MEC 654
Polytechnique-UPMC-Caltech
Year 2014-2015

Turbulence

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chapter 13

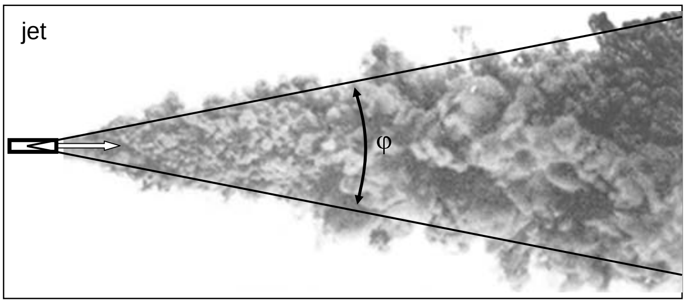
statistics in physical space

13.1 statistics in physical space

- **observations**

  ✓ in terms of **trajectories**, turbulence is characterized by an **unstable** and **chaotic** behavior of fluid particles

  ✓ in terms of **ensemble**, it appears that the turbulent behavior becomes **stable** and **steady**

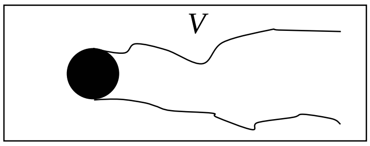


$\varphi = const.$
universal

⇨ ✓ irregular trajectories
✓ statistical regularity

13.1 statistics in physical space (…)

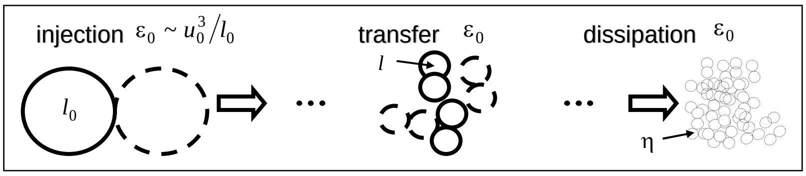
✓ dissipation rate



$$\varepsilon_V \equiv \overline{\langle \epsilon \rangle_V}^T = \underbrace{\frac{1}{V}\iiint_V [\overbrace{\frac{1}{T}\int_T 2\nu \underline{\underline{d}}.\underline{\underline{d}}\, dt}^{\text{time average}}\,]dV}_{\text{spatial average}}$$

⇨ ε_V is a statistical quantities

✓ energy cascade model


injection $\varepsilon_0 \sim u_0^3/l_0$ transfer ε_0 dissipation ε_0

⇨ $\varepsilon_0, u_0, l_0, l, \eta$ are statistical quantities

13.1 statistics in physical space (…)

- **ensemble average**

✓ $u = u\left(\underline{x}, t\right)$ = random variable

✓ $\langle u \rangle \equiv \dfrac{1}{N}\sum_{j=1}^{N} u^{(j)}$ = average over N realisations

✓ $\langle u \rangle =$ mathematical expectation $\equiv \int u\, p(u)\, du$ where $p(u) =$ probability density function

✓ $\langle (\,.\,) \rangle$ commutes with all space or time derivatives and integrals

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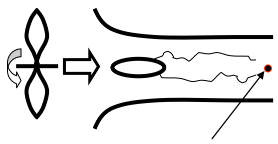
13.1 statistics in physical space (…)

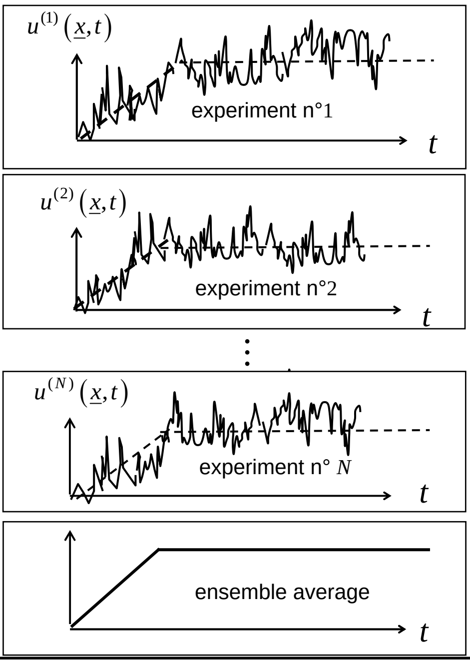
- **ensemble average (…)**

$\langle u \rangle \equiv \dfrac{1}{N}\sum_{j=1}^{N} u^{(j)}$ = average over N realisations



mesurement of $u\left(\underline{x}, t\right)$ at $\underline{x}$ repeating the experiment N times

$$\boxed{\langle \underline{u} \rangle \left(\underline{x}, t\right) = \frac{1}{N}\sum_{j=1}^{N} u^{(j)}\left(\underline{x}, t\right)}$$ ⇦



13.1 statistics in physical space (…)

- **fluctuation** $\boxed{\underline{u}^{'}(\underline{x},t)=\underline{u}-\langle\,\underline{u}\,\rangle}$

$$\Rightarrow \quad \langle\,\underline{u}^{'}\,\rangle=\langle\,\underline{u}-\langle\,\underline{u}\,\rangle\,\rangle \quad =\langle\,\underline{u}\,\rangle-\langle\,\underline{u}\,\rangle=0$$

ensemble average
is idempotent

- **Reynolds decomposition**

$$\boxed{\begin{cases}\underline{u}(\underline{x},t)=\langle\,\underline{u}\,\rangle+\underline{u}^{'}\\ p(\underline{x},t)=\langle\,p\,\rangle+p^{'}\\ \langle\,\underline{u}'\rangle=\langle\,p'\rangle=0\end{cases}}$$

13.1 statistics in physical space (…)

- **statistical stationarity**

✓ when the boundary conditions are stationary, we observe that the turbulence is statistically stationary, meaning that :

$$\boxed{\frac{\partial\, p(u)}{\partial t}=0}$$

$p(u)$ = probability density function of *u*

✓ in this case, the ensemble average becomes equivalent to a time average

$$\Rightarrow \quad \boxed{\langle\,(\;.\;)\,\rangle=\lim_{T\to\infty}\frac{1}{T}\int_{t}^{t+T}(\;.\;)dt^{'}}$$

independant on *t*

⇒ a single flow realization (an experience) is enough because the variable goes through all possible states over time

⇒ fortunately for the experimenters ...

13.1 statistics in physical space (…)

- **statistical homogeneity**

✓ some flows are also statistically homogeneous in one space direction x_i in space, or more, meaning that :

$$\boxed{\frac{\partial\, p(\, u' = u - \langle u \rangle\,)}{\partial x_i} = 0}$$

$p(u')$ = probability density function of u’

Note : $\partial\, p(\, u\,)/\partial t = 0$ was required for having statistical stationarity. We only need to have $\partial\, p(\, u'\,)/\partial x_i = 0$ for statistical homogeneity. This will be explained later.

✓ in this case, ensemble average become equivalent to spatial averages along each homogeneity direction :

$$\boxed{\langle(\,.\,)'\rangle = \lim_{L\to\infty}\frac{1}{L}\int_{x_i}^{x_i+L}(\,.\,)'dx}$$

independant on x_i

⇨ a single flow realization is enough because the variable goes through all possible states along each statistical homogeneity direction

⇨ fortunately for the numericicists

13.3 Reynolds’ decomposition

- **Reynolds equation**

✓ continuity

$$\langle div\,\underline{u}\rangle = 0 = div\,\langle\underline{u}\rangle$$

$$div\,\underline{u}' = div\big(\underline{u} - \langle\underline{u}\rangle\big) = div\,\underline{u} - div\langle\underline{u}\rangle = 0 - 0 = 0$$

⇨ the mean field and the fluctuating field are both solenoidal

- **momentum**

$$\langle\ \frac{\partial}{\partial t}\underline{u} + \nabla\underline{u}.\underline{u} = -\frac{1}{\rho}\underline{grad\ p} + div(\ 2\nu\,\underline{\underline{d}}\ )\ \rangle$$

case of the flow of a newtonian incompressible homogeneous fluid with no volumic force

$$\Rightarrow\ \frac{\partial\langle\underline{u}\rangle}{\partial t} + \enclose{circle}{\langle\nabla\underline{u}.\underline{u}\rangle} = -\frac{1}{\rho}\underline{grad}\,\langle\, p\rangle + div(\ 2\nu\langle\underline{\underline{d}}\rangle\ )$$

- **non-linear term**

$$\begin{aligned}\langle\nabla\underline{u}.\underline{u}\rangle &= \langle\nabla(\ \langle\underline{u}\rangle + u'\ ).(\ \langle\underline{u}\rangle + u'\ )\rangle \\ &= \langle\nabla\langle\underline{u}\rangle.\langle\underline{u}\rangle + \nabla\langle\underline{u}\rangle.u' + \nabla u'.\langle\underline{u}\rangle + \nabla u'.u'\rangle \\ &= \nabla\langle\underline{u}\rangle.\langle\underline{u}\rangle\ + \nabla\langle\underline{u}\rangle.\cancel{\langle\underline{u}'\rangle} + \nabla\cancel{\langle\underline{u}'\rangle}.\langle\underline{u}\rangle + \langle\nabla\underline{u}'.\underline{u}'\rangle\end{aligned}$$

$$\Rightarrow\ \langle\nabla\underline{u}.\underline{u}\rangle = \nabla\langle\underline{u}\rangle.\langle\underline{u}\rangle\ + \langle\nabla\underline{u}'.\underline{u}'\rangle = \enclose{circle}{\nabla\langle\underline{u}\rangle.\langle\underline{u}\rangle} + \langle\overbrace{\enclose{circle}{div\,\underline{u}'\otimes\underline{u}'}}^{\text{new term}} - \underline{u}'\ .\ \cancel{div\,\underline{u}'}\rangle$$

$$\Rightarrow\ \boxed{\underbrace{\frac{\partial\langle\underline{u}\rangle}{\partial t} + \nabla\langle\underline{u}\rangle.\langle\underline{u}\rangle}_{D\langle\underline{u}\rangle/Dt\ =\ \text{material derivative following the mean field}} = -\frac{1}{\rho}\underline{grad}\,\langle\, p\,\rangle + div\big(\ 2\nu\langle\underline{\underline{d}}\rangle - \underbrace{\langle\underline{u}'\otimes\underline{u}'\rangle}_{\text{turbulent stress = Reynolds stress tensor}}\ \big)}$$

13.3 Reynolds' decomposition (…)

- **Reynolds equation (…)** $$\boxed{\frac{D\langle \underline{u}\rangle}{Dt} = -\frac{1}{\rho}\underline{grad}\,\langle p\rangle + div\left(\underbrace{2\nu\langle \underline{\underline{d}}\rangle}_{\text{viscous stress tensor}} - \underbrace{\langle \underline{u}'\otimes\underline{u}'\rangle}_{\text{turbulent stress tensor = Reynolds stress tensor}}\right)} \quad (1)$$

- **Reynolds stress tensor** $$\begin{cases} \underline{\underline{R}}(\underline{x},t) = \langle \underline{u}'\otimes\underline{u}'\rangle \\ R_{ij}(\underline{x},t) = \langle u_i'\, u_j'\rangle \end{cases}$$

⇨ additional diffusion due to turbulent stresses $\langle \underline{u}'\otimes\underline{u}'\rangle$

⇨ in turbulent regions $\left|\langle \underline{u}'\otimes\underline{u}'\rangle\right| >> \left|2\nu\langle \underline{\underline{d}}\rangle\right|$

- **a closure problem :** to détermine $\langle \underline{u}\rangle$ from (1) , we must know $\langle \underline{u}'\otimes\underline{u}'\rangle$

⇨ we use « constitutive laws » of the kind $\langle \underline{u}'\otimes\underline{u}'\rangle = f(\langle \underline{\underline{d}}\rangle)$

⇨ this is what we mean usually by « modelling turbulence »

⇨ in engineering softwares, this is usually that way the « turbulence problem » is closed

95% of the softwares used in industry

13.3 kinetic energy budget

- **kinetic energy** $\langle e_k\rangle = \langle \tfrac{1}{2}\underline{u}^2\rangle = \underbrace{\tfrac{1}{2}\langle \underline{u}\rangle^2}_{K} + \underbrace{\tfrac{1}{2}\langle \underline{u}'^2\rangle}_{k}$

- **equation of the mean kinetic energy** $K = \tfrac{1}{2}\langle \underline{u}\rangle^2$

✓ contract the vectorial equation of $\langle \underline{u}\rangle$ with vector $\langle \underline{u}\rangle$

$$\langle \underline{u}\rangle .\left[\frac{D\langle \underline{u}\rangle}{Dt} = -\frac{1}{\rho}\underline{grad}\,\langle p\rangle + div\left(2\nu\langle \underline{\underline{d}}\rangle - \langle \underline{u}'\otimes\underline{u}'\rangle\right)\right]$$

$$\langle u_i\rangle\left[\frac{\partial\langle u_i\rangle}{\partial t} + \langle u_j\rangle\frac{\partial\langle u_i\rangle}{\partial x_j} = -\frac{1}{\rho}\frac{\partial\langle p\rangle}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\nu\left(\frac{\partial\langle u_i\rangle}{\partial x_j} + \frac{\partial\langle u_j\rangle}{\partial x_i}\right)\right] - \frac{\partial}{\partial x_j}\langle \underline{u}_i' u_j'\rangle\right]$$

$$\Rightarrow \overbrace{\frac{\partial K}{\partial t} + \langle u_j\rangle\frac{\partial K}{\partial x_j}}^{DK/Dt} = -\frac{1}{\rho}\frac{\partial\langle p\rangle\langle u_i\rangle}{\partial x_i} + \langle p\rangle\cancel{\frac{\partial\langle u_i\rangle}{\partial x_i}} + \nu\frac{\partial}{\partial x_j}\left(\langle u_i\rangle\frac{\partial\langle u_i\rangle}{\partial x_j}\right) - \nu\left(\frac{\partial\langle u_i\rangle}{\partial x_j}\right)^2$$
$$+\nu\langle u_i\rangle\frac{\partial}{\partial x_i}\cancel{\frac{\partial\langle u_j\rangle}{\partial x_j}} - \frac{\partial}{\partial x_j}\left(\langle u_i\rangle\langle \underline{u}_i' u_j'\rangle\right) + \langle \underline{u}_i' u_j'\rangle\frac{\partial\langle u_i\rangle}{\partial x_j}$$

$$\Rightarrow \frac{DK}{Dt} = \underbrace{\frac{\partial}{\partial x_j}\left[-\langle p\rangle\langle u_j\rangle + \nu\frac{\partial K}{\partial x_j} - \langle u_i\rangle\langle \underline{u}_i' u_j'\rangle\right]}_{div\,\underline{\phi}_K} \underbrace{+\langle \underline{u}_i' u_j'\rangle\frac{\partial\langle u_i\rangle}{\partial x_j}}_{-P=\langle \underline{u}'\otimes\underline{u}'\rangle:\nabla\langle \underline{u}\rangle} - \underbrace{\nu\left(\frac{\partial\langle u_i\rangle}{\partial x_j}\right)^2}_{\varepsilon_K = \varepsilon_{1K} = \nu\left|\nabla\langle \underline{u}\rangle\right|^2}$$

13.3 kinetic energy budget (…)

- **equation of the mean kinetic energy** $K = \frac{1}{2}\langle \underline{u} \rangle^2$ **(…)**

$$\boxed{\frac{DK}{Dt} = -P + div\,\underline{\phi}_K - \varepsilon_K}$$

$$\begin{cases} P = -\langle \underline{u}' \otimes \underline{u}' \rangle : \nabla \langle \underline{u} \rangle & \text{- production} \\ \underline{\phi}_K = \left[-\frac{\langle p \rangle}{\rho}\underline{\underline{1}} - \langle \underline{u}' \otimes \underline{u}' \rangle \right] . \langle \underline{u} \rangle + \nu\, \underline{grad}\, K & \text{- flux} \\ \varepsilon_K = 2\nu \left| \nabla \langle \underline{u} \rangle \right|^2 \geq 0 & \text{- dissipation} \left(\text{pseudo - dissipation } \varepsilon_{1K} \right) \end{cases}$$

- **equation of the mean turbulent kinetic energy (TKE)** $k = \frac{1}{2}\langle u_i'^2 \rangle$ (homework)
  - ✓ compose an equation for $u_i' = u_i - \langle u_i \rangle$
  - ✓ multiply it by u_i'
  - ✓ find :

$$\boxed{\frac{Dk}{Dt} = P + div\,\underline{\phi}_k - \varepsilon_k}$$

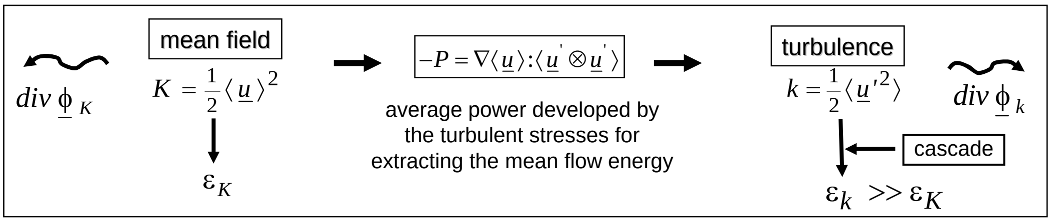
$$\begin{cases} P = -\langle \underline{u}' \otimes \underline{u}' \rangle : \nabla \langle \underline{u} \rangle & \text{- production} \\ \underline{\phi}_k = -\frac{1}{\rho}\langle p' \underline{u}' \rangle - \frac{1}{2}\langle \underline{u}'^2 \underline{u}' \rangle + \nu\, \underline{grad}\, k & \text{- flux} \\ \varepsilon_k = 2\nu \langle \left| \nabla \underline{u}' \right|^2 \rangle \geq 0 & \text{- dissipation (pseudo - dissipation } \varepsilon_{1k}) \end{cases}$$

13.3 kinetic energy budget (…)

- **kinetic energy** $\langle e_k \rangle = \langle \frac{1}{2}\underline{u}^2 \rangle = \underbrace{\frac{1}{2}\langle \underline{u} \rangle^2}_{K} + \underbrace{\frac{1}{2}\langle \underline{u}'^2 \rangle}_{k}$

- **balance**

$$\begin{array}{lllll} \frac{DK}{Dt} = & div\,\underline{\phi}_K & -\ \varepsilon_K & -P & \\ & \text{diffusion} & \text{dissipation} & \boxed{\text{exchange}} & \boxed{P = -\nabla\langle \underline{u} \rangle : \langle \underline{u}' \otimes \underline{u}' \rangle} \\ \frac{Dk}{Dt} = & div\,\underline{\phi}_k & -\ \varepsilon_k & +P & > 0 \text{ in most cases} \end{array}$$



mean field

$K = \frac{1}{2}\langle \underline{u} \rangle^2$

ε_K

$div\,\underline{\phi}_K$

$-P = \nabla\langle \underline{u} \rangle : \langle \underline{u}' \otimes \underline{u}' \rangle$

average power developed by
the turbulent stresses for
extracting the mean flow energy

turbulence

$k = \frac{1}{2}\langle \underline{u}'^2 \rangle$

$div\,\underline{\phi}_k$

cascade

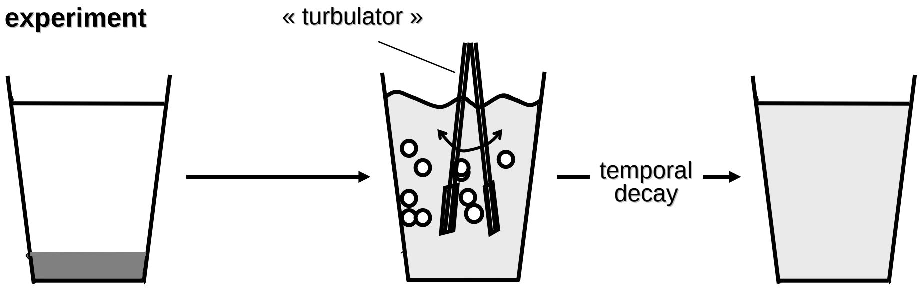
$\varepsilon_k >> \varepsilon_K$

- **dissipation** $\varepsilon = \nu \langle \left| \nabla \underline{u} \right|^2 \rangle = \underbrace{\nu \left| \nabla \langle \underline{u} \rangle \right|^2}_{\varepsilon_K} + \underbrace{\nu \langle \left| \nabla \underline{u}' \right|^2 \rangle}_{\varepsilon_k}$ (pseudo-dissipation ε_1)

$$\varepsilon_K << \varepsilon_k$$

13.4 temporal decay of homogeneous turbulence

- **an experiment**



✓ turbulence is produced by an agitator

✓ we stop forcing and turbulence proceeds until complete dissipation

- **hypotheses** **H1 -** no mean velocity

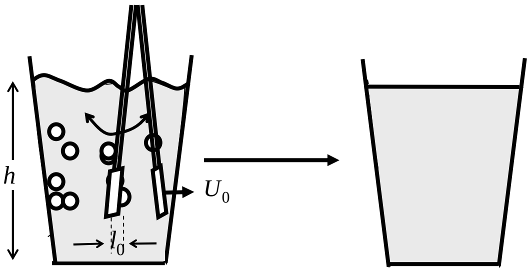
  H2 - homogeneous turbulence

$$\Rightarrow \quad \frac{Dk}{Dt}=\frac{\partial k}{\partial t}+\underset{\textbf{H1}}{\cancel{\langle \underline{u}\rangle}}.\underset{\textbf{H2}}{\underline{\cancel{grad}}}\;k=\underset{\textbf{H2}}{\underline{\cancel{grad}}}\;k+\underset{\substack{\uparrow\\ P=-\langle \underline{u}'\otimes\underline{u}'\rangle:\nabla\cancel{\langle\underline{u}\rangle}\quad \textbf{H1}}}{P}-\varepsilon_k \quad \Rightarrow \quad \boxed{\frac{\partial k}{\partial t}=-\varepsilon_k(t)}$$

temporal decay

13.4 temporal decay of homogeneous turbulence (…)

- **temporal decay** $\boxed{\frac{\partial k}{\partial t}=-\varepsilon_k(t)}$
- **time scales ?**



✓ «turbulator» size l_0

✓ characteristic forcing velocity : U_0

✓ energy : $k(0)\sim U_0^2$

✓ dissipation : $\varepsilon_k(0)\sim U_0^3/l_0$

✓ time scale : $\tau_\varepsilon \sim k(0)/\varepsilon_k(0)\sim l_0/U_0$

- **numbers**

✓ $U_0 \approx 1 m.s^{-1}$

✓ $h \approx 10 cm$

✓ $l_0 \approx 10^{-2} m$

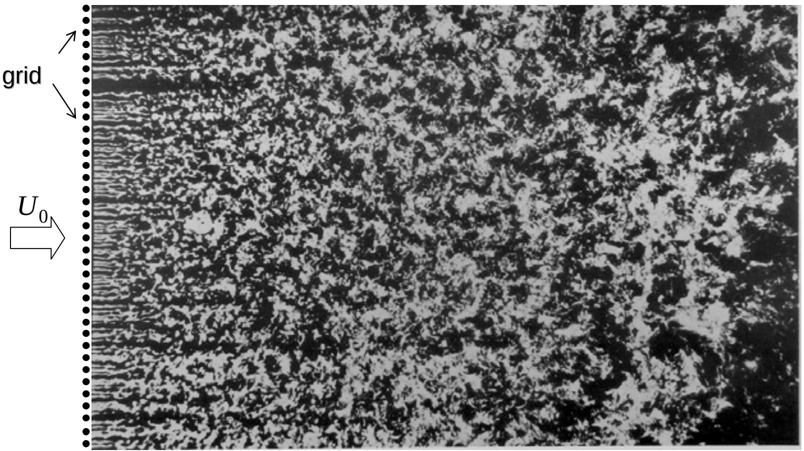
✓ $\nu \approx 10^{-6} m^2 s^{-1}$

$$\tau_\varepsilon \sim \frac{l_0}{U_0}\sim 10^{-2}s$$ to be compared to viscous time scale $$\tau_\nu \sim \frac{h^2}{\nu}\sim 10^3 s$$

⇨ calm sweatty warm coffee thanks to turbulence : increasing mixing, then rapid dissipation

13.5 spatial decay of homogeneous turbulence

- **grid tubulence**



Van Dyke 1983

- **hypotheses** **H1 -** uniform mean velocity $\langle \underline{u} \rangle = \underline{U}_0$

  H2 - statistically steady turbulence

  H3 - homogeneous turbulence in directions normal to the flow

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13.5 spatial decay of homogeneous turbulence (…)

- **hypotheses** **H1 -** uniform mean velocity $\langle \underline{u} \rangle = \underline{U}_0$

  H2 - statistically steady turbulence

  H3 - homogeneous turbulence in directions normal to the flow

$$\Rightarrow \quad \frac{Dk}{Dt} = \underset{\textbf{H2}}{\cancel{\frac{\partial k}{\partial t}}} + \langle \underline{u} \rangle . \underline{grad}\; k = div\, \underline{\phi}_k + \cancel{P} - \varepsilon_k \qquad P = -\langle \underline{u}' \otimes \underline{u}' \rangle : \nabla \cancel{\langle \underline{u} \rangle} \quad \textbf{H1}$$

$$\Rightarrow \quad U_0 \frac{\partial k}{\partial x} = \frac{\partial}{\partial x}(\underline{\phi}_k)_x + \underset{\textbf{H3}}{\cancel{\frac{\partial}{\partial y}(\underline{\phi}_k)_y}} + \underset{\textbf{H3}}{\cancel{\frac{\partial}{\partial z}(\underline{\phi}_k)_z}} - \varepsilon_k$$

with $\underline{\phi}_k = -\frac{1}{\rho}\langle p' \underline{u}' \rangle - \frac{1}{2}\langle \underline{u}'^2 \underline{u}' \rangle + \nu\, \underline{grad}\; k \quad \Rightarrow \quad (\underline{\phi}_k)_x = -\frac{1}{\rho}\langle p' u' \rangle - \langle \frac{1}{2} u_i'^2 u' \rangle + \nu \frac{\partial k}{\partial x}$

- **orders of magnitude**

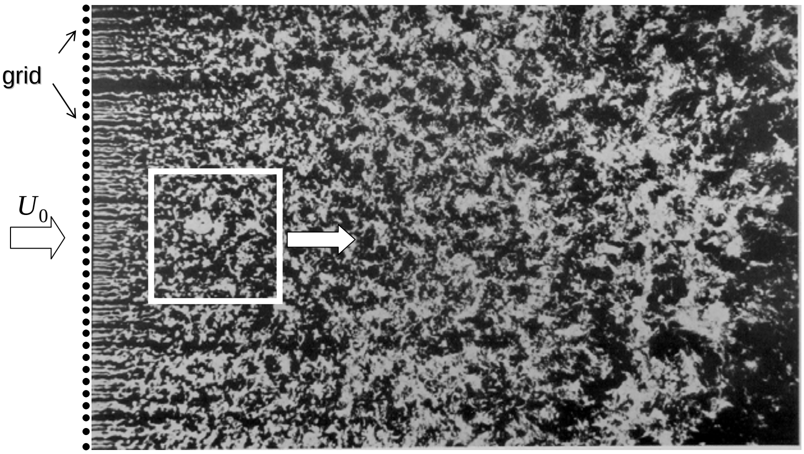
if $\sqrt{k}/U_0 << 1$ - turbulence rate

$$\left.\begin{array}{l} \text{convection}: U_0 \dfrac{\partial k}{\partial x} = O\left(\dfrac{U_0 k}{L}\right) \\ \text{turbulent diffusion}: \dfrac{\partial}{\partial x}\langle \frac{1}{2} u_i'^2 u' \rangle = O\left(\dfrac{k^{3/2}}{L}\right) \end{array}\right\} \Rightarrow \frac{\text{diffusion}}{\text{convection}} \sim \frac{k^{3/2}}{U_0 k} = \frac{\sqrt{k}}{U_0} \Rightarrow \boxed{U_0 \frac{\partial k}{\partial x} \approx -\varepsilon_k}$$

spatial decay

✓ homework : evaluate orders of magnitude of the other flux terms

13.5 spatial decay of homogeneous turbulence



Van Dyke 1983

<u>Note</u> :
turbulent scales increase downstream (?)

we will explain why in few minutes

✓ spatial decay $\boxed{U_0 \frac{\partial k}{\partial x} \approx -\ \varepsilon_k}$ $\longleftarrow t = \frac{x}{U_0} \longrightarrow$ $\boxed{\frac{\partial k}{\partial t} = -\ \varepsilon_k}$ ✓ temporel decay

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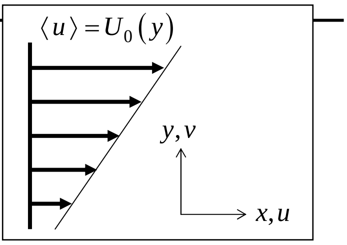
13.6 homogeneous shear flow



✓ a model flow for turbulent boundary and mixing layers

- **hypotheses** **H1 –** constant shear $\langle \underline{u} \rangle = U_0(y)\,\underline{e}_x$
  H2 - statistically steady flow
  H3 - statistically homogeneous turbulence

$$\Rightarrow \quad \frac{Dk}{Dt} = \underbrace{\cancel{\frac{\partial k}{\partial t}}}_{\textbf{H2}} + U \underbrace{\cancel{\frac{\partial k}{\partial y}}}_{\textbf{H3}} = \underbrace{\cancel{div\,\underline{\phi}_k}}_{\textbf{H3}} + \underset{\uparrow \atop P = -\langle u' v' \rangle \frac{d\langle U \rangle}{dy}\ \textbf{H1}}{P} -\ \varepsilon_k \quad \Rightarrow \quad \boxed{P = \ \varepsilon_k}$$

turbulence
in equilibrium

- **homogeneous turbulence** $\boxed{\langle \underline{u} \rangle \overset{\rightarrow}{\cancel{\leftarrow}} \underline{u}'}$

✓ Reynolds equation

$$\frac{D\langle \underline{u} \rangle}{Dt} = -\frac{1}{\rho} \underline{grad}\,\langle p \rangle + div\Big(\underbrace{2\nu \langle \underline{\underline{d}} \rangle}_{\text{viscous stress tensor}}\Big) \overset{\textbf{H3}}{\cancel{- div \underbrace{\langle \underline{u}' \otimes \underline{u}' \rangle}_{\text{turbulent stress tensor = Reynolds stress tensor}}}}$$

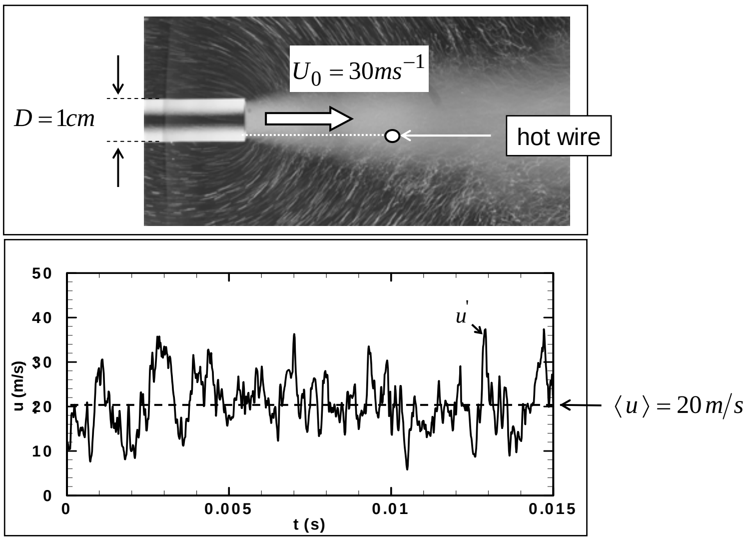
$\Rightarrow$ turbulence is produced by the mean flow gradients but there is no feedback on the mean field

chapter 14

statistics in Fourier space – introduction

14.1 example : a jet

- **hot wire measurements**



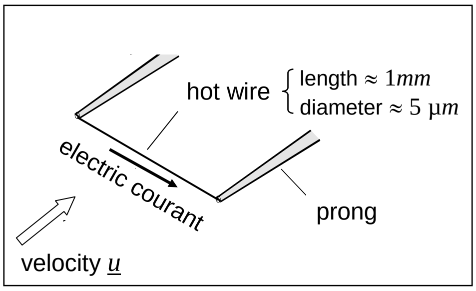
14.1 example : a jet (…)

• hot wire : principle

✓ a thin metal wire is crossed by an electric current which heats it

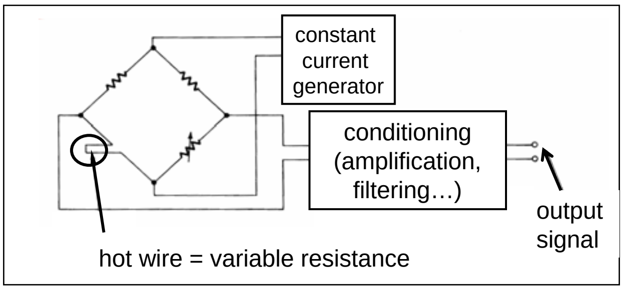
✓ its resistance varies (linearly) with temperature

✓ when the wire is immersed in a flow, it is cooled by forced convection. Thus its resistance varies with the fluid velocity. If the current is kept constant : a voltage variation is measured



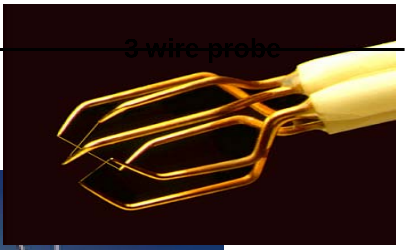
• hot wire : constant current mode

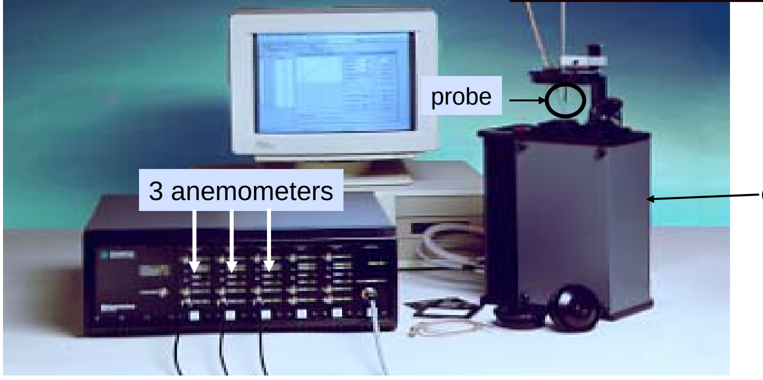
✓ voltage variations due to wire resistance variation for a constant current supply are measured by a Wheastone bridge.



14.1 example : a jet (…)

• hot wire : 3 velocity components

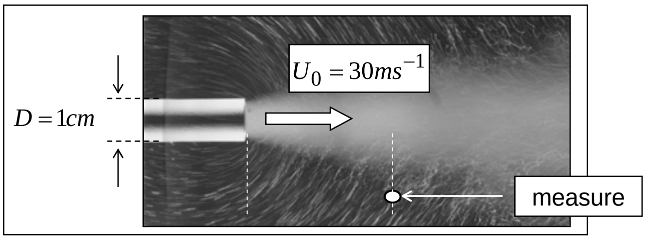


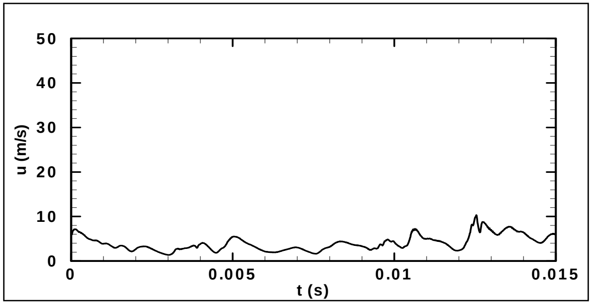


Dantec

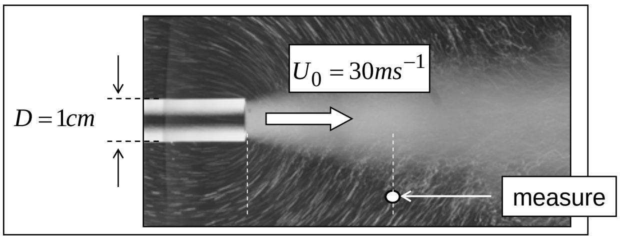
hot-wire electronic set-up

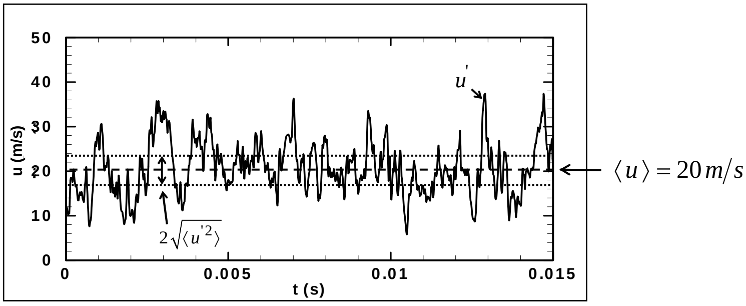
14.1 example : a jet (…)



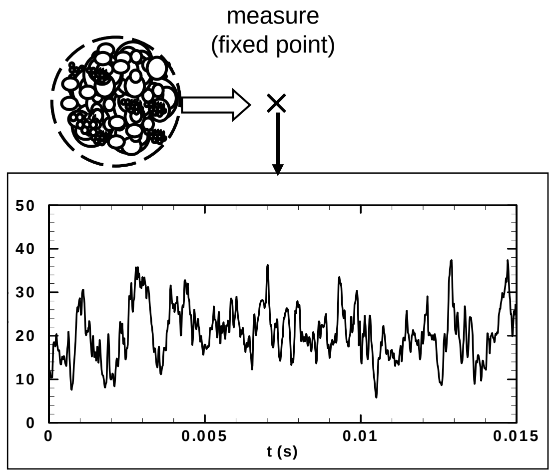


14.1 example : a jet (…)

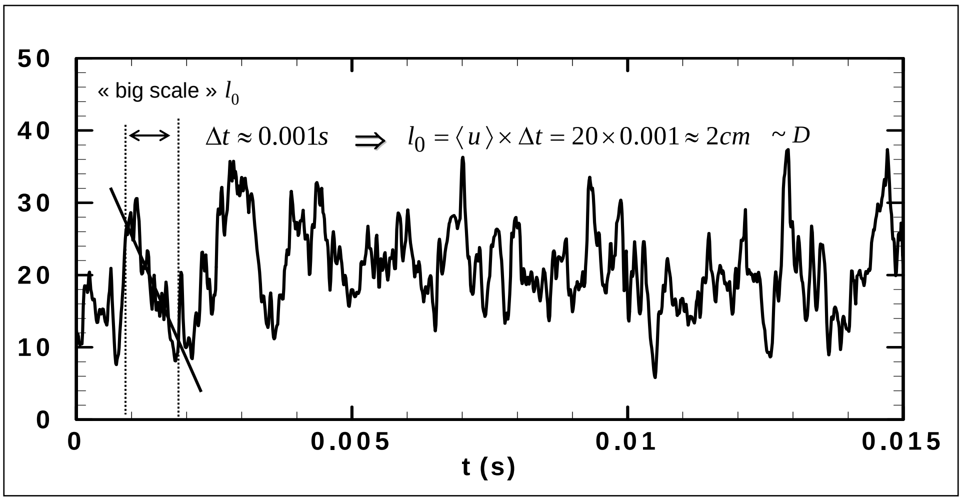




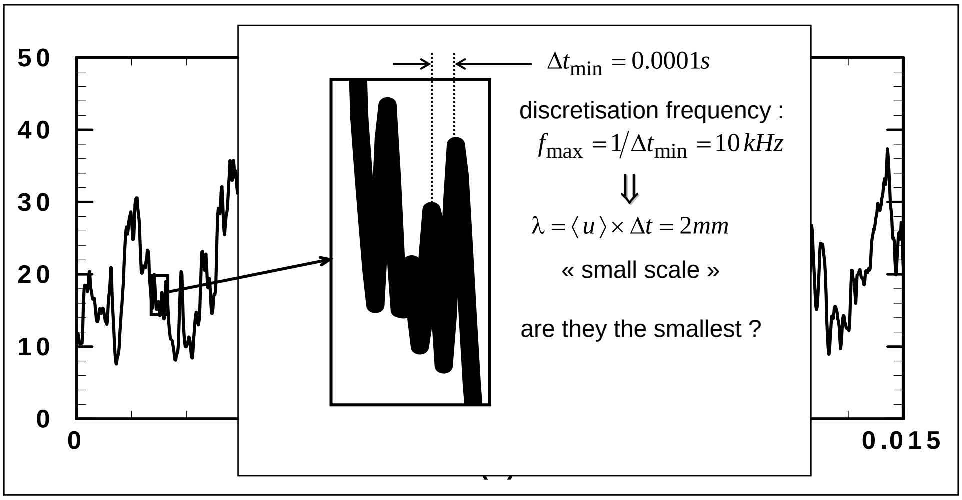
14.1 example : a jet (…)



14.1 example : a jet (…)



14.1 example : a jet (…)



$\Delta t_{\min} = 0.0001s$

discretisation frequency :

$f_{\max} = 1/\Delta t_{\min} = 10\,kHz$

$\Downarrow$

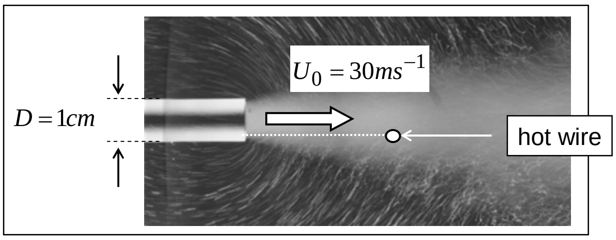
$\lambda = \langle u \rangle \times \Delta t = 2mm$

« small scale »

are they the smallest ?

14.1 example : a jet (…)

• the smaller scales

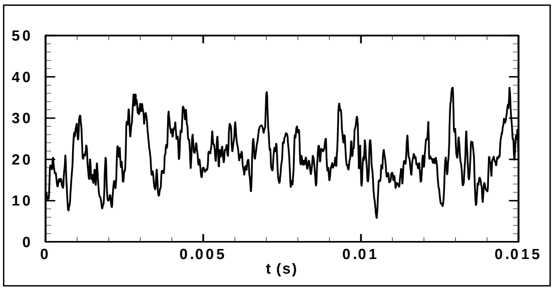


✓ injection of energy into the cascade $\begin{cases} l_0 \sim D \\ u_0 \sim U_0 \end{cases} \Rightarrow \mathrm{Re}_0 = \frac{U_0 l_0}{\nu} \sim 2\,10^4 \left(\nu \approx 1.5\,10^{-5} m^2.s^{-1}\right)$

✓ Kolmogorov scale $\frac{\eta}{l_0} = \left(\mathrm{Re}_0\right)^{-3/4} \sim 6\,10^{-4} \Rightarrow \boxed{\eta = 6\,\mu m}$

✓ Kolmogorov frequency $\boxed{f_\eta = U_0/\eta \sim 5MHz \text{ !!!}}$

14.2 the Fourier transform



⇨ turbulence scales may be characterized by means of Fourier analyses of its signals

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14.2 the Fourier transform

✓ wave vector $\boxed{\underline{x} \quad \Leftrightarrow \quad \underline{\kappa} \ \text{(wave vector)}}$

✓ Fourier mode

$$\boxed{\underline{\hat{u}}(\underline{\kappa}) = TF\left\{\underline{u}(\underline{x})\right\} = \frac{1}{(2\pi)^3}\int_{R^3} \underline{u}(\underline{x})\, e^{-i\,\underline{\kappa}\cdot\underline{x}}\, d^3\underline{x}}$$

nota :

- time dependence is implicit
- we forget the « primes »

$$\Updownarrow$$

✓ inverse Fourier transform

$$\boxed{\underline{u}(\underline{x}) = TF^{-1}\left\{\underline{\hat{u}}(\underline{\kappa})\right\} = \int_{R^3} \underline{\hat{u}}(\underline{\kappa})\, e^{i\,\underline{\kappa}\cdot\underline{x}}\, d^3\underline{\kappa}}$$

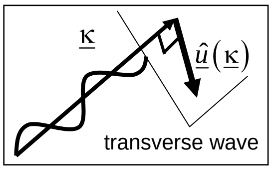
✓ interpretation : $\underline{\hat{u}}(\underline{\kappa})$ = amplitude of the sinusoidal component $e^{i\,\underline{\kappa}\cdot\underline{x}}$ of wave vector $\underline{\kappa}$ in $\underline{u}(\underline{x})$

✓ property : derivation

$$\boxed{TF\left\{\frac{\partial^n \underline{u}(\underline{x})}{\partial x_i^n}\right\} = (i\kappa_i)^n\, \underline{\hat{u}}(\underline{\kappa})}$$

⇨ in Fourier space, space derivation becomes a simple algebraic product

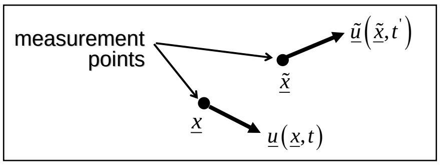
14.2 the Fourier transform (…)



- **incompressible flows**

✓ continuity $\boxed{TF\left\{ \frac{\partial u_i}{\partial x_i} \right\} = i\,\kappa_i\,\hat{u}_i(\underline{\kappa}) = i\,\underline{\kappa}\,.\,\underline{\hat{u}}(\underline{\kappa}) = 0}$

$\underline{u}(\underline{x}) = \int\limits_{R^3} \underline{\hat{u}}(\underline{\kappa}) e^{i\,\underline{\kappa}\,.\,\underline{x}}\, d^3\underline{\kappa}$ ⇒ turbulence is decomposed into transverse waves of wavelengths $l = \frac{2\pi}{\|\underline{\kappa}\|}$



- **double correlation tensor**

$\boxed{\underline{\underline{R}}(\underline{x},\underline{\tilde{x}};t,t') = \langle \underline{u}(\underline{x},t) \otimes \underline{u}(\underline{\tilde{x}},t') \rangle}$

✓ components

$R_{ij}(\underline{x},\underline{\tilde{x}};t,t') = \langle u_i(\underline{x},t)\, u_j(\underline{\tilde{x}},t') \rangle$

✓ statistically steady and fully homogeneous turbulence

$R_{ij}(\underline{x},\underline{\tilde{x}};t,t') = R_{ij}(\underline{l} = \underline{\tilde{x}} - \underline{x}; \tau = t - t')$ independant of t or $\underline{x}$

14.2 the Fourier transform (…)

- **the spectral tensor of the double correlations**

✓ homogeneous turbulence : spatial correlation at a given time $\boxed{R_{ij}(\underline{x},\underline{\tilde{x}};t) = \langle u_i(\underline{x},t)\, u_j(\underline{\tilde{x}},t) \rangle = R_{ij}(\underline{l} = \underline{\tilde{x}} - \underline{x}; t)}$

✓ Fourier transform

$$R_{ij}(\underline{l}) = TF^{-1}\left\{ \phi_{ij}(\underline{\kappa}) \right\} = \int_{R^3} \phi_{ij}(\underline{\kappa})\; e^{i\underline{\kappa}\,.\,\underline{l}}\; d^3\underline{\kappa}$$

$$\Updownarrow$$

$$\phi_{ij}(\underline{\kappa}) = TF\left\{ R_{ij}(\underline{l}) \right\} = \frac{1}{(2\pi)^3} \int\limits_{R^3} R_{ij}(\underline{l})\; e^{-i\underline{\kappa}\,.\,\underline{l}}\; d^3\underline{l}$$

✓ one can show that $\boxed{\langle \hat{u}_i^*(\underline{\kappa})\; \hat{u}_j(\underline{p}) \rangle = \phi_{ij}(\underline{\kappa})\, \delta(\underline{\kappa} - \underline{p})}$ see annex

$\hat{u}_i^*(\underline{\kappa})$ = complex conjugate of $\hat{u}_i(\underline{\kappa})$

⇨ the Fourier basis is orthogonal : the product of 2 Fourier modes of different wave numbers is nil

⇨ $\phi_{ij}(\underline{\kappa})$ is the **spectral tensor of the double correlations**

⇨ shorter : this is the **second order spectral tensor**

⇨ it evalutes the correlation between two Fourier modes

annex – second order spectral tensor : demonstration

$$\phi_{ij}(\underline{\kappa})\,\delta(\underline{\kappa}-\underline{p})=\langle\hat{u}_i^*(\underline{\kappa})\;\hat{u}_j(\underline{p})\rangle$$

✓ **2 Fourier modes**
$$\underline{\hat{u}}(\underline{\kappa})=\frac{1}{(2\pi)^3}\int_{R^3}\underline{u}(\underline{x})e^{-i\underline{\kappa}\cdot\underline{x}}\,d^3\underline{x}$$

$$\underline{\hat{u}}(\underline{p})=\frac{1}{(2\pi)^3}\int_{R^3}\underline{u}(\underline{\tilde{x}})\,e^{-i\underline{p}\cdot\underline{\tilde{x}}}\,d^3\underline{\tilde{x}}$$

✓ **correlation**
$$\langle\hat{u}_i^*(\underline{\kappa})\,\hat{u}_j(\underline{p})\rangle=\frac{1}{(2\pi)^6}\int_{R^6}\langle\underbrace{u_i(\underline{x})\,u_j(\underline{\tilde{x}})\rangle}_{R_{ij}(\underline{\tilde{x}}-\underline{x}=\underline{l})}\,e^{i\left(\underline{\kappa}\cdot\underline{x}-\underline{p}\cdot\underline{\tilde{x}}\right)}d^3\underline{x}\,d^3\underline{\tilde{x}}$$

homogeneous turbulence

⇨ $$\langle\hat{u}_i^*(\underline{\kappa})\,\hat{u}_j(\underline{p})\rangle=\frac{1}{(2\pi)^6}\int_{R^3}R_{ij}(\underline{l})\,[\,\underbrace{\int_{R^3}e^{i\left(\underline{\kappa}-\underline{p}\right)\cdot\underline{x}}d^3\underline{x}}_{(2\pi)^3\,\delta\left(\underline{\kappa}-\underline{p}\right)}\,]e^{-i\underline{p}\cdot\underline{l}}\;\;d^3\underline{l}$$

Fourier = orthogonal basis →

⇨ $$\langle\hat{u}_i^*(\underline{\kappa})\,\hat{u}_j(\underline{p})\rangle=\underbrace{\frac{1}{(2\pi)^3}\int_{R^3}R_{ij}(\underline{l})e^{-i\underline{p}\cdot\underline{l}}\;\;d^3\underline{l}}_{\phi_{ij}(\underline{p})}\;\;]\delta\left(\underline{k}-\underline{p}\right)\;=\phi_{ij}(\underline{\kappa})\,\delta\left(\underline{k}-\underline{p}\right)$$

14.3 the energy spectrum

$$\boxed{R_{ij}(\underline{l})=\langle u_i(\underline{x})\,u_j(\underline{x}+\underline{l})\rangle=\int_{R^3}\phi_{ij}(\underline{\kappa})\;\;e^{i\underline{\kappa}\cdot\underline{l}}\;d^3\underline{\kappa}}$$

- **kinetic energy**

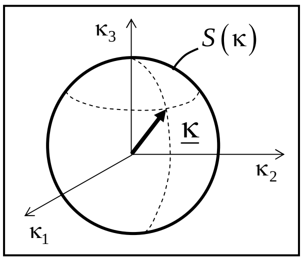
$i=j,\underline{l}=0$ ⇨ $\frac{1}{2}\langle\underline{u}^2\rangle\;=\frac{1}{2}R_{ii}(0)\;\;=\int_{R^3}\frac{1}{2}\phi_{ii}(\underline{\kappa})\;\;d^3\underline{\kappa}$

where $\boxed{\frac{1}{2}\phi_{ii}(\underline{\kappa})=\langle\frac{1}{2}\|\underline{\hat{u}}\|^2\rangle(\underline{\kappa})}$ ⇨ kinetic energy = sum over all wave vectors $\underline{\kappa}$ of the energy of the waves

- **energy spectrum** $\boxed{\frac{1}{2}\langle\underline{u}^2\rangle\equiv\int_0^\infty E(\kappa)\,d\kappa}$ $\kappa=\|\underline{\kappa}\|$

where $\boxed{E(\kappa)=\int_{S(\kappa=\|\underline{\kappa}\|)}\frac{1}{2}\phi_{ii}(\underline{\kappa})\;\;da}$

energy spectrum

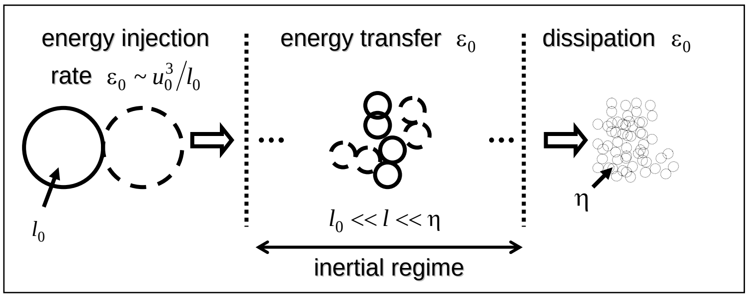


⇨ energy spectrum $E(\kappa)$ = average on a spherical shell of radius $\kappa=\|\underline{\kappa}\|$ of the wave energy

⇨ $E(\kappa)$ sums energy of the waves of wave number $\kappa=\|\underline{\kappa}\|$ in all possible directions

14.3 the energy spectrum

- **the « - 5/3 » law** (remainder)



✓ independant of ν

✓ only depends on ε_0 and l_0

$$\left.\right\} \Rightarrow \quad E(\kappa) = f(\varepsilon_0, \kappa)$$

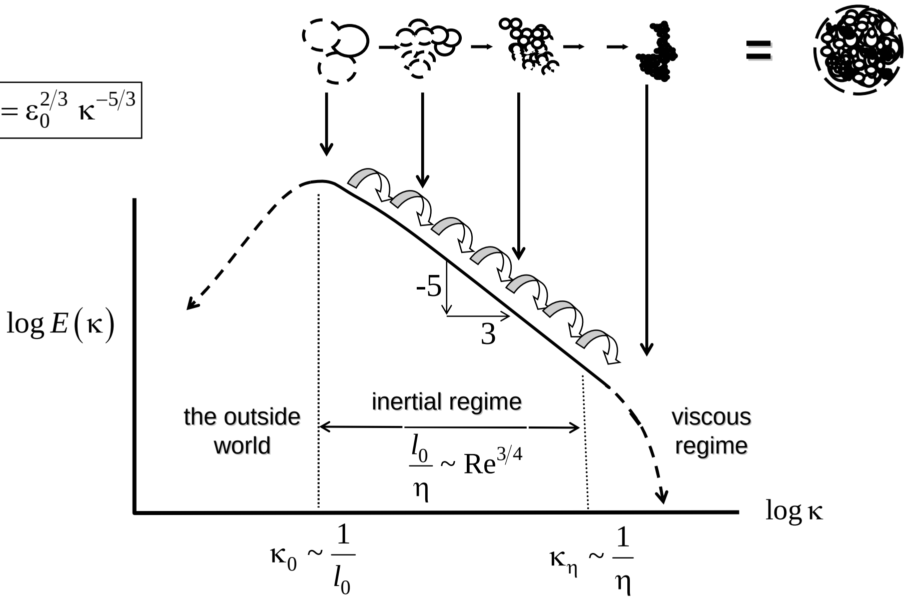
✓ dimensional analysis

$$\begin{cases} \left[E(\kappa)\right] = L^3\, T^{-2} & \Leftarrow \quad \frac{1}{2}\langle \underline{u}^2 \rangle = \int_0^\infty E(\kappa)\, d\kappa \\ \left[\varepsilon_0\right] = L^2\, T^{-3} & \Leftarrow \quad \varepsilon \sim \partial k / \partial t \\ \left[\kappa\right] = L^{-1} & \end{cases} \qquad \Rightarrow \quad \boxed{E(\kappa) \sim \varepsilon_0^{2/3}\ \kappa^{-5/3}}$$

14.3 the energy spectrum

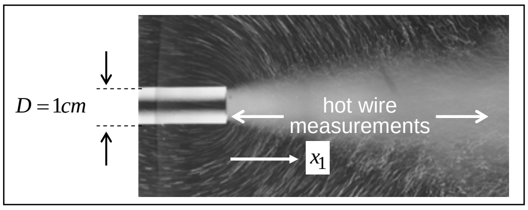
- **the « - 5/3 » law** (remainder : lesson 2, chapter 5, § 5.5)

$$\boxed{E(\kappa) = \varepsilon_0^{2/3}\ \kappa^{-5/3}}$$



14.3 the energy spectrum

- **example : a jet**

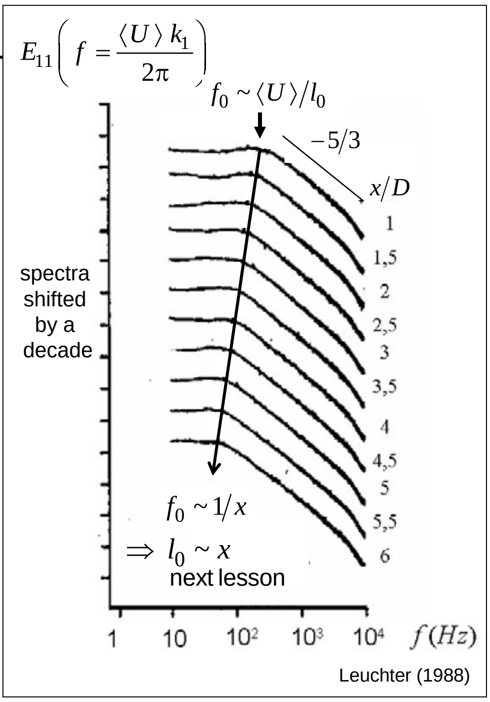


- **remarks**

✓ $E_{11}(k_1)$ is a one-dimensionnal energy spectrum : this notion will be explained later on

✓ « -5/3 » law still valid (same dimensions)

⇨ the Richardson-Kolmogorov's phenomenological theory works remarkably well !



Leuchter (1988)

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chapter 15

statistics in Fourier space – extensive

15.1 energy spectrum of isotropic incompressible turbulence

- **isotropy** means : invariance by rotation and mirror symetry
- **second order spectral tensor** $\underline{\underline{\phi}}(\underline{\kappa})$ **: isotropic form**

✓ one seeks an expression that ensures isotropy of the bilinear form

$[\ \underline{\underline{\phi}}(\underline{\kappa}).\underline{a}\ ].\underline{b} = \phi_{ij}(\underline{\kappa})a_ib_j = \psi(\underline{\kappa},\underline{a},\underline{b})$ where $\underline{a}$ and $\underline{b}$ are two arbitrary vectors

✓ possible isotropic quantities are scalars built with arguments $\underline{\kappa},\underline{a},\underline{b}$:

$\kappa^2 = \kappa_i\kappa_i,\ a_ib_i\ ,(a_i\kappa_i)(\ b_j\kappa_j\ ),...$

⇨ the seeked bilinear form can only be : $\phi_{ij}(\underline{\kappa})a_ib_j \ = \alpha(\kappa)(a_ib_i)+\beta(\kappa)\ (a_i\kappa_i)(b_j\kappa_j)$

⇨ $\boxed{\phi_{ij}(\underline{\kappa})\text{ isotropic} \quad \Leftrightarrow \quad \phi_{ij}(\underline{\kappa}) \ = \alpha(\kappa)\ \delta_{ij} + \beta(\kappa)\ \kappa_i\kappa_j}$

- **continuity :** $\kappa_i\ \phi_{ij}(\underline{\kappa})\,\delta(\ \underline{\kappa}-\underline{p}\ ) = \underbrace{\kappa_i\ \langle\hat{u}_i^*(\underline{\kappa})}_{=0}\ \hat{u}_j(\ \underline{p}\ )\rangle = 0$

⇨ $\left[\alpha(\kappa)+\beta(\kappa)\kappa^2\right]\kappa_j = 0$ ⇨ $\beta(\kappa) = -\dfrac{\alpha(\kappa)}{\kappa^2}$ ⇨ $\boxed{\phi_{ij}(\underline{\kappa}) \ = \alpha(\kappa)\ (\ \delta_{ij} - \dfrac{\kappa_i\kappa_j}{\kappa^2}\ )}$

✓ we have $E(k) = \frac{1}{2}\int_{S(\kappa)}\phi_{ii}(\underline{\kappa})\ dS = \frac{1}{2}\int_{S(\kappa)}\alpha(\kappa)[3-1]\ dS = 4\pi\kappa^2\alpha(\kappa)$ ⇨ $\alpha(\kappa) = \dfrac{E(\kappa)}{4\pi\kappa^2}$

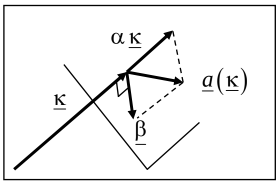
- **conclusion** $\boxed{\phi_{ij}(\underline{\kappa}) \ = \dfrac{E(\kappa)}{4\pi\kappa^2}\ (\ \delta_{ij} - \dfrac{\kappa_i\kappa_j}{\kappa^2}\ )}$

15.1 energy spectrum of isotropic icompressible turbulence

- **isotropic second order spectral tensor** $\boxed{\phi_{ij}(\underline{\kappa}) \ = \dfrac{E(\kappa)}{4\pi\kappa^2}\ (\ \delta_{ij} - \dfrac{\kappa_i\kappa_j}{\kappa^2}\ )}$
- **remark : the incompressible sub-space**

✓ we have : $\phi_{ij}(\underline{\kappa}) \ = \dfrac{E(\kappa)}{4\pi\kappa^2}\ P_{ij}(\underline{\kappa})$ where $P_{ij}(\underline{\kappa}) \ = \delta_{ij} - \dfrac{\kappa_i\kappa_j}{\kappa^2}$ = projector onto the space normal to $\underline{\kappa}$ (incompressible sub-space)

- **proof**



let : $\begin{cases}\underline{a}(\underline{\kappa}) = \alpha\ \underline{\kappa} + \underline{\beta} \\ \underline{\beta}.\underline{\kappa} = 0\end{cases}$

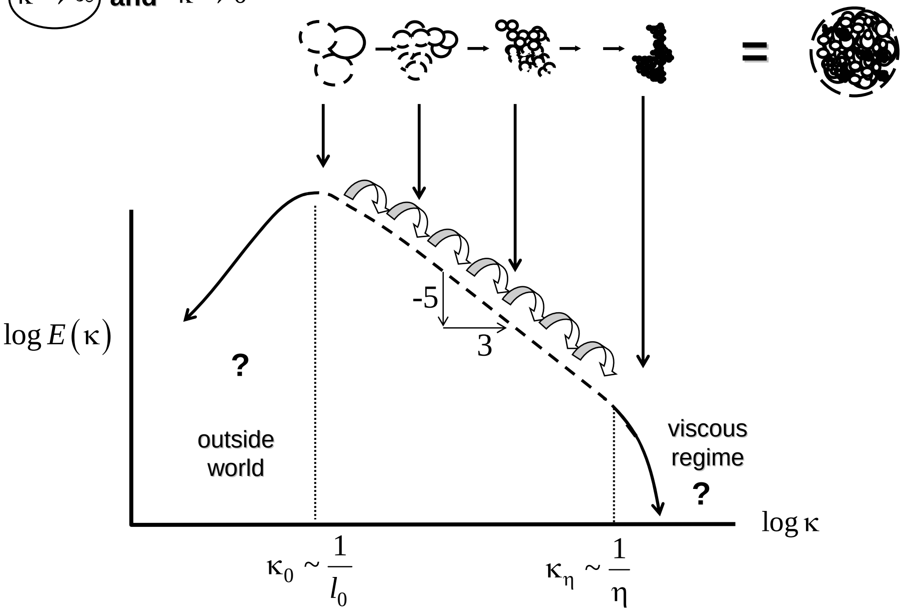
then : $(\ \underline{\underline{P}}.\underline{a}\ )_i = P_{ij}\ a_j = (\ \delta_{ij} - \dfrac{\kappa_i\kappa_j}{\kappa^2}\ )(\ \alpha\,\kappa_j + \beta_j\ )$

$$= \alpha\underbrace{[\ \kappa_i - \dfrac{\kappa_i(\ \kappa_j\kappa_j\ )}{\kappa^2}\ ]}_{=0} + [\ \beta_i - \dfrac{\kappa_i(\ \overbrace{\kappa_j\beta_j}^{=0}\ )}{\kappa^2}\ ]$$

$$= \beta_i$$

15.2 limiting shapes of $E(\kappa)$

- **laws for** $\kappa \to \infty$ **and** $\kappa \to 0$



15.2 limiting shapes of $E(\kappa)$ (…)

- **limiting shape** $E(\kappa \to \infty)$

✓ after the inertial regime (singular), the flow recovers infinite differentiability (C^∞)

⇨ $E(\kappa\to\infty)$ must decrease exponentially, that is faster than any power of κ

- **proof**

for the the n-order derivative of a scalar function, we have :

$$u^{(n)} = \frac{\partial^n u}{\partial x^n} \quad \Leftrightarrow \quad \widehat{u^{(n)}}(\kappa) = TF\{\, u^{(n)} \,\} = (i\,\kappa)^n\, \hat{u}(\kappa)$$

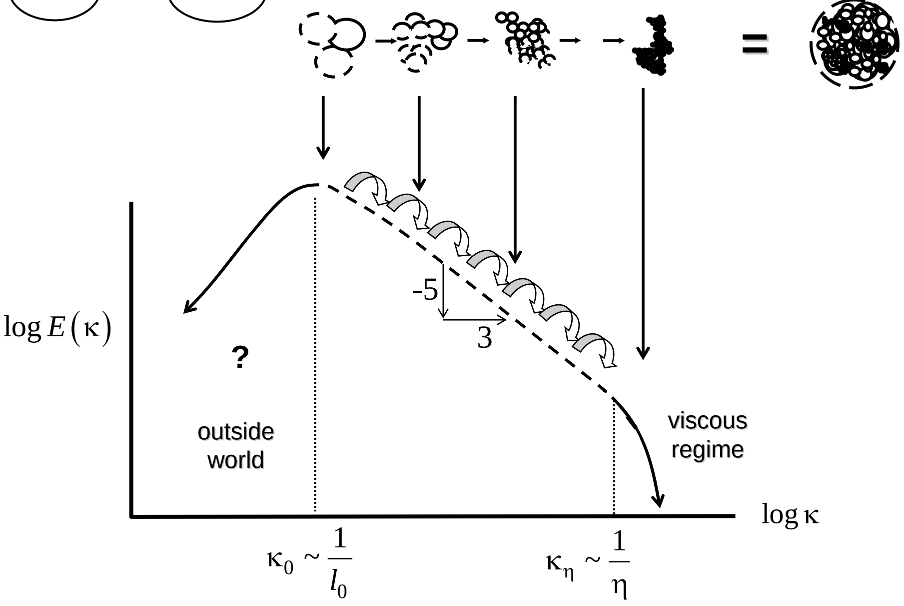
⇨ $$\langle\, \widehat{u^{(n)}}^{*}(\kappa)\,\widehat{u^{(n)}}(\kappa)\,\rangle = \kappa^{2n}\langle\, \hat{u}^{*}(\kappa)\,\hat{u}(\kappa)\rangle = \kappa^{2n}E(\kappa)$$

⇨ $$\boxed{\langle\, {u^{(n)}}^2\,\rangle = \int_0^\infty \langle\, \widehat{u^{(n)}}^{*}(\kappa)\,\widehat{u^{(n)}}(p)\,\rangle\, d\kappa, = \int_0^\infty \kappa^{2n}E(\kappa)\,d\kappa,\ \forall n}$$

⇨ $\forall n,\ \langle\, {u^{(n)}}^2\,\rangle$ is finite ⟺ $\boxed{E(\kappa \to \infty) \sim e^{-\kappa}}$ $E(\kappa \to \infty)$ must decrease faster than κ^{2n} whatever n

15.2 limiting shapes of $E(\kappa)$

- **laws for** $\kappa \to \infty$ **and** $\kappa \to 0$



15.2 limiting shapes of $E(\kappa)$ (…)

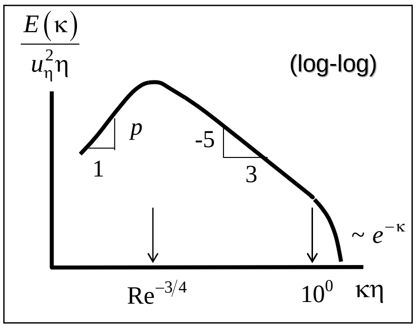
- **limiting shape** $E(\kappa \to 0)$
  - ✓ numerical simulations of isotropic turbulence suggest $\boxed{E(\kappa \to 0) \sim \kappa^p, 2 \le p \le 4}$

- **a model** (see Pope, 2000)

$$\boxed{E(\kappa) = const. \times \varepsilon^{2/3} \kappa^{-5/3} f_L(\kappa l_0)\, f_\eta(\kappa \eta)}$$

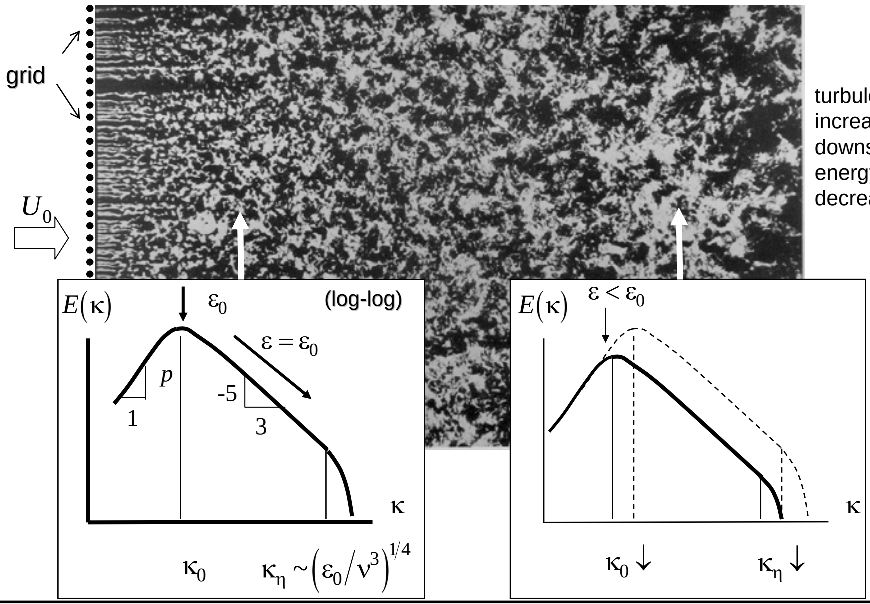
where
$$\begin{cases} f_L(\kappa l_0) = \left(\dfrac{\kappa l_0}{\sqrt{(\kappa l_0)^2 + c_L}} \right)^{5/3+p} \\ f_\eta(\kappa \eta) = \exp\{ -\beta[\, [\, (\kappa\eta)^4 + c_\eta^4 \,]^{1/4} - c_\eta \,]\, \} \end{cases}$$

✓ properties
$$\begin{cases} \lim_{\kappa l_0 \to \infty} f_L(\kappa l_0) = 1 \\ \lim_{\kappa l_0 \to 0} f_L(\kappa l_0) \sim \kappa^{5/3+p} \\ \lim_{\kappa \eta \to \infty} f_\eta(\kappa \eta) \sim e^{-\beta\kappa} \\ \lim_{\kappa \eta \to 0} f_\eta(\kappa \eta) = 1 \end{cases}$$



15.3 how does isotropic turbulence decay ?

✓ how does the energy spectrum deforms ?



turbulent scales
increase
downstream as
energy
decreases