\left(x-x'\right)^2+\left(y-y'\right)^2}\,dx'dy' \\ v\left(x,y\right) = \dfrac{1}{2\pi}\iint_S \dfrac{\left(x-x'\right)\omega\left(x',y'\right)}{\left(x-x'\right)^2+\left(y-y'\right)^2}\,dx'dy' \end{cases}$$

$$\Rightarrow \iint_S 2\omega\left(xu+yv\right)dxdy = \frac{1}{\pi}\iint_S \iint_S \frac{x\,y'-x'\,y}{\left(x-x'\right)^2+\left(y-y'\right)^2}\,\omega\left(x,y\right)\omega\left(x',y'\right)dx'dy'\,dx\,dy = 0$$

$$\Rightarrow \frac{dM}{dt} = -\frac{1}{2}\nu\iint_S \left(x^2+y^2\right)\Delta\omega\,dxdy = -\frac{1}{2}\nu\iint_S \left(x^2+y^2\right)\frac{\partial^2\omega}{\partial x_i^2}\,dxdy$$
$$= -\frac{1}{2}\nu\iint_S \frac{\partial}{\partial x_i}\left[\left(x^2+y^2\right)\frac{\partial\omega}{\partial x_i}\right]dxdy + \frac{1}{2}\nu\iint_S \frac{\partial\omega}{\partial x_i}\frac{\partial}{\partial x_i}\left(x^2+y^2\right)dxdy$$
$$= -\frac{1}{2}\nu\iint_S \frac{\partial}{\partial x_i}\left[\left(x^2+y^2\right)\frac{\partial\omega}{\partial x_i}\right]dxdy + \nu\iint_S \frac{\partial\omega}{\partial x_i}x_i\,dxdy$$
$$= -\frac{1}{2}\nu\,\cancel{\oint_C \left(x^2+y^2\right)\frac{\partial\omega}{\partial x_i}n_i dl} + \nu\,\cancel{\oint_C \omega\,x_i}\,n_i dl - \nu\iint_S \omega\frac{\partial x_i}{\partial x_i}\,dxdy = -2\nu\iint_S \omega\,dxdy = -2\nu\,\Gamma$$

annex - invariants : demonstrations (…)

- **angular momentum (3D)** $\boxed{\underline{M} = \frac{1}{3}\iiint_V \underline{x}\wedge(\underline{x}\wedge\underline{\omega})\,dV}$

✓ vorticity $\Rightarrow \frac{d}{dt}\underline{M} = \frac{1}{3}\iiint_V \underline{x}\wedge\left(\underline{x}\wedge\frac{\partial\underline{\omega}}{\partial t}\right)dV = \frac{1}{3}\iiint_V \underline{x}\wedge\left(\underline{x}\wedge \underline{rot}\,\underbrace{(\underline{u}\wedge\underline{\omega}-\nu\,\underline{rot}\,\underline{\omega})}_{\underline{A}}\right)dV \iint dS$

✓ identity $\underline{x}\wedge(\underline{x}\wedge\underline{rot}\,\underline{A}) = (\underline{x}.\underline{rot}\,\underline{A})\underline{x} - \underline{x}^2\,\underline{rot}\,\underline{A} = \underbrace{(\underline{x}\otimes\underline{x}-\underline{x}^2\underline{\underline{1}})}_{\underline{\underline{P}}}.\underline{rot}\,\underline{A}$

✓ $\underline{\underline{P}} = \underline{x}\otimes\underline{x}-\underline{x}^2\underline{\underline{1}}$ is the projector in the plane normal to $\underline{x}$ .Its components are $P_{ij} = x_i x_j - \underline{x}^2\delta_{ij}$

✓ for the i^{th} componant one gets : $\left[\underline{x}\wedge(\underline{x}\wedge\underline{rot}\,\underline{A})\right]_i = P_{ij}\,\varepsilon_{jpq}\,\partial A_q/\partial x_p$

$$\Rightarrow \frac{dM_i}{dt} = \frac{1}{3}\varepsilon_{jpq}\iiint_V P_{ij}\frac{\partial A_q}{\partial x_p}dV = \frac{1}{3}\varepsilon_{jpq}\left(\iiint_V \frac{\partial P_{ij}A_q}{\partial x_p}dV - \iiint_V A_q\frac{\partial P_{ij}}{\partial x_p}dV\right)$$

✓ the first integral gives a surface integral which vanishes, because $\underline{\omega}$, so $\underline{A} = (\underline{u}\wedge\underline{\omega}-\nu\,\underline{rot}\,\underline{\omega})$, are nil on the bounding surface.

$$\Rightarrow \frac{dM_i}{dt} = -\frac{1}{3}\varepsilon_{jpq}\iiint_V A_q\frac{\partial}{\partial x_p}\left(x_i x_j - \underline{x}^2\delta_{ij}\right)dV = -\frac{1}{3}\varepsilon_{jpq}\iiint_V A_q\left(\delta_{ip}x_j + \delta_{jp}x_i - 2x_p\delta_{ij}\right)dV$$

where we used $\frac{\partial \underline{x}^2}{\partial x_p} = 2x_l\frac{\partial x_l}{\partial x_p} = 2x_l\delta_{lp} = 2x_p$

annex - invariants : demonstrations (…)

$$\frac{dM_i}{dt} = -\frac{1}{3}\varepsilon_{jpq}\iiint_V A_q\left(\delta_{ip}x_j + \delta_{jp}x_i - 2x_p\delta_{ij}\right)dV$$

✓ identities $\begin{cases}\varepsilon_{jpq}\delta_{jp} = 0\\ \varepsilon_{jpq}\delta_{ip}x_j = \varepsilon_{pjq}\delta_{ij}x_p = -\varepsilon_{jpq}\delta_{ij}x_p\end{cases}$

$$\Rightarrow \frac{dM_i}{dt} = \frac{1}{3}\varepsilon_{jpq}\iiint_V 3A_q x_p\delta_{ij}\,dV = \varepsilon_{ipq}\iiint_V A_q x_p\,dV = \iiint_V(\underline{x}\wedge\underline{A})_i\,dV$$

✓ $\underline{A} = (\underline{u}\wedge\underline{\omega}-\nu\,\underline{rot}\,\underline{\omega})$

$$\Rightarrow \frac{dM_i}{dt} = \left[\iiint_V \underline{x}\wedge(\underline{u}\wedge\underline{\omega})\,dV - \nu\iiint_V \underline{x}\wedge\underline{rot}\,\underline{\omega}\,dV\right]_i$$

✓ $\underline{u}\wedge\underline{\omega} = \underline{grad}\left(\underline{u}^2/2\right) - \nabla\underline{u}.\underline{u}$ (see Lamb decomposition) $\Rightarrow (\underline{u}\wedge\underline{\omega})_i = \frac{\partial}{\partial x_j}\left(\frac{1}{2}\underline{u}^2\delta_{ij} - u_i u_j\right)$

$$\Rightarrow \frac{dM_i}{dt} = \varepsilon_{ipq}\iiint_V x_p\frac{\partial}{\partial x_j}\left(\frac{1}{2}\underline{u}^2\delta_{qj} - u_q u_j\right)dV - \nu\left[\iiint_V \underline{x}\wedge\underline{rot}\,\underline{\omega}\,dV\right]_i$$

$$= \varepsilon_{ipq}\iiint_V \frac{\partial}{\partial x_j}\left[\left(\frac{1}{2}\underline{u}^2\delta_{qj} - u_q u_j\right)x_p\right]dV - \varepsilon_{ipq}\iiint_V\left(\frac{1}{2}\underline{u}^2\delta_{qj} - u_q u_j\right)\delta_{jp}\,dV$$

$$-\nu\left[\iiint_V \underline{x}\wedge\underline{rot}\,\underline{\omega}\,dV\right]_i$$

$$= \varepsilon_{ipq}\oint\!\!\oint_S\left(\frac{1}{2}\underline{u}^2\delta_{qj} - u_q u_j\right)x_p n_j\,dS - \varepsilon_{ipq}\iiint_V\left(\frac{1}{2}\underline{u}^2\cancel{\delta_{qp}} - u_q u_p\right)dV - \nu\left[\iiint_V \underline{x}\wedge\underline{rot}\,\underline{\omega}\,dV\right]_i$$

annex - invariants : demonstrations (…)

$$\boxed{\frac{dM_i}{dt} = \varepsilon_{ipq} \oiint_S \left(\tfrac{1}{2}\underline{u}^2 \delta_{qj} - u_q u_j \right) x_p n_j \, dS + \varepsilon_{ipq} \iiint_V u_q u_p \, dV - \nu \iiint_V \left(\underline{x} \wedge \underline{rot}\ \underline{\omega} \right)_i dV}$$

✓ in a 3D flow $\|\underline{u}\| \sim r^{-3}$ (3D Biot-Savart), the two first integral vanish.

$$\Rightarrow \quad \frac{dM_i}{dt} = -\nu \iiint_V \left(\underline{x} \wedge \underline{rot}\ \underline{\omega} \right)_i dV$$

✓ $$\left(\underline{x} \wedge \underline{rot}\ \underline{\omega}\right)_i = \varepsilon_{ijk} x_j \, \varepsilon_{kpq} \frac{\partial \omega_q}{\partial x_p} = \left(\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp} \right) x_j \frac{\partial \omega_q}{\partial x_p} = x_j \frac{\partial \omega_j}{\partial x_i} - x_j \frac{\partial \omega_i}{\partial x_j}$$

where we uses the identity : $\varepsilon_{ijk}\varepsilon_{kpq} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}$

$$\Rightarrow \quad \frac{dM_i}{dt} = -\nu \iiint_V \left[x_j \frac{\partial \omega_j}{\partial x_i} - x_j \frac{\partial \omega_i}{\partial x_j} \right]_i dV$$

$$\Rightarrow \quad \frac{dM_i}{dt} = -\nu \iiint_V \left[\frac{\partial x_j \omega_j}{\partial x_i} - \frac{\partial x_j \omega_i}{\partial x_j} + 3\omega_i \right]_i dV$$

✓ for exponentially decaying vorticity : $\dfrac{dM_i}{dt} = -3\nu \iiint_V \omega_i \, dV$

$$\Rightarrow \quad \frac{d\underline{M}}{dt} = -3\nu \iiint_V \underline{\omega} \, dV$$

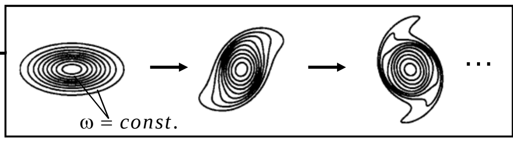
annex - invariants : demonstrations (…)

$$\frac{d\underline{M}}{dt} = -3\nu \iiint_V \underline{\omega} \, dV = -3\nu \iiint_V \underline{rot}\ \underline{u} \, dV = -3\nu \oiint_S \underline{u} \wedge \underline{n} \, dS$$

✓ in a 3D flow : $\|\underline{u}\| \sim r^{-3} \Rightarrow \boxed{\underline{M}^{3D} = const.}$

$\Rightarrow$ in a 3D flow angular momentum is conserved (in the absence of external forces)

11.3 vortex patches



- **so, how explain the following figure ?**

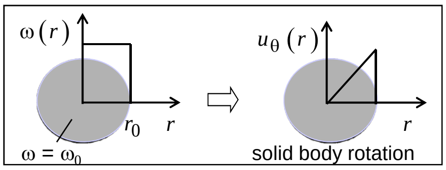
✓ circulation

$$\Gamma = \oint_C \underline{u} \cdot \underline{dl} = \iint_S \omega \, dS \quad \Rightarrow \quad \Gamma(r) = \int_0^{2\pi} u_\theta(r,\theta)\, r d\theta = \int_0^{2\pi}\int_0^{r} \omega(r',\theta)\, r' dr' d\theta \qquad (*)$$

✓ if ω were cylindrical : $\Gamma(r) = 2\pi\, u_\theta(r) = 2\pi \int_0^r \omega(r')\, r' dr' \quad \Rightarrow \quad \boxed{u_\theta(r) = \int_0^r \omega(r')\, dr'}$

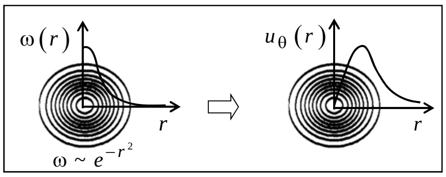
➢ case 1 : constant vorticity (Rankine vortex)

$$\boxed{\begin{cases} \omega(r) = \omega_0, r \le r_0 \\ \omega(r) = 0, r > r_0 \end{cases}}$$



➢ case 2 : gaussian vorticity (Lamb-Oseen vortex)

$$\boxed{\omega(r) = \omega_0 \, e^{-(r/r_0)^2}}$$

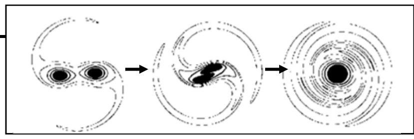


✓ then explain figure on the top (non cylindrical)

✓ training : calculate $\Gamma_0 = \lim_{r\to\infty} \Gamma(r)$ for the 2 cases above

(*) note that $d\Gamma(r)/dt \neq 0$ if r lies in the patch (see annex on circulation)

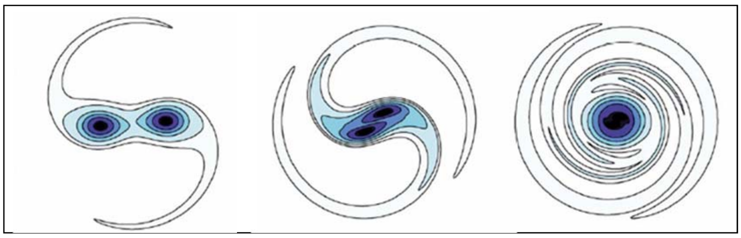
11.2 vortex patches



- **how explain this one ?**

✓ conservation of angular momentum

$$\lim_{\mathrm{Re}>>1} dM/dt = 0 \quad \Rightarrow \quad \boxed{M = \iint_S \| \underline{x} - \underline{x}_c \|^2 \, \omega \, dS = const.}$$



induced rotation of the vortices (Biot-Savart) producing filaments

convection of this vorticity outside leads to inner vorticity concentration, so as to maintain angular momentum

the two vortex cores merge

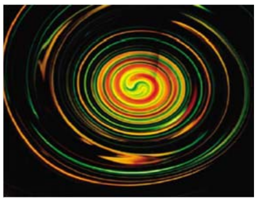
11.3 vortex patches

- **vortex pairing process**

✓ differential rotation : filamentation

✓ conservation of angular momentum : the vortex cores go closer

✓ viscous diffusion (on the long term) leads to the merger



Leweke (IRPHE)

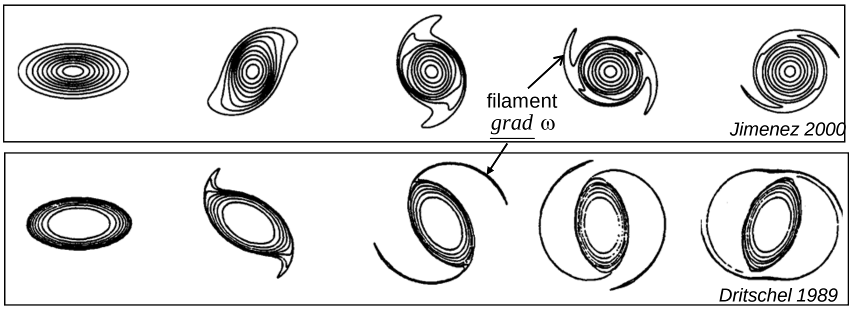
11.3 vortex patches

Turbulence_in_periodic_box_long_simulation.mp4

11.4 cascade of enstrophy – conservation of energy

- **enstrophy** $\underline{\omega}^2$ $\Rightarrow$ variance = measure of vorticity amplitude dispersion
- **equation** $\omega \times \left[\dfrac{d\omega}{dt} = \nu \Delta\omega \right]$ $\Rightarrow$ $\boxed{\dfrac{d\omega^2}{dt} = \underbrace{div\,[\,\underline{grad}\,(\,\tfrac{1}{2}\omega^2\,)\,]}_{\text{flux}} \underbrace{-2\nu \left\| \underline{grad}\ \omega \right\|^2}_{\text{destruction} <0}}$ see annex
- **observations**

$\mathrm{Re} \to \infty$



- **equation** $\boxed{\dfrac{d}{dt}\iint_S \omega^2 dxdy = -2\nu \iint_S \left\| \underline{grad}\ \omega \right\|^2 dxdy < 0}$ $\Rightarrow$ enstrophy decreases

see annex

annex – equation of enstrophy : demonstration

- equation of ω^2: $\omega \times \left[\dfrac{d\omega}{dt} = \nu \Delta\omega \right]$

✓ $\omega \dfrac{d\omega}{dt} = \dfrac{d}{dt}(\,\tfrac{1}{2}\omega^2\,)$

✓ $\nu\,\omega\,\Delta\omega = \nu\,\omega \dfrac{\partial^2\omega}{\partial x_i^2} = \nu \dfrac{\partial}{\partial x_i}\left(\,\omega \dfrac{\partial\omega}{\partial x_i}\,\right) - \nu \left(\,\dfrac{\partial\omega}{\partial x_i}\,\right)^2 = \nu \dfrac{\partial}{\partial x_i}\left[\,\dfrac{\partial}{\partial x_i}\left(\,\tfrac{1}{2}\omega^2\,\right)\,\right] - \nu \left(\,\dfrac{\partial\omega}{\partial x_i}\,\right)^2$

$= \nu\,div\ \underline{grad}\left(\,\tfrac{1}{2}\omega^2\,\right) - \nu \left(\,\dfrac{\partial\omega}{\partial x_i}\,\right)^2 = \nu\,\Delta\left(\,\tfrac{1}{2}\omega^2\,\right) - \nu \left|\nabla\omega\right|^2$

$\Rightarrow$ $\boxed{\dfrac{d\omega^2}{dt} = \dfrac{\partial\omega^2}{\partial t} + \underline{u}\,.\,\underline{grad}\ \omega^2\ . = \underbrace{\nu\,div\,[\,\underline{grad}\,(\,\tfrac{1}{2}\omega^2\,)\,]}_{\text{diffusion}} \underbrace{-2\nu \left\| \underline{grad}\ \omega \right\|^2}_{\text{destruction} <0}}$

- equation of $\iint_S \omega^2 dxdy$

✓ $\dfrac{d}{dt}\iint_S \omega^2 dxdy = \iint_S \dfrac{\partial}{\partial t}\omega^2 dxdy = -\iint_S [\,\overbrace{\underline{u}.\underline{grad}\ \omega^2}^{div(\,\omega^2 \underline{u}\,) - \omega^2 div\,\underline{u}} - \nu\ div\,[\,\underline{grad}\,(\,\tfrac{1}{2}\omega^2\,)\,] + 2\nu \left\| \underline{grad}\ \omega \right\|^2\,]dxdy$

$= \cancel{-\oint_C [\,\omega^2 \underline{u} + \nu\,\underline{grad}\,(\,\tfrac{1}{2}\omega^2\,)\,].\underline{n}\ dl} + \iint_S \omega^2\ \cancel{div\,\underline{u}}\ dxdy - 2\nu \iint_S \left\| \underline{grad}\ \omega \right\|^2 dxdy$

$\Rightarrow$ $\boxed{\dfrac{d}{dt}\iint_S \omega^2 dxdy = -2\nu \iint_S \left\| \underline{grad}\ \omega \right\|^2 dxdy}$

11.4 cascade of enstrophy – conservation of energy (…)

- **enstrophy in 2D flows**

$$\frac{d\omega^2}{dt} = \underbrace{\nu\, div\left[\, \underline{grad}\left(\tfrac{1}{2}\omega^2\right)\right]}_{\text{diffusion}} \underbrace{-2\nu \left\| \underline{grad}\ \omega \right\|^2}_{\text{destruction } <0} \qquad (1)$$

✓ one observed that

$$\boxed{\lim_{\nu\to 0} 2\nu \iint_S \left\| \underline{grad}\ \omega \right\|^2 \approx const \quad \Rightarrow \quad \text{filamentation : a cascade of enstrophy in 2D flows}}$$

⇨ following (1) $\boxed{\lim_{t\to\infty} \iint_S \omega^2\, dS = 0}$ ⇨ enstrophy vanishes in unforced 2D flows

- **analogy : the 3D cascade of energy** $e_k = \frac{1}{2}\underline{u}^2$ **in 3D flows** (remainder)

$$\frac{de_k}{dt} = \underbrace{div\left[\left(p/\rho + \varphi\right)\underline{u} - \nu\, \underline{\omega} \wedge \underline{u}\right]}_{\text{diffusion}} \underbrace{-\nu\, \underline{\omega}^2}_{\text{destruction } <0} \qquad (2)$$

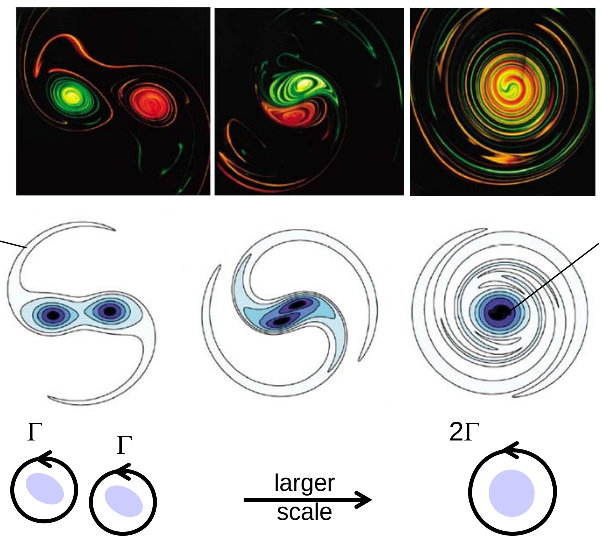
✓ one observes that $\lim_{\nu\to 0} \nu \iiint_V \underline{\omega}^2 dV \approx const$ ⇨ cascade of energy in 3D flows (see paradox of turbulence and model of Richardson-Kolmogorov)

- **energy in 2D flows ?** $\forall \nu,\ \lim_{t\to\infty} \nu \iint_S \omega^2\, dS = 0$

(2) ⇨ $\boxed{\text{energy} \iint_S e_k\, dS \quad \text{is conserved in 2D flows !}}$

11.5 an inverse cascade of energy

- **observation**

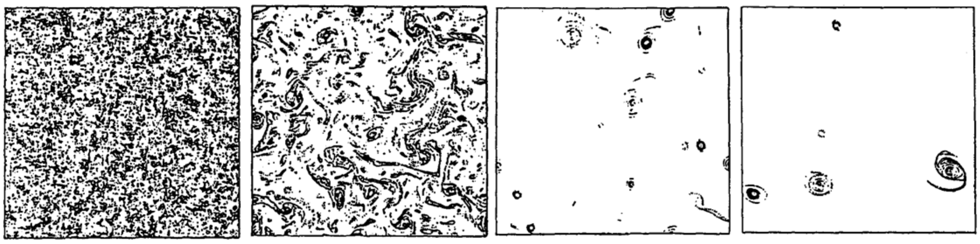


Meunier et al. 2005

⇨ inverse cascade : energy containing scales become larger

11.5 an inverse cascade of energy (…)

✓ numerical simulation of a 2D isotropic turbulence decay



Mc Williams 1982

11.5 an inverse cascade of energy (…)

numerical simulation
of a 2D isotropic turbulence

? MATLAB Simulation of 2D Turbulence in a Periodic Box (1024 x 1024) - YouTube [720p].mp4

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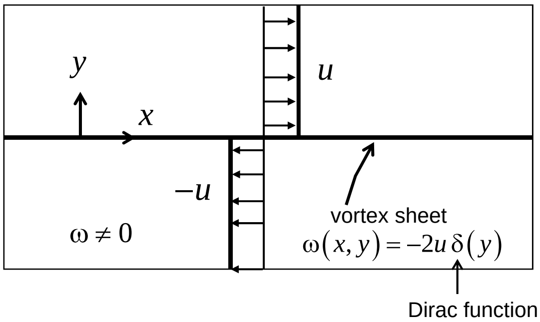
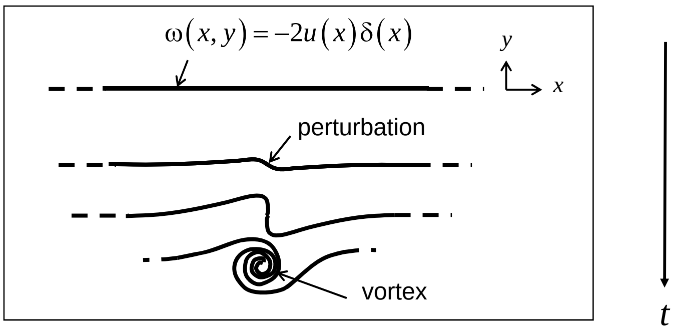
11.5 an inverse cascade of energy (…)

- **another example : the mixing layer**



11.5 an inverse cascade of energy (…)

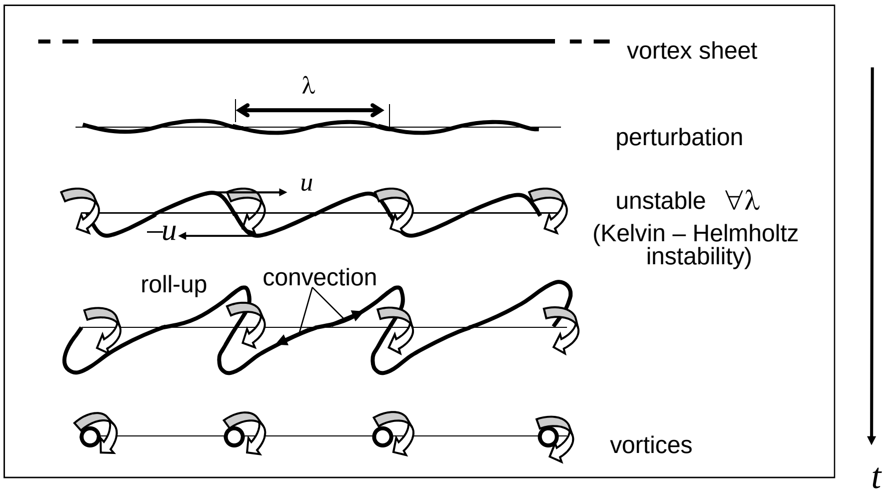
- **mixing layer : an unstable flow**



⇨ vorticity sheets are unstable and produce vortices

11.5 an inverse cascade of energy (…)

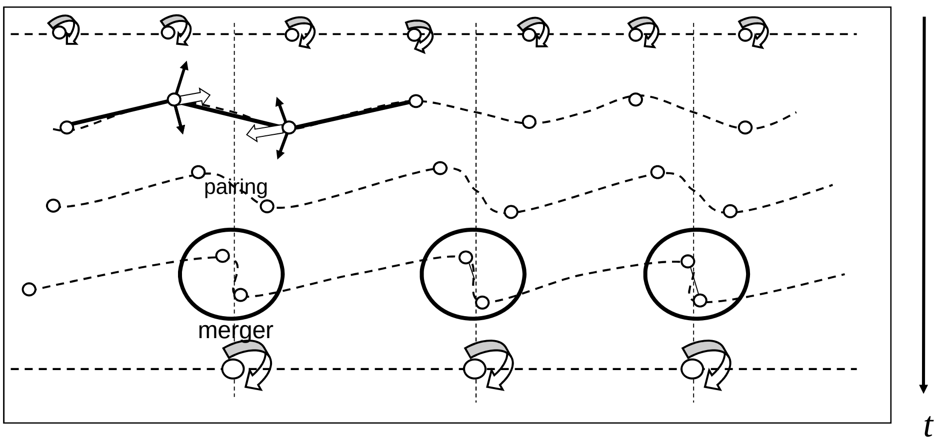
• **mixing layer : the Kelvin-Helmholtz instability**



11.5 an inverse cascade of energy (…)

• **vortex sheet : the Kelvin-Helmholtz instability**

✓ replace the vorticity sheet by a periodic vortex alley



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11.5 an inverse cascade of energy (…)

- **observations**

numerical simulation
of a temporal plane
mixing layer

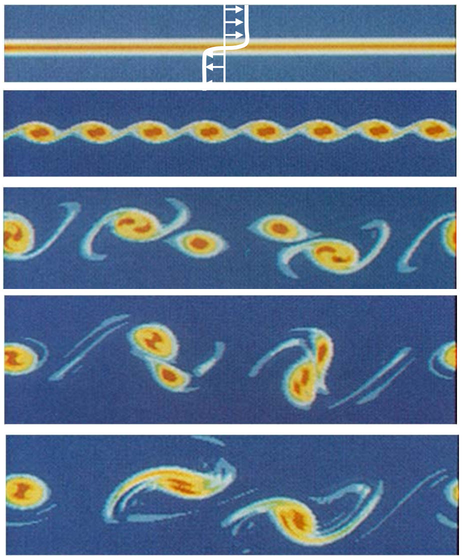
tribute

Pierre Comte (2010)

harmonic forcing

pairing

merger



t

Pierre Comte (1999)

11.5 an inverse cascade of energy (…)

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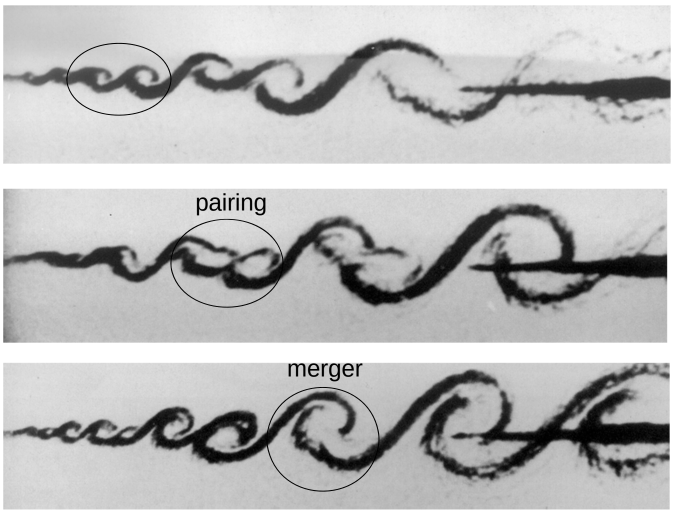
- **observations**

spatial development of a mixing layer separating H2 et He



t

Van Dyke : An album of fluid motion, 1983

11.6 3D → 2D dynamics : a first inventory

✓ 3D → 2D : the fundamental change is elimimination of the distorsion term

$$\boxed{\frac{d\,\underline{\omega}}{dt} = \cancel{\nabla\underline{u}.\underline{\omega}} + \nu\,\Delta\underline{\omega}}$$

⇨ $$\boxed{\frac{d\,\underline{\omega}}{dt} = \nu\,\Delta\underline{\omega}}$$ convection – diffusion equation

✓ minimization of enstrophy : vorticity concentration

✓ inversion of the energy cascade : from small scales to large scales

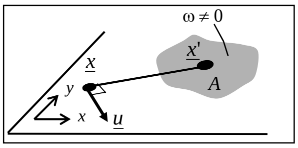
✓ at equilibrium :
- a minimum enstrophy state
- a maximum free energy state (not easy to define)

11.7 chaos

⇨ we look now to the collective interaction of point vortices

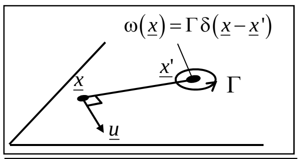
- **Biot-Savart**

✓ vortex patch



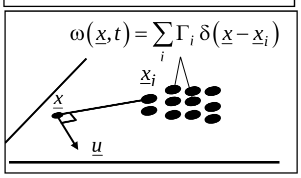
$$\underline{u}(\underline{x}) = \frac{1}{2\pi}\iint_S \frac{\omega(\underline{x}')\wedge(\underline{x}-\underline{x}')}{\|\underline{x}-\underline{x}'\|^2}\,d\underline{x}'$$

✓ point vortex



$$\underline{u}(\underline{x}) = \frac{\Gamma}{2\pi}\frac{\underline{x}-\underline{x}'}{\|\underline{x}-\underline{x}'\|^2}$$

✓ *n* point vortices



$$\underline{u}(\underline{x}) = \sum_i \frac{\Gamma_i}{2\pi}\frac{\underline{x}-\underline{x}_i}{\|\underline{x}-\underline{x}'\|^2}$$

⇩

⇨ each vortex moves as a material particle under the combined effects of other vortices. It has no effect on itself

$$\boxed{\underline{u}(\underline{x}_i) = \frac{d\,\underline{x}_i}{dt} = \sum_{j\neq i}\frac{\Gamma_j}{2\pi}\frac{\underline{x}_j-\underline{x}_i}{\|\underline{x}-\underline{x}'\|^2}}$$

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11.7 chaos (…)

• **movement equation** $\underline{u}\left(\underline{x}_i\right)=\frac{d\,\underline{x}_i}{dt}=\sum_{j\neq i}\frac{\Gamma_j}{2\pi}\frac{\underline{x}_j-\underline{x}_i}{\left\|\underline{x}-\underline{x}'\right\|^2}$

• **one can set this equation under an hamiltonian form** (homework)

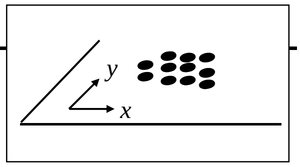
$2n$ équations $\boxed{\begin{cases}\Gamma_i\dfrac{dx_i}{dt}=\dfrac{\partial E}{\partial y_i}\\ \Gamma_i\dfrac{dy_i}{dt}=-\dfrac{\partial E}{\partial x_i}\end{cases}}$ Hamiltonian form Nota : $\underline{x}_i=\left(x_i\,,y_i\right)$

with $\boxed{E=-\frac{1}{4\pi}\sum_{ij}\Gamma_i\Gamma_j\log\left\|\underline{x}_j-\underline{x}_i\right\|}$ vortex interaction energy

• **energy conservation** $\frac{\partial E}{\partial t}=\frac{\partial E}{\partial x_i}\frac{dx_i}{dt}+\frac{\partial E}{\partial y_i}\frac{dy_i}{dt}=0$

$\Rightarrow$ $\boxed{E=const.}$ invariant

11.7 chaos (…)



✓ let be a collection of point vortices of circulation

$\Gamma=\iint_S\omega\;dS=\sum_i\Gamma_i=const.$

• **momentum equations :** $\Gamma_i\frac{dx_i}{dt}=\frac{\partial E}{\partial y_i},\Gamma_i\frac{dy_i}{dt}=-\frac{\partial E}{\partial x_i},i=1,n$ Hamiltonian form

• **movement invariants :**

✓ linear momentum $\begin{cases}I_x=\iint_S y\,\omega\,dS\\ I_y=-\iint_S x\,\omega\;dS\end{cases}$ $\Rightarrow$ $\boxed{\begin{cases}I_x=\sum_i y_i\Gamma_i=const.\\ I_y=-\sum_i x_i\Gamma_i=const.\end{cases}}$

✓ angular momentum $\begin{cases}\underline{M}=-M\,\underline{e}_z\\ M=\iint_S\left(x^2+y^2\right)\omega\,dS\end{cases}$ $\Rightarrow$ $\boxed{M=\sum_i\left(x_i+y_i\right)^2\Gamma_i=const.}$

✓ interaction energy $\Rightarrow$ $\boxed{E=-\frac{1}{4\pi}\sum_{ij}\Gamma_i\Gamma_j\log\left\|\underline{x}_j-\underline{x}_i\right\|=const.}$

• **inventory** ✓ $2n$ équations

✓ 4 invariants $\left(I_x,I_y,M,E\right)$

• **in the 2D space :**

si $2n$ - $4\leq 2$	regular solutions	
si $2n$ - $4>2$	chaotic solutions	$(n\geq 4)$

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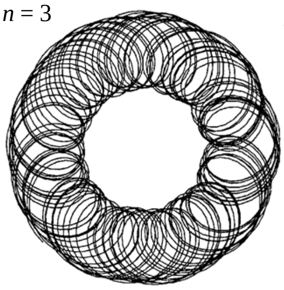
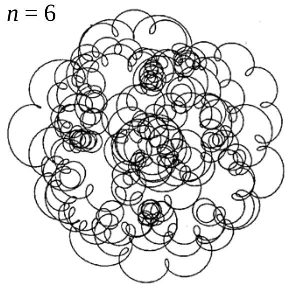
11.7 chaos (…)

- **in the 2D space :**

si $2n - 4 \leq 2$	regular solutions	
si $2n - 4 > 2$	chaotic solutions	$(n \geq 4)$

$n = 3$ $n = 6$





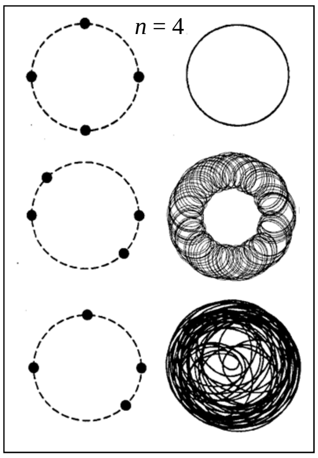
Jimenez 2000

quasi - periodic chaotic

11.7 chaos (…)

- **in the 2D space :**

si $2n - 4 \leq 2$	regular solutions	
si $2n - 4 > 2$	chaotic solutions	$(n \geq 4)$



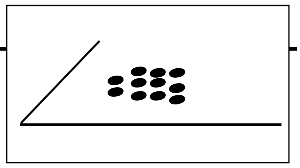
periodic

quasi - periodic

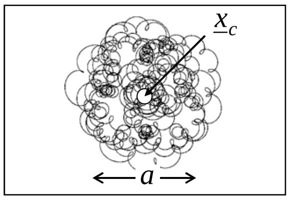
chaotic

Boatto & Pierrehumbert 1999

11.7 chaos (…)



$\Rightarrow$ one may characterize the spatial evolution of the system by considering the centroid and the dispersion radius



- **centroid**

✓ linear moment
$$\begin{cases} I_x = \sum_i y_i \Gamma_i = const. \\ I_y = -\sum_i x_i \Gamma_i = const. \end{cases}$$

$$\Rightarrow \begin{cases} x_c = \sum_i x_i \Gamma_i / \sum_i \Gamma_i = -I_y / \Gamma = const. \\ y_c = \sum_i y_i \Gamma_i / \sum_i \Gamma_i = I_x / \Gamma = const. \end{cases}$$

✓ centroid
$$\boxed{\underline{x}_c\left(x_c, y_c\right) = \underline{x}_c\left(\frac{-I_y}{\Gamma}, \frac{I_x}{\Gamma}\right)}$$

the centroid of the vortex system is conserved

$\Rightarrow$ take it as the origin

- **dispersion radius**

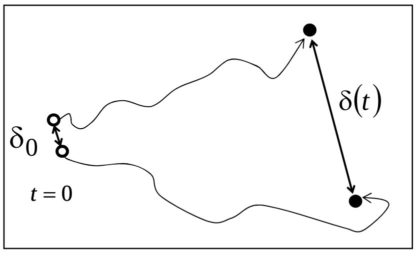
✓ angular momentum $M = \sum_i \left\| \underline{x}_i - \underline{x}_c \right\|^2 \Gamma_i = const.$

✓ dispersion radius
$$\boxed{a^2 = \frac{\sum_i \left\| \underline{x}_i - \underline{x}_c \right\|^2 \Gamma_i}{\Gamma} = const.}$$

the dispersion of the vortices from the centroid is characterized by radius a

11.7 chaos (…)

- **chaos index**



✓ finite time Lyapunov exponent

$$\boxed{\lambda\left(t\right) \equiv \lim_{\delta_0 \to 0} \frac{1}{t} \log \frac{\delta\left(t\right)}{\delta_0}}$$

✓ Lyapunov exponent

$$\boxed{\lambda \equiv \lim_{t \to \infty} \lambda\left(t\right)} \quad \lambda > 0 \quad \Rightarrow$$ exponentially divergent trajectories

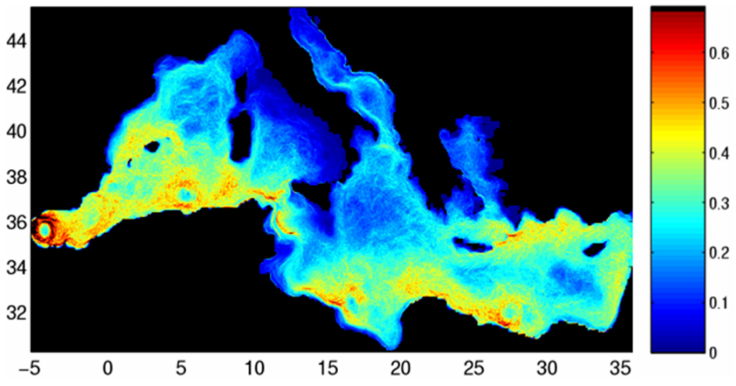
11.7 chaos (…)

✓ finite time Lyapunov exponent $\lambda(t) = \lim_{\delta_0 \to 0} \frac{1}{t} \log \frac{\delta(t)}{\delta_0}$

✓ finite size Lyapunov exponent $\lambda(\underline{x}, \delta_0, t) \equiv \frac{1}{t} \log \frac{\delta(t)}{\delta_0}$

Hernandez-Garcia, 2004.
« Mixing in the mediterranean sea from finite size Lyapunov exponents ». *Trends in Ecology*

Film : trajmovie.gif



11.8 2D dynamics : conclusion

• **notions**

✓ vortices
✓ chaos
✓ filamentation of vorticity
✓ vortex merger
✓ inverse cascade of energy

⇨ **the start of complexity**