MEC 654
Polytechnique-UPMC-Caltech
Year 2014-2015

Turbulence

chapter 17

eddy viscosity models

17.1 the eddy viscosity concept (…)

• **eddy viscosity : a first level approach for closing the Reynolds equations**

case of the flow of a newtonian incompressible and homogeneous fluid without external force

✓ Navier-Stokes $\begin{cases} div\,\underline{u} = 0 \\ \dfrac{d\,\underline{u}}{dt} = div\ \underline{\underline{\sigma}} \end{cases}$

✓ Cauchy stress tensor (see chapter 3) $\underline{\underline{\sigma}} = -\underbrace{p}_{\text{pressure}}\,\underline{\underline{1}} + \underbrace{\underline{\underline{\tau}}}_{\text{viscous stress tensor}}$

✓ newtonian constitutive law (see chapter 3) $\underline{\underline{\tau}} = \kappa\ \cancel{div\,\underline{u}}\ \underline{\underline{1}} + 2\,\eta\ [\ \underline{\underline{d}} - \frac{1}{3}\ \cancel{div(\underline{u})}\,\underline{\underline{1}}\ ]$

$$\Rightarrow \boxed{\underline{\underline{\sigma}} = -\,p\,\underline{\underline{1}} + 2\eta\ \underline{\underline{d}}}$$

✓ the Reynolds equation (see chapter 12)

$$\begin{cases} \rho\dfrac{D\langle \underline{u}\rangle}{Dt} = div\big(\ \langle \underline{\underline{\sigma}}\rangle - \rho\,\underline{\underline{R}}\ \big) \\ \langle \underline{\underline{\sigma}}\rangle = -\langle\ p\ \rangle\,\underline{\underline{1}} +\ 2\eta\,\langle \underline{\underline{d}}\rangle \\ \underline{\underline{R}} = \langle\,\underline{u}'\otimes\underline{u}'\,\rangle \quad \text{- Reynolds stress tensor} \end{cases}$$

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17.1 the eddy viscosity concept (…)

✓ the Reynolds equation (see chapter 12)

$$\begin{cases} \dfrac{D\langle \underline{u}\rangle}{Dt} = div\Big(\dfrac{1}{\rho}\langle \underline{\underline{\sigma}}\rangle - \underline{\underline{R}}\ \Big) \\ \langle \underline{\underline{\sigma}}\rangle = -\langle\ p\ \rangle\,\underline{\underline{1}} +\ 2\eta\,\langle \underline{\underline{d}}\rangle \\ \underline{\underline{R}} = \langle\,\underline{u}'\otimes\underline{u}'\,\rangle \quad \text{- Reynolds stress tensor} \end{cases}$$

• **Boussinesq's eddy viscosity relationship** $-\rho\,\underline{\underline{R}} \leftrightarrow \langle \underline{\underline{\sigma}}\rangle \Rightarrow \boxed{-\underline{\underline{R}} = -\alpha\,\underline{\underline{1}} + \beta\,\langle \underline{\underline{d}}\rangle}$

• **property** $trace\{\ \underline{\underline{R}}\ \} = \langle u_i^2\rangle = \langle u_1^2\rangle + \langle u_2^2\rangle + \langle u_3^2\rangle = 2\,k = 3\alpha - \beta\ \underbrace{\cancel{div\langle\underline{u}\rangle}}_{trace\{\ \langle \underline{\underline{d}}\rangle\ \}}$

$$\Rightarrow \alpha = \frac{2}{3}k \quad \Rightarrow \quad -\underline{\underline{R}} = -\frac{2}{3}k\,\underline{\underline{1}} + \beta\,\langle \underline{\underline{d}}\rangle \quad \Rightarrow \quad \frac{2}{3}k = \text{"turbulent pressure"}$$

• **analogy** $-\underline{\underline{R}} = -\frac{2}{3}k\,\underline{\underline{1}} +\ \beta\,\langle \underline{\underline{d}}\rangle \Rightarrow \boxed{\underline{\underline{R}} = \frac{2}{3}k\,\underline{\underline{1}} - 2\nu_\varepsilon\,\langle \underline{\underline{d}}\rangle}$

$$\boxed{\begin{array}{l}\beta = 2\nu_\varepsilon \\ \nu_\varepsilon = \text{ dynamic eddy viscosity}\end{array}}$$

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17.1 the eddy viscosity concept (…)

$$
\begin{cases}
\dfrac{D\langle \underline{u}\rangle}{Dt} = div\left(\dfrac{\langle \underline{\underline{\sigma}}\rangle}{\rho} - \underline{\underline{R}} \right) \\
\langle \underline{\underline{\sigma}}\rangle = -\langle p\rangle \underline{\underline{1}} + 2\eta \langle \underline{\underline{d}}\rangle \\
-\underline{\underline{R}} = \langle \underline{u}' \otimes \underline{u}' \rangle \quad \text{- Reynolds stress tensor} \\
-\underline{\underline{R}} + \dfrac{2}{3} k\, \underline{\underline{1}} = 2\nu_\varepsilon \langle \underline{\underline{d}}\rangle \quad \text{- Boussinesq's "eddy viscosity" closure relationship}
\end{cases}
$$

$$\Rightarrow \quad \frac{D\langle \underline{u}\rangle}{Dt} = div\left[-\left(\frac{1}{\rho}\langle p\rangle + \frac{2}{3}k \right)\underline{\underline{1}} + 2(\nu + \nu_\varepsilon)\langle \underline{\underline{d}}\rangle \right]$$

$$\Rightarrow \quad \boxed{\begin{array}{l} \dfrac{D\langle \underline{u}\rangle}{Dt} = -\dfrac{1}{\rho}\underline{grad}\langle \tilde{p}\rangle + div\left[2(\nu + \nu_\varepsilon)\langle \underline{\underline{d}}\rangle \right] \\ \langle \tilde{p}\rangle = \langle p\rangle + \dfrac{2}{3}\rho k \longrightarrow \text{turbulent pressure} \end{array}}$$

• **a closed equation** $\begin{cases} \|\underline{u}\| \sim U \\ p \sim \rho U^2 \end{cases} \Rightarrow \dfrac{\langle \tilde{p}\rangle}{\langle p\rangle} = 1 + \dfrac{2}{3}\dfrac{k}{U} \approx 1$ if $\dfrac{k}{U} << 1$ (turbulent rate)

17.1 the eddy viscosity concept (…)

• **physical viscosity** $\boxed{\nu \sim v \times l}$

thermal velocity ← → free molecular path

• **« eddy viscosity »** $\boxed{\nu_\varepsilon \sim u_0 \times l_0}$

velocity of the "big eddies" conveying energy ← → scale of the « big eddies »

• **zero- equation closure model**

⇒ expressing u_0, l_0 as functions of the mean field : the mixing length model

• **one - equation closure model**

⇒ one can write down an equation for the « eddy viscosity » ν_ε

• **two-equation closure model**

⇒ an equation for u_0 , another for l_0

✓ a popular approach is the k - ε model

$$\begin{cases} u_0 \sim \sqrt{k} \\ l_0 \sim \dfrac{u_0^3}{\varepsilon} \sim \dfrac{k^{3/2}}{\varepsilon} \end{cases} \Rightarrow \boxed{\nu_\varepsilon = const. \times \frac{k^2}{\varepsilon}} \Leftarrow \begin{cases} \text{need of a } k \text{ - equation} \\ \text{need of an } \varepsilon \text{ - equation} \end{cases}$$

17.2 criticism

- **Boussinesq** $\boxed{\langle \underline{u}' \otimes \underline{u}' \rangle - \frac{2}{3} k \, \underline{\underline{1}} = -2\nu_\varepsilon \langle \underline{\underline{d}} \rangle}$

✓ through the Boussinesq's « eddy viscosity » concept, turbulence is treated as a new **viscosity**, that is as a new **physical** property of the **fluid**. this is not justified because turbulence is a **dynamical** property of the flow

✓ through this relationship, the turbulence has **no autonomy** : it follows rigidly and instantaneouly the mean flow distortions

✓ introducing $\begin{cases} \tau_d = \left| \langle \underline{\underline{d}} \rangle \right|^{-1} & \text{- mean distorsion time scale} \\ \tau_\varepsilon = k/\varepsilon & \text{- turbulence turnover time scale} \end{cases}$

the hypothesis is **admissible** when turbulence is "in equilibrium", namely when $\tau_d >> \tau_\varepsilon$ that is in flows whose average properties only change little along the mean streamlines, as in **equilibrium boundary layers**, in **mixing layers**, in **jets**…

✓ this is **unsatisfactory** when $\tau_d \sim \tau_\varepsilon$ as for instance in **separated flows**, in **strongly curved flows** or in **shocked flows**…

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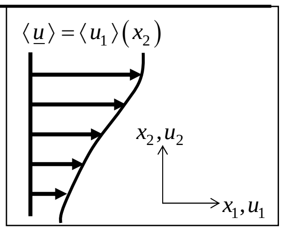
17.3 application : quasi-parallel shear flows



- **hypotheses**

H1 - statistically steady turbulence

H2 - quasi-parallel flow : $\dfrac{\partial}{\partial x_1} \sim \dfrac{\partial}{\partial x_3} << \dfrac{\partial}{\partial x_2}$

- **Boussinesq** $\boxed{-\langle \underline{u}' \otimes \underline{u}' \rangle + \frac{2}{3} k \, \underline{\underline{1}} = 2\nu_\varepsilon \langle \underline{\underline{d}} \rangle}$

$$\Rightarrow \quad \langle \underline{\underline{d}} \rangle = \frac{1}{2}\left(\nabla \langle \underline{u} \rangle + {}^t\nabla \langle \underline{u} \rangle \right) \approx \frac{1}{2} \frac{\partial \langle u_1 \rangle}{\partial x_2} \left(\underline{e}_1 \otimes \underline{e}_2 + \underline{e}_2 \otimes \underline{e}_1 \right)$$

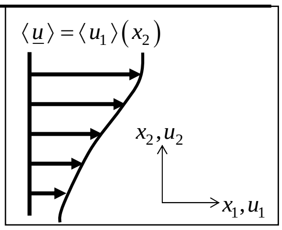
$$\Rightarrow \quad \langle \underline{\underline{d}} \rangle_{ij} \approx \frac{1}{2} \frac{\partial \langle u_1 \rangle}{\partial x_2} \left(\delta_{i1}\delta_{j2} + \delta_{i2}\delta_{j1} \right)$$

$$\Rightarrow \quad -\langle \underline{u}' \otimes \underline{u}' \rangle + \frac{2}{3} k \, \underline{\underline{1}} = \nu_\varepsilon \frac{\partial \langle u_1 \rangle}{\partial x_2} \left(\underline{e}_1 \otimes \underline{e}_2 + \underline{e}_2 \otimes \underline{e}_1 \right)$$

$$\Rightarrow \quad -\langle \underline{u}_i' \underline{u}_j' \rangle + \frac{2}{3} k \, \delta_{ij} = \nu_\varepsilon \frac{\partial \langle u_1 \rangle}{\partial x_2} \left(\delta_{i1}\delta_{j2} + \delta_{i2}\delta_{j1} \right)$$

$$\Rightarrow \quad \begin{cases} \langle \underline{u}_1'^2 \rangle = \langle \underline{u}_2'^2 \rangle = \langle \underline{u}_3'^2 \rangle \approx \frac{2}{3} k \\ \boxed{-\langle \underline{u}_1' \underline{u}_2' \rangle \approx \nu_\varepsilon \dfrac{\partial \langle u_1 \rangle}{\partial x_2}} \\ \langle \underline{u}_1' \underline{u}_3' \rangle = \langle \underline{u}_2' \underline{u}_3' \rangle \approx 0 \end{cases}$$

17.3 application : quasi-parallel shear flows (…)



- **hypotheses** **H1 -** statistically steady turbulence

  H2 - quasi-parallel flow : $\dfrac{\partial}{\partial x_1} \sim \dfrac{\partial}{\partial x_3} << \dfrac{\partial}{\partial x_2}$

- **Reynolds equation**

$$\boxed{\begin{cases} div\, \underline{u} = 0 \\ D\langle \underline{u} \rangle / Dt = -\underline{grad}\, \langle \tilde{p} \rangle / \rho \; + \; div\left[\, 2\nu \langle \underline{\underline{d}} \rangle - \langle \underline{u}' \otimes \underline{u}' \rangle \,\right] \\ \langle \tilde{p} \rangle = \langle p \rangle + \; 2\rho k/3 \end{cases}}$$

✓ following Prandtl's boundary layer analysis and introducing the previous Boussinesq's relationships, one can show that this reduces to (see annex) :

$$\boxed{\begin{cases} \langle u_1 \rangle \dfrac{\partial \langle u_1 \rangle}{\partial x_1} + \langle u_2 \rangle \dfrac{\partial \langle u_1 \rangle}{\partial x_2} = (\; \nu + \nu_\varepsilon \;) \dfrac{\partial^2 \langle u_1 \rangle}{\partial x_2^2} \\ \langle p \rangle + \dfrac{4}{3} \rho\, k \approx const. \end{cases}}$$

note - turbulence contributes to a second order variation of the mean pressure, we neglect usually because the turbulence rate is small (see above, §16.1)

annex - Reynolds equation for a quasi parallel shear flow

- **the different terms of the Reynolds system of equations given above read :**

✓ $$\frac{D\langle \underline{u} \rangle}{Dt} = \underbrace{\cancel{\frac{\partial \langle \underline{u} \rangle}{\partial t}}}_{H1} + \langle u_1 \rangle \frac{\partial \langle \underline{u} \rangle}{\partial x_1} + \langle u_2 \rangle \frac{\partial \langle \underline{u} \rangle}{\partial x_2} + \underbrace{\cancel{\langle u_3 \rangle \frac{\partial \langle \underline{u} \rangle}{\partial x_3}}}_{H2}$$

✓ $$\underline{grad}\, \langle \tilde{p} \rangle = \underline{grad} \left[\, \langle p \rangle + \tfrac{2}{3} \rho\, k \,\right] \overset{H2}{\approx} \frac{\partial}{\partial x_2} \left[\, \langle p \rangle + \tfrac{2}{3} \rho\, k \,\right] \underline{e}_2$$

✓ $$\langle \underline{\underline{d}} \rangle = \tfrac{1}{2} \left(\nabla \langle \underline{u} \rangle + {}^t\nabla \langle \underline{u} \rangle \right) \overset{H2}{\approx} \frac{1}{2} \frac{\partial \langle u_1 \rangle}{\partial x_2} \left(\, \underline{e}_1 \otimes \underline{e}_2 + \underline{e}_2 \otimes \underline{e}_1 \,\right) \Rightarrow div\, \langle \underline{\underline{d}} \rangle \approx \frac{1}{2} \frac{\partial^2 \langle u_1 \rangle}{\partial x_2^2} \underline{e}_1 + \underbrace{\cancel{\frac{1}{2} \frac{\partial}{\partial x_1} \frac{\partial \langle u_1 \rangle}{\partial x_2} \underline{e}_2}}_{H2}$$

✓ $$\langle \underline{u}' \otimes \underline{u}' \rangle - \tfrac{2}{3} k\, \underline{\underline{1}} = -\nu_\varepsilon \frac{\partial \langle u_1 \rangle}{\partial x_2} \left(\, \underline{e}_1 \otimes \underline{e}_2 + \underline{e}_2 \otimes \underline{e}_1 \,\right) \Rightarrow div\, \langle \underline{u}' \otimes \underline{u}' \rangle \overset{H2}{\approx} \frac{2}{3} \frac{\partial k}{\partial x_2} \underline{e}_2 - \nu_\varepsilon \frac{\partial^2 \langle u_1 \rangle}{\partial x_2^2} \underline{e}_1$$

$$div\left(\, k\, \underline{\underline{1}} \,\right) = \underline{grad}\, k$$

$$\Rightarrow \begin{cases} \dfrac{\partial \langle u_1 \rangle}{\partial x_1} + \dfrac{\partial \langle u_2 \rangle}{\partial x_2} = 0 \\ \langle u_1 \rangle \dfrac{\partial \langle u_1 \rangle}{\partial x_1} + \langle u_2 \rangle \dfrac{\partial \langle u_1 \rangle}{\partial x_2} = (\; \nu + \nu_\varepsilon \;) \dfrac{\partial^2 \langle u_1 \rangle}{\partial x_2^2} \\ \langle u_1 \rangle \dfrac{\partial \langle u_2 \rangle}{\partial x_1} + \langle u_2 \rangle \dfrac{\partial \langle u_2 \rangle}{\partial x_2} = -\dfrac{\partial}{\partial x_2} \left[\, \langle p \rangle + \tfrac{4}{3} \rho\, k \,\right] \\ \langle u_1 \rangle \dfrac{\partial \langle u_3 \rangle}{\partial x_1} + \langle u_2 \rangle \dfrac{\partial \langle u_3 \rangle}{\partial x_2} = 0 \end{cases}$$

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annex - Reynolds equation for a quasi parallel shear flow (…)

- **orders of magnitude analysis**

✓ putting
$$\begin{cases} \partial/\partial x_1 = O(\ L^{-1}\ ) \\ \partial/\partial x_2 = O(\ \delta^{-1} >> L^{-1}\ ) \\ \langle u_1 \rangle = O(U) \\ \langle u_2 \rangle = O(V) \\ \langle\ p \rangle = O(\ \rho U^2\ ) \end{cases}$$

✓ the continuity equation imposes $\dfrac{\partial\langle u_1\rangle}{\partial x_1} + \dfrac{\partial\langle u_2\rangle}{\partial x_2} = 0 \quad \Rightarrow \quad \dfrac{U}{L} \sim \dfrac{V}{\delta} \quad \Rightarrow \quad \dfrac{V}{U} \sim \dfrac{\delta}{L} << 1$

✓ orders of magnitude in the momentum equation then are :

$$\Rightarrow \begin{cases} \underbrace{\langle u_1\rangle \dfrac{\partial\langle u_1\rangle}{\partial x_1}}_{O\left(U^2/L\right)} + \underbrace{\langle u_2\rangle \dfrac{\partial\langle u_1\rangle}{\partial x_2}}_{O\left(UV/\delta = U^2/L\right)} = \underbrace{(\ \nu + \nu_\varepsilon\ ) \dfrac{\partial^2\langle u_1\rangle}{\partial x_2^2}}_{O\left(\max\{\nu,\nu_\varepsilon\}\, U/\delta^2\right)} \\ \underbrace{\langle u_1\rangle \dfrac{\partial\langle u_2\rangle}{\partial x_1}}_{O\left(UV/L = U^2/L \times \boxed{\delta/L}\right)} + \underbrace{\langle u_2\rangle \dfrac{\partial\langle u_2\rangle}{\partial x_2}}_{O\left(V^2/\delta = U^2/L \times \boxed{\delta/L}\right)} = \underbrace{-\dfrac{\partial}{\partial x_2}\left[\ \dfrac{1}{\rho}\langle\ p\rangle + \dfrac{4}{3}k\ \right]}_{O\left(U^2/\delta = U^2/L \times \boxed{L/\delta}\right)} \end{cases}$$

annex - Reynolds equation for a quasi parallel shear flow (…)

$$\Rightarrow \begin{cases} \underbrace{\langle u_1\rangle \dfrac{\partial\langle u_1\rangle}{\partial x_1}}_{O(1)} + \underbrace{\langle u_2\rangle \dfrac{\partial\langle u_1\rangle}{\partial x_2}}_{O(1)} = \underbrace{(\ \nu + \nu_\varepsilon\ ) \dfrac{\partial^2\langle u_1\rangle}{\partial x_2^2}}_{O\left(\frac{\max\{\nu,\nu_\varepsilon\}}{U\,L} \times (\frac{\delta}{L})^2\right)} \\ \underbrace{\langle u_1\rangle \dfrac{\partial\langle u_2\rangle}{\partial x_1}}_{O(1)} + \underbrace{\langle u_2\rangle \dfrac{\partial\langle u_2\rangle}{\partial x_2}}_{O(1)} = \underbrace{-\dfrac{\partial}{\partial x_2}\left[\ \dfrac{1}{\rho}\langle\ p\rangle + \dfrac{4}{3}k\ \right]}_{O\left((\ \delta/L\ )^{-2}\right)} \end{cases}$$

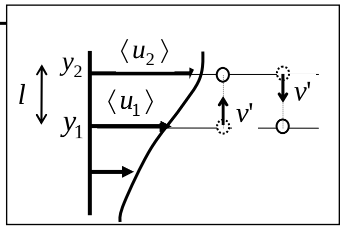
✓ putting $\mathrm{Re} = \dfrac{U\,L}{\max\{\nu,\nu_\varepsilon\}} >> 1$, a viscous and/or turbulent flow regime requires :

$$\mathrm{Re}^{-1}(\ \frac{\delta}{L}\ )^2 = 1 \quad \Rightarrow \quad \frac{\delta}{L} \sim \sqrt{\mathrm{Re}}$$

✓ the u_2- equation then reads :

$$\langle\ p\rangle + \frac{4}{3}\rho k \approx const.$$

17.4 a displaced particle argument



- **quasi parallel shear flow**

✓ Boussinesq's relationship reads $\boxed{\langle u'v' \rangle \approx -\nu_\varepsilon \dfrac{\partial \langle u \rangle}{\partial y}}$

- **eddy viscosity** $\boxed{\nu_\varepsilon \sim v\, l}$

- **a displaced particle argument**

✓ let a fluid particle being displaced in a shear flow of rate $\dfrac{d\langle u \rangle}{dy} \approx \dfrac{\langle u_2 \rangle - \langle u_1 \rangle}{l}$

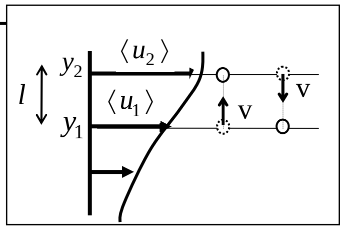
✓ supposing the horizontal momentum of the particle is conserved during the process :

➢ if $v' > 0$ the fluctuation u' at y_2 is : $u' = \langle u_1 \rangle - \langle u_2 \rangle = -l \dfrac{\partial \langle u \rangle}{\partial y}$

➢ if $v' < 0$ the fluctuation u' at y_1 is : $u' = \langle u_2 \rangle - \langle u_1 \rangle = l \dfrac{\partial \langle u \rangle}{\partial y}$

$\Rightarrow \boxed{\langle u' v' \rangle = -\underbrace{(\,|v|\, l\,)}_{\nu_\varepsilon (>0)} \dfrac{\partial \langle u \rangle}{\partial y}}$

17.4 a displaced particle argument



- **physical viscosity versus eddy viscosity**

$$\boxed{\langle u' v' \rangle = -\underbrace{(\,|v|\, l\,)}_{\nu_\varepsilon (>0)} \frac{\partial \langle u \rangle}{\partial y}}$$

✓ physical viscosity $\left\{ \begin{array}{l} \text{thermal velocity } v \sim 300\, m.s^{-1} \\ \text{free molecular path } l \sim 10^{-7}\, m \text{ (gaz)} \end{array} \right\} \Rightarrow \boxed{\nu \sim v \times l} \sim 10^{-5} - 10^{-4}\, m^2.s^{-1}$

✓ « eddy viscosity » $\left\{ \begin{array}{l} \text{« big eddies » velocity } u_0 \\ \text{« big eddies » scale } l_0 \end{array} \right\} \Rightarrow \boxed{\nu_\varepsilon \sim u_0 \times l_0}$ such as $\mathrm{Re}_\varepsilon = \dfrac{u_0 \times l_0}{\nu_\varepsilon} \sim 1$

✓ mixing layer $\left\{ \begin{array}{l} u_0 \approx 0.1\, \Delta U \\ l_0 \approx \delta(x) \end{array} \right. \Rightarrow \mathrm{Re}_\varepsilon = \dfrac{\Delta U \times \delta(x)}{\nu} \dfrac{\nu}{\nu_\varepsilon} = \mathrm{Re} \dfrac{\nu}{\nu_\varepsilon} \sim 1 \Rightarrow \boxed{\nu_\varepsilon \sim \mathrm{Re} \times \nu}$

✓ boundary layer $\left\{ \begin{array}{l} u_0 \approx u_\tau \\ l_0 \approx y \end{array} \right. \Rightarrow \mathrm{Re}_\varepsilon = \dfrac{u_\tau\, y}{\nu} \dfrac{\nu}{\nu_\varepsilon} = y^+ \dfrac{\nu}{\nu_\varepsilon} \sim 1 \Rightarrow \boxed{\nu_\varepsilon \sim y^+ \times \nu}$

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17.5 a zéro equation model : the mixing length model (*)

- **quasi parallel shear flow** $\boxed{\langle u' v' \rangle = -\underbrace{(\,|\mathrm{v}|\,l\,)}_{\nu_\varepsilon\,(>0)} \dfrac{\partial \langle u \rangle}{\partial y}}$

$\Rightarrow$ the two scales v and l must be expressed directly as a function of the mean field

- **the velocity scale** v

✓ in a shear flow $|\,u'\,| \sim |\langle u_2 \rangle - \langle u_1 \rangle| = l\,\dfrac{\partial \langle u \rangle}{\partial y} \;\Rightarrow\; \langle u'^2 \rangle \sim l^2 \left(\dfrac{\partial \langle u \rangle}{\partial y}\right)^2$

✓ suppose now that $\langle u'^2 \rangle \approx \langle v'^2 \rangle$ (isotropy) : $\mathrm{v} \sim \sqrt{\langle v'^2 \rangle} \sim \sqrt{\langle u'^2 \rangle} \sim l \left| \dfrac{\partial \langle u \rangle}{\partial y} \right|$

$$\Rightarrow \boxed{\langle u' v' \rangle = -(\,|\mathrm{v}|\,l\,)\frac{\partial \langle u \rangle}{\partial y} = -l^2 \left|\frac{\partial \langle u \rangle}{\partial y}\right| \frac{\partial \langle u \rangle}{\partial y}} \Rightarrow \boxed{\nu_\varepsilon = l^2 \left|\frac{\partial \langle u \rangle}{\partial y}\right|}$$

- **the “mixing length”** l

✓ boundary layers : $l = \kappa\, y, \quad \kappa = 0,4$ Von-Karman (1930)

✓ mixing layers : $l \sim (\,\Delta U / \overline{U}\,)\, x$

(*) Kolmogorov (1942), Prandtl (1945)

16.5 a zéro equation model : the mixing length model (…)

- **quasi parallel shear flow**

$$\boxed{\begin{aligned}&\langle u' v' \rangle = -\nu_\varepsilon \frac{\partial \langle u \rangle}{\partial y}\\ &\nu_\varepsilon = l^2 \left|\frac{\partial \langle u \rangle}{\partial y}\right|\end{aligned}}$$

- **generalisation**

$$\boxed{\begin{aligned}&\langle \underline{u}' \otimes \underline{u}' \rangle = -\nu_\varepsilon\, \underline{\underline{d}}\\ &\nu_\varepsilon = \nu_\varepsilon(l, \nabla \underline{u})\end{aligned}}$$

Smagorinsky (1963) $\boxed{\nu_\varepsilon = l^2 \sqrt{\langle \underline{\underline{d}} \rangle : \langle \underline{\underline{d}} \rangle}}$ $\quad \langle \underline{\underline{d}} \rangle : \langle \underline{\underline{d}} \rangle = 2\left(\dfrac{\partial u_i}{\partial x_j}\right)^2 + 2\dfrac{\partial u_i}{\partial x_j}\dfrac{\partial u_j}{\partial x_i}$

see annex

Baldwin & Lomax (1978) $\boxed{\nu_\varepsilon = l^2 \sqrt{\langle \underline{\underline{\Omega}} \rangle : \langle \underline{\underline{\Omega}} \rangle}}$ $\quad \langle \underline{\underline{\Omega}} \rangle : \langle \underline{\underline{\Omega}} \rangle = 2\left(\dfrac{\partial u_i}{\partial x_j}\right)^2 - 2\dfrac{\partial u_i}{\partial x_j}\dfrac{\partial u_j}{\partial x_i}$

- **… and many other formulations, all being empirical**

annex – Smagorinsky versus Baldwin-Lomax

$$\langle \underline{\underline{d}} \rangle : \langle \underline{\underline{d}} \rangle = \frac{1}{4}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} \right)\left(\underline{e}_i \otimes \underline{e}_j \right) : \left(\underline{e}_l \otimes \underline{e}_k \right)$$

$$= \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} \right)\delta_{jl}\delta_{ik} = \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

$$= \left(\frac{\partial u_i}{\partial x_j} \right)^2 + \left(\frac{\partial u_j}{\partial x_i} \right)^2 + 2\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i} = 2\left(\frac{\partial u_i}{\partial x_j} \right)^2 + 2\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}$$

$$\langle \underline{\underline{\Omega}} \rangle : \langle \underline{\underline{\Omega}} \rangle = \frac{1}{4}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right)\left(\frac{\partial u_k}{\partial x_l} - \frac{\partial u_l}{\partial x_k} \right)\left(\underline{e}_i \otimes \underline{e}_j \right) : \left(\underline{e}_l \otimes \underline{e}_k \right)$$

$$= \ldots = 2\left(\frac{\partial u_i}{\partial x_j} \right)^2 - 2\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}$$

17.6 a one-equation model : the Spalart-Allmaras model (*)

• a transport equation for the eddy viscosity

$$\boxed{\frac{D\nu_\varepsilon}{Dt} = \underbrace{S_{\nu_\varepsilon}}_{\text{source of } \nu_\varepsilon} + div\left[\underbrace{\chi_\varepsilon \, \underline{grad}\left(\nu_\varepsilon \right)}_{\text{flux of } \nu_\varepsilon} \right]}$$

✓ source term $S_{\nu_\varepsilon} = f\left(\nu_\varepsilon, \underline{grad}\left(\nu_\varepsilon \right), \Omega, l_{\min} \right)$

where $\begin{cases} \Omega = \text{rotation rate} \\ l_{\min} = \text{minimum distance from the wall} \end{cases}$

✓ flux term $\begin{cases} \chi_\varepsilon = \nu_\varepsilon / \sigma_\varepsilon = \text{diffusivity of } \nu_\varepsilon \\ \sigma_\varepsilon = \text{turbulent Prandtl number} \end{cases}$

⇨ a “UFO”, 100% empirical…that **works remarkably well** for the modeling of wall flows for which it has been conceived

(*) Spalart & Allmaras (1994)

17.7 two - equation models : the k - ε model (…)

- **a transport equation for each of the two variables v and l of** $\nu_\varepsilon \sim |\mathrm{v}|\, l$

✓ velocity : $\mathrm{v} \sim \sqrt{k}$

✓ lengthscale : using $\varepsilon = \dfrac{\mathrm{v}^3}{l}$ leads to $l = \dfrac{\mathrm{v}^3}{\varepsilon} \sim \dfrac{k^{3/2}}{\varepsilon}$ (*)

⇨ eddy viscosity : $\nu_\varepsilon \sim \dfrac{k^2}{\varepsilon} = C_\eta \dfrac{k^2}{\varepsilon}$ ⇨ need for a k - equation and for a ε - equation

- **at first, let's check the validity of** $\nu_\varepsilon = C_\eta \dfrac{k^2}{\varepsilon}$

✓ let's consider quasi-parallel shear flows ⇨ next slide

(*) many contributors. Jones & Lauder (1972) for the foundations

17.7 two - equation models : the k - ε model (…)

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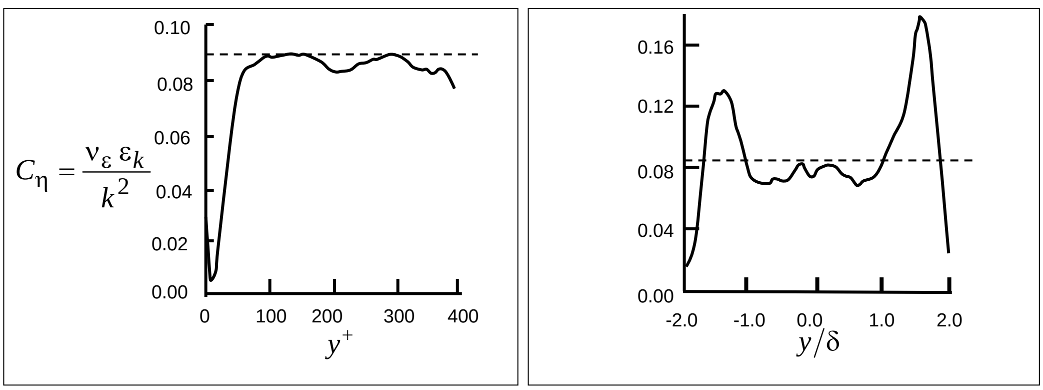
channel flow : DNS Re=13 750 (*) temporal mixing layer : DNS (**)



⇨ correct in regions where turbulence is in equilibrium (logarithmic region, mixing layer center region)

⇨ in the standard k - ε model: $\boxed{C_\eta \approx 0.09}$

(*) Rogers & Moser (1994) ; (**) Kim et al. (1987)

17.7 two - equation models : the k - ε model (...)

• k - equation $\quad \boxed{\frac{Dk}{Dt} = P + div(\phi_k) - \varepsilon_k}$

$$\begin{cases} P \approx -\langle \underline{u}' \otimes \underline{u}' \rangle : \nabla \underline{u} = \nu_\varepsilon \; \nabla\langle \underline{u}\rangle : \langle \nabla \underline{u}\rangle \\ \underline{\phi}_k = -\frac{1}{\rho}\langle p' \underline{u}' \rangle - \frac{1}{2}\langle \underline{u}'^2 \underline{u}' \rangle + \nu \; \underline{grad}\, k \\ \varepsilon_k = \varepsilon_{1k} = 2\nu \langle |\nabla \underline{u}'|^2 \rangle \end{cases}$$

• ε - equation $(\varepsilon = \varepsilon_k)$: far too complex !

⇨ one « mimics» the k-equation (**) : $\boxed{\frac{D\varepsilon}{Dt} = \underbrace{P_\varepsilon}_{\text{production}} + div(\underbrace{\phi_\varepsilon}_{\text{flux}}) - \underbrace{\varepsilon_\varepsilon}_{\text{destruction}}}$

⇨ k - ε modelling

$$\begin{cases} \langle \underline{u}' \otimes \underline{u}' \rangle = -\nu_\varepsilon \; \underline{\underline{d}} \\ \nu_\varepsilon \sim \frac{k^2}{\varepsilon} = C_\mu \frac{k^2}{\varepsilon} \\ P = -\langle \underline{u}' \otimes \underline{u}' \rangle : \nabla \underline{u} = \nu_\varepsilon \; \underline{\underline{d}} : \langle \nabla \underline{u} \rangle \quad \text{- production} \\ \left.\begin{array}{l} \frac{Dk}{Dt} = P + div \boxed{\phi_k} - \varepsilon \\ \frac{D\varepsilon}{Dt} = \boxed{P_\varepsilon} + div \boxed{\phi_\varepsilon} - \boxed{\varepsilon_\varepsilon} \end{array}\right\} \Leftarrow \text{framed terms must be closed} \end{cases}$$

17.7 two - equation models : the k - ε model (...)

• k - ε modelling (...)

$$\left.\begin{array}{l} \frac{Dk}{Dt} = P + div \boxed{\phi_k} - \varepsilon \\ \frac{D\varepsilon}{Dt} = \boxed{P_\varepsilon} + div \boxed{\phi_\varepsilon} - \boxed{\varepsilon_\varepsilon} \end{array}\right\} \Leftarrow \text{framed terms must be closed}$$

• **fluxes : first gradient formulation**

$$\begin{cases} \underline{\phi}_a = div(\chi_\varepsilon \; \underline{grad}\; a) \\ \chi_\varepsilon = \nu_\varepsilon / \sigma_\varepsilon \quad \text{- diffusivity} \\ \sigma_\varepsilon \quad \text{- turbulent Prandtl number} \end{cases}$$

• **dissipation rate : production - destruction**

$$\begin{cases} P_\varepsilon = C_{\varepsilon 1} \frac{\varepsilon}{k} P \quad \text{- production} \\ \varepsilon_\varepsilon = C_{\varepsilon 2} \frac{\varepsilon}{k} \varepsilon \quad \text{- destruction} \end{cases}$$

✓ characteristic time scale $\tau = k/\varepsilon$

the k - ε model

$$\begin{cases} \underline{u}' \otimes \underline{u}' \rangle = -\nu_\varepsilon \; \underline{\underline{d}} \\ \nu_\varepsilon = C_\mu k^2/\varepsilon \\ P = -\langle \underline{u}' \otimes \underline{u}' \rangle : \nabla \underline{u} = \nu_\varepsilon \; \underline{\underline{d}} : \nabla \langle \underline{u} \rangle \\ \begin{cases} \frac{Dk}{Dt} = P + div\left(\frac{\nu_\varepsilon}{\sigma_k} \underline{grad}\, k\right) - \varepsilon \\ \frac{D\varepsilon}{Dt} = C_{\varepsilon 1} \frac{\varepsilon}{k} P + div\left(\frac{\nu_\varepsilon}{\sigma_\varepsilon} \underline{grad}\, \varepsilon\right) - C_{\varepsilon 2} \frac{\varepsilon}{k} \varepsilon \end{cases} \end{cases}$$

⇨ 5 constants : $C_\eta, C_{\varepsilon 1}, C_{\varepsilon 2}, \sigma_k, \sigma_\varepsilon$

17.7 two - equation models : the k - ε model (...)

- **basic flows are used to fix the constants**

- **flow 1** : temporal decay of isotropic turbulence

$$\Rightarrow \begin{cases} \text{no mean flow}: \nabla\langle \underline{u} \rangle = P = 0 \\ \text{homogeneous turbulence}: \underline{grad}\, k = \underline{grad}\, \varepsilon = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \dfrac{dk}{dt} = -\varepsilon \\ \dfrac{d\varepsilon}{dt} = -C_{\varepsilon 2}\dfrac{\varepsilon^2}{k} \end{cases} \Rightarrow \boxed{\frac{k(t)}{k(0)} \sim \left(\frac{t}{t_0}\right)^{-\alpha}} \quad \alpha = \frac{1}{C_{\varepsilon 2} - 1}$$

(see training lecture)

✓ experiments $\alpha \approx 1.3 \Rightarrow C_{\varepsilon 2} = 1.77$

✓ standart k - ε model $\alpha \approx 1.09 \Leftarrow \boxed{C_{\varepsilon 2} = 1.92}$

17.7 two - equation models : the k - ε model (...)

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- **flow 2** : homogeneous shear flow $\dfrac{\partial\langle u_i \rangle}{\partial x_j} = S\,\delta_{i1}\,\delta_{j2}$

$\Rightarrow$ homogeneous turbulence: $\underline{grad}\, k = \underline{grad}\, \varepsilon = 0$ $\Rightarrow$

$$\begin{cases} P = -S(t)\langle u_1' u_2' \rangle \\ \dfrac{dk}{dt} = P - \varepsilon & (1) \\ \dfrac{d\varepsilon}{dt} = C_{\varepsilon 1}\dfrac{\varepsilon}{k}P - C_{\varepsilon 2}\dfrac{\varepsilon^2}{k} & (2) \end{cases}$$

✓ introducing the time scale $\tau = \dfrac{k}{\varepsilon}$

(1) $\Rightarrow$ (1') $\boxed{\dfrac{\tau}{k}\dfrac{dk}{dt} = \dfrac{P}{\varepsilon} - 1}$ homework

(1) + (2) $\Rightarrow$ (2') $\boxed{\dfrac{d\tau}{dt} = (C_{\varepsilon 2} - 1) - (C_{\varepsilon 1} - 1)\dfrac{P}{\varepsilon}}$ homework

✓ experiments and DNS show that: $\boxed{S\,\tau \approx const.}$ $\boxed{\dfrac{P}{\varepsilon} \approx const. \approx 1.7}$

(1') $\Rightarrow$ $\dfrac{k(t)}{k(0)} = e^{\frac{t}{\tau}\left(\frac{P}{\varepsilon} - 1\right)} \to \infty$

<u>interpretation</u> : turbulence extracts energy in the reservoir of the mean flow energy which is infinite if the flow is statistically homogeneous (no feedback on the mean flow, see §13.6)

(2') $\Rightarrow$ $\dfrac{C_{\varepsilon 2} - 1}{C_{\varepsilon 1} - 1} = \dfrac{P}{\varepsilon} \approx 1,7$ $\quad C_{\varepsilon 2} = 1.77 \Rightarrow \boxed{C_{\varepsilon 1} = 1.45}$

17.7 two - equation models : the k - ε model (…)

✓ **other flows :** boundary layers (log region), jet, …

⇨ provide the turbulent Prandtl numbers constants $\sigma_k, \sigma_\varepsilon$

⇨ suggest other compromises concerning the values of the constants obtained in «pure » situations (isotropy, homogeneity…)

⇨ **standard values**

$$\left(C_\eta, C_{\varepsilon 1}, C_{\varepsilon 2}, \sigma_k, \sigma_\varepsilon\right) = \left(0.09, 1.44, 1.92, 1.0, 1.3\right)$$

<u>Note</u> - with such values of $C_{\varepsilon 1}, C_{\varepsilon 2}$ one gets $\dfrac{P}{\varepsilon} = \dfrac{C_{\varepsilon 2} - 1}{C_{\varepsilon 1} - 1} \approx 2{,}1$, instead of $\dfrac{P}{\varepsilon} = 1.7$ as found in homogenous shear flows

⇨ other ajustements are possible from case to case

⇨ no turbulence model can pretend to be universal

17.8 the case of a scalar quantity

- **temperature**

✓ equation of temperature for a heated flow of newtonian incompressible homogeneous fluid

$$\begin{cases} \rho c \dfrac{dT}{dt} = 2\eta \, \underline{\underline{d}} : \underline{\underline{d}} + \mathrm{K} \Delta T \\ \epsilon = 2\eta \, \underline{\underline{d}} : \underline{\underline{d}} \ \text{ - dissipation rate} \\ c \text{ - heat capacity, } \mathrm{K} \text{ - conductivity} \end{cases}$$

✓ neglecting dissipation (see annex)

$$\begin{cases} \dfrac{dT}{dt} = \chi \, \Delta T \\ \chi = \dfrac{\mathrm{K}}{\rho c} \ = \text{temperature diffusivity} \end{cases}$$

⇨ a « heat equation » valid for describing the **diffusion of any passive scalar** $\theta(\underline{x}, t)$ (temperature ,mass, …) in a flow

$$\boxed{\begin{cases} \dfrac{d\theta}{dt} = \chi \, \Delta \theta \\ \chi = \text{diffusivity} \end{cases}}$$

annex – neglecting dissipation in the heat equation

$$\begin{cases} \rho c \dfrac{dT}{dt} = \cancel{2\eta \underline{\underline{d}}:\underline{\underline{d}}} + K\Delta T \\ \epsilon = 2\eta\ \underline{\underline{d}}:\underline{\underline{d}} \text{ - dissipation rate} \\ c \text{ - heat capacity, } K \text{ - conductivity} \end{cases}$$

newtonian incompressible homogeneous fluid

- **orders of magnitude**

$$x = L\,\bar{x},\ t = \frac{L}{U}\bar{t}, \underline{u} = U\,\bar{\underline{u}}, T = T_0 + \delta T\,\bar{T} \quad \Rightarrow \quad \left(\frac{\rho c\,\delta T\,U}{L}\right)\frac{d\bar{T}}{d\bar{t}} = \left(\frac{2\eta U^2}{L^2}\right)\bar{\underline{\underline{d}}}:\bar{\underline{\underline{d}}} + \left(\frac{K\,\delta T}{L^2}\right)\Delta T$$

$$\Rightarrow \quad \frac{d\bar{T}}{d\bar{t}} = 2\left(\frac{\mathrm{Ec}}{\mathrm{Re}}\right)\bar{\underline{\underline{d}}}:\bar{\underline{\underline{d}}} + \left(\frac{1}{\mathrm{Pr\,Re}}\right)\Delta\bar{T} \qquad \begin{cases} \mathrm{Re} = UL/\nu \text{ - Reynolds} \\ \mathrm{Ec} = U^2/c\delta T \text{ - Eckert} \\ \mathrm{Pr} = \nu/(K/\rho c) \text{ - Prandtl} \end{cases}$$

⇨ dissipation term negligible if : $\delta T >> U^2/c$

⇨ meaning : small velocity U and temperature variations δT sufficiently large (*)

(*) however, δT must remain in the limit of incompressibility, $\alpha\ \delta\,\mathrm{T} << 1$, where α denotes the dilatation coefficient ($\alpha = 1/T$ for a perfect gaz)

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17.8 the case of a scalar quantity (…)

- **diffusion equation**

$$\boxed{\begin{cases} \dfrac{d\theta}{dt} = \chi\ \Delta\theta \\ \chi = \text{coefficient of diffusion} \end{cases}}$$

✓ the transport equation of the mean scalar $\langle\theta\rangle$ reads

$$\frac{D\langle\theta\rangle}{Dt} = div\left(\ \chi\ \underline{grad}\ \langle\theta\rangle - \underbrace{\langle \underline{u}'\ \theta'\rangle}_{\text{turbulent flux of } \theta}\ \right)$$

- **Boussinesq's « eddy diffusivity » relationship**

$$\boxed{\langle \underline{u}'\ \theta'\rangle = -\underbrace{\chi_\varepsilon}_{\text{turbulent diffusivity}}\ \underline{grad}\ \langle\theta\rangle}$$

$$\Rightarrow \quad \boxed{\rho c \frac{D\langle\theta\rangle}{Dt} = div\left[\ (\ \chi + \chi_\varepsilon\ )\ \underline{grad}\ \langle\theta\rangle\ \right]}$$

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17.9 conclusion

- **closing the Reynolds equations**

✓ a forty years' effort

✓ no universal model

✓ a choice of models and recommendations

- **« eddy viscosity » models**

✓ 95% of the industrial softwares

✓ limits: flows strongly out of equilibrium (in strong spectral imbalance)

✓ consecutive to rapid mean flow distorsions (curvatures, rotations, waves, separations, impacts...)

chapter 18

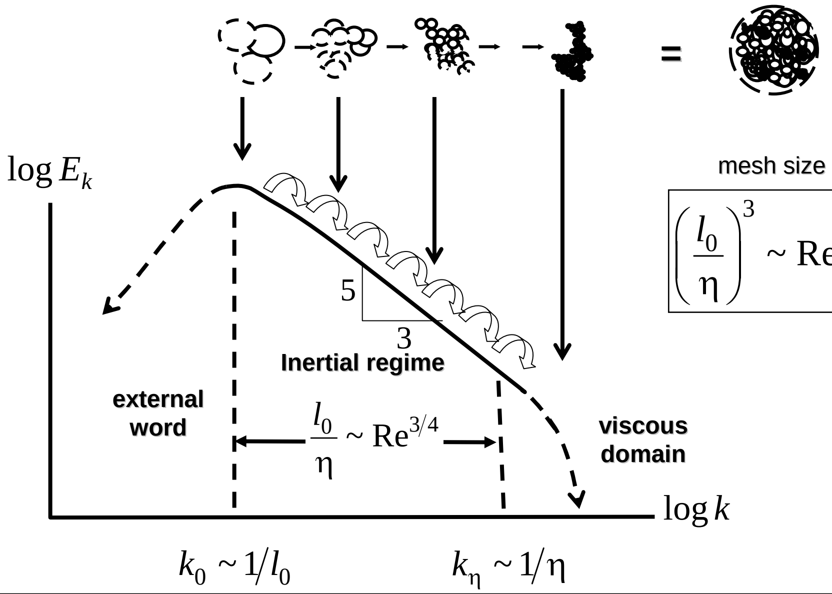
simulations : DNS and LES

18.1 Direct Numerical Simulation (DNS)

18.2 Large Eddy Simulation (LES)

18.1 Direct Numerical Simulation (DNS)

- **back to chapter 5**



mesh size

$$\boxed{\left(\frac{l_0}{\eta}\right)^3 \sim \mathrm{Re}^{9/4}}$$

18.1 Direct Numerical Simulation (DNS) (…)

- **the cost of a DNS**

✓ spatial mesh size : $N_p = \left(\frac{l_0}{\eta}\right)^3 \sim \mathrm{Re}_0^{9/4}$

✓ time mesh size : $\tau_l \sim \frac{l}{u_l} \sim \frac{l}{(\varepsilon_0 l)^{1/3}} \Rightarrow N_\tau \sim \frac{\tau_0}{\tau_\eta} \sim \frac{\left(l_0^2/\varepsilon_0\right)^{1/3}}{\left(\eta^2/\varepsilon_0\right)^{1/3}} = \left(\frac{l_0}{\eta}\right)^{\frac{2}{3}} = \left(\mathrm{Re}_0^{3/4}\right)^{\frac{2}{3}} = \mathrm{Re}_0^{1/2}$

✓ total cost: $\boxed{N_p \times N_\tau \sim \mathrm{Re}_0^{11/4}}$

$$\mathrm{Re}_0 = 10^6 \begin{cases} N_p \sim 10^{54/4} \simeq 310^{13} \\ N_\tau \sim 10^3 \\ N \sim 310^{16} \end{cases}$$

- **warning** : precise (high orders) numerical schemes are required.

18.1 Direct Numerical Simulation (DNS) (…)

- **DNS : state of the art**

✓ homogeneous turbulence
✓ unseparated boundary layers
✓ channel, pipes
✓ mixing layers, jets, wakes, trailing vortices

} at moderate Reynolds number

- **why do we need DNS ?**

✓ a DNS is it is the most complete experience that we can achieve. It provides information inaccessible to measurement

✓ a successful DNS always becomes a standard reference for physical interpretation and modelling

- **perspectives**

✓ an airplane … in 2070

✓ shall we close the wind tunnels in the next century ?

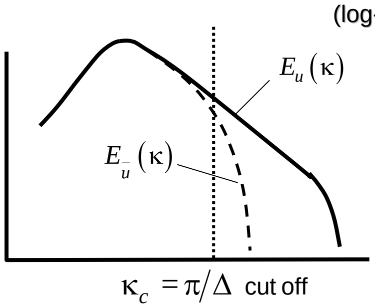
18.2 Large Eddy Simulation (LES)

- **spatial filtering**

$$\boxed{\overline{u}_i\left(\underline{x},t\right)=G_\Delta \underset{\uparrow}{*} u_i\left(\underline{x},t\right)=\int G_\Delta\left(\underline{x},\underline{\tilde{x}}\right)u_i\left(\underline{x}-\underline{\tilde{x}},t\right)d\underline{\tilde{x}}} \qquad \int G_\Delta\left(\underline{x},\underline{\tilde{x}}\right)d\underline{\tilde{x}}=1$$

convolution

(log-log)



- **properties**

✓ decomposition : $u_i=\overline{u}_i+u_i'$

✓ non idempotent : $\overline{\overline{u}_i}=\overline{\left(\overline{u}_i+u_i'\right)}=\overline{\overline{u}}_i+\underbrace{\overline{u_i'}}_{\neq 0}$

$\Rightarrow \overline{\overline{u}}_i\neq\overline{u}_i$

✓ commutation with time derivation

✓ commutation with space derivation at the condition where **:**

$$\boxed{G_\Delta\left(\underline{x},\underline{\tilde{x}}\right)=G_\Delta\left(\underline{x}-\underline{\tilde{x}}\right)}$$

homogeneous filter
(see annex)

annex – LES : space filtering and derivation

✓ space derivation of the convolution product $\overline{u}_i\left(\underline{x},t\right)=\int G_\Delta\left(\underline{x},\underline{\tilde{x}}\right)u_i\left(\underline{x}-\underline{\tilde{x}},t\right)d\underline{\tilde{x}}$

$$\frac{\partial\overline{u}_i\left(\underline{x},t\right)}{\partial x_j}=\int G_\nabla\left(\underline{x},\underline{\tilde{x}}\right)\frac{\partial u_i\left(\underline{x}-\underline{\tilde{x}},t\right)}{\partial x_j}d\underline{\tilde{x}}+\int\frac{\partial G_\nabla\left(\underline{x},\underline{\tilde{x}}\right)}{\partial x_j}u_i\left(\underline{x}-\underline{\tilde{x}},t\right)d\underline{\tilde{x}}+$$

$$=\overline{\left(\frac{\partial u_i}{\partial x_j}\right)}\left(\underline{x},t\right)+\int\frac{\partial G_\nabla\left(\underline{x},\underline{\tilde{x}}\right)}{\partial x_j}u_i\left(\underline{x}-\underline{\tilde{x}},t\right)d\underline{\tilde{x}}$$

$$\frac{\partial\overline{u}_i\left(\underline{x},t\right)}{\partial x_j}=\overline{\left(\frac{\partial u_i}{\partial x_j}\right)}\left(\underline{x},t\right)\quad\Longleftrightarrow\quad\boxed{G_\Delta\left(\underline{x},\underline{\tilde{x}}\right)=G_\Delta\left(\underline{l}=\underline{x}-\underline{\tilde{x}}\right)}$$

homogeneous filter

18.2 Large Eddy Simulation (LES) (…)

• filters : examples

✓ « box filter »

$$G_\Delta\left(\underline{x}-\underline{\tilde{x}}\right)=\begin{cases}1/\Delta^3 & \text{if } \left\|x_i-\tilde{x}_i\right\|\leq\dfrac{\Delta}{2}\\ 0 & \text{otherwise}\end{cases}$$

$$\Delta=\left(\Delta x_1\Delta x_2\Delta x_3\right)^{1/3}$$

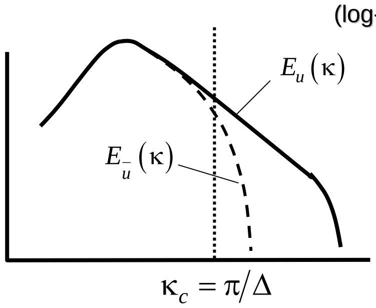
✓ gaussian filter

$$G_\Delta\left(\underline{x}-\underline{\tilde{x}}\right)=\left(\frac{6}{\pi}\right)^{3/2}\frac{1}{\Delta^3}\exp\left(-6\frac{\left(\underline{x}-\underline{\tilde{x}}\right)^2}{\Delta^2}\right)$$

✓ spectral filter

$$G_\Delta\left(\underline{\kappa}\right)=\begin{cases}1 & \text{if } \left\|\underline{\kappa}\right\|\leq\kappa_c=\dfrac{\pi}{\Delta}\\ 0 & \text{otherwise}\end{cases}$$

(log-log)



• resolution

✓ L = size of the computational domain

✓ N_p points in each directions

⇨ mesh size : $h=\dfrac{L}{N_p}$

⇨ maximum resolution : $\kappa_{\max}=\dfrac{2\pi}{h}$

⇨ $\boxed{\kappa_c\leq\dfrac{\kappa_{\max}}{2}=\dfrac{\pi}{h}}$ (Nyquist theorem)

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18.2 Large Eddy Simulation (LES) (…)

- **space filtered Navier- Stokes equations**

✓ same as the Reynolds decomposition : $\boxed{\underline{u}(\underline{x},t)=\overline{\underline{u}}(\underline{x},t)+\underline{u}'(\underline{x},t)}$

✓ however now : $\begin{cases}\overline{\overline{\underline{u}}}\neq\overline{\underline{u}}\\ \overline{\underline{u}'}\neq 0\end{cases}$

✓ continuity $\begin{cases}\overline{div\,\underline{u}}=\boxed{0=div\,\overline{\underline{u}}}\\ div\,\underline{u}'=div(\underline{u}-\overline{\underline{u}})=0\end{cases}$ **nota** : only homogeneous filters are considered

✓ momentum $\boxed{\frac{\overline{D}\overline{\underline{u}}}{\overline{D}t}=\frac{\partial\overline{\underline{u}}}{\partial t}+div(\overline{\underline{u}}\otimes\overline{\underline{u}})=-\frac{1}{\rho}\underline{grad}\,\overline{p}+div(\nu\nabla\overline{\underline{u}}-\boxed{\underline{\underline{\tau}}^L})}$

- **Leonard's tensor**

$$\boxed{\underline{\underline{\tau}}^L=\underbrace{[(\overline{\overline{\underline{u}}\otimes\overline{\underline{u}}}-\overline{\underline{u}}\otimes\overline{\underline{u}})+\overline{\overline{\underline{u}}\otimes\underline{u}'}+\overline{\underline{u}'\otimes\overline{\underline{u}}}]}_{\text{additional terms}}+\overline{\underline{u}'\otimes\underline{u}'}}$$

$\overline{\underline{u}'\otimes\underline{u}'} \Leftrightarrow \langle\underline{u}'\otimes\underline{u}'\rangle$ Reynolds stress tensor

18.2 Large Eddy Simulation (LES) (…)

- **Leonard's tensor (…)**

$$\boxed{\underline{\underline{\tau}}^L=\left[\underbrace{\left(\overline{\overline{\underline{u}}\otimes\overline{\underline{u}}}-\overline{\underline{u}}\otimes\overline{\underline{u}}\right)}_{\text{resolved scales}}+\underbrace{\overline{\overline{\underline{u}}\otimes\underline{u}'}+\overline{\underline{u}'\otimes\overline{\underline{u}}}}_{\text{strain due to scale interactions}}\right]+\underbrace{\overline{\underline{u}'\otimes\underline{u}'}}_{\text{« subgrid scale » strain}}}$$

strain due to scale interactions: $\kappa<\kappa_c$ and $\kappa>\kappa_c$

« subgrid scale » strain $\kappa>\kappa_c$

- **filtered kinetic energy** $\boxed{\overline{e_c}=\underbrace{\tfrac{1}{2}\overline{\underline{u}}.\overline{\underline{u}}}_{K_c}+\underbrace{\tfrac{1}{2}(\overline{\underline{u}.\underline{u}}-\overline{\underline{u}}.\overline{\underline{u}})}_{k_c}}$

resolved scale energy $=K_c$ $\qquad k_c=$ residual energy

$$P_c=\tau_{il}{}^R\frac{\partial\overline{u}_i}{\partial x_l}$$

exchanges

annex – LES : equation of the filtered kinertic energy

✓ the $K_c = \frac{1}{2}\overline{u}_i \overline{u}_i$ - equation comes from the multiplication of the $\overline{u_i}$ - equation by $\overline{u_i}$

$$\overline{u}_i \times \left(\frac{\partial \overline{u}_i}{\partial t} + \overline{u}_l \frac{\partial \overline{u}_i}{\partial x_l} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_l} + \frac{\partial}{\partial x_l}\left(\nu \frac{\partial \overline{u}_i}{\partial x_l} - \tau_{il}^R \right) \right)$$

✓ this gives an equation analogous tor that obtained using Reylnolds averaging

$$\underbrace{\frac{\overline{D} K_c}{\overline{D}t} = \frac{\partial K_c}{\partial t} + \overline{u}_l \frac{\partial K_c}{\partial x_l}}_{C = \text{ convection}} = \underbrace{-\tau_{il}^R \frac{\partial \overline{u}_i}{\partial x_l}}_{P = \text{ échange avec } k_c} + \underbrace{\frac{\partial}{\partial x_l}\left(-\frac{\overline{p}\overline{u}_i}{\rho} + \nu \frac{\partial \overline{u}_i}{\partial x_l} - \overline{u}_i \tau_{il}^R \right)}_{D = \text{ diffusion}} \underbrace{- \nu \left(\frac{\partial \overline{u}_i}{\partial x_l} \right)^2}_{\varepsilon_{Kc} = \text{ dissipation}}$$

18.2 Large Eddy Simulation (LES) (…)

• **closure : the eddy viscocity concept**

✓ space filtering $\quad \frac{\overline{D}\,\overline{\underline{u}}}{\overline{D}t} = div\left(\overline{\underline{\underline{\sigma}}} - \underline{\underline{\tau}}^R \right) \quad \begin{cases} \overline{\underline{\underline{\sigma}}} = -\frac{1}{\rho}\overline{p}\,\underline{\underline{1}} + \nu\,\overline{\underline{\underline{d}}} \\ \underline{\underline{\tau}}^R \text{ - Leonard's tensor} \end{cases}$

✓ ensemble average $\quad \frac{D\langle \underline{u} \rangle}{Dt} = div\left(\langle \underline{\underline{\sigma}} \rangle - \underline{\underline{R}} \right) \quad \begin{cases} \langle \underline{\underline{\sigma}} \rangle = -\frac{1}{\rho}\langle p \rangle \underline{\underline{1}} + \nu \langle \underline{\underline{d}} \rangle \\ \underline{\underline{\tau}}^R \text{ - Reynolds' tensor} \end{cases}$

• **analogy** $\quad \boxed{\underline{\underline{\tau}}^R - \frac{1}{3} Tr\left\{ \underline{\underline{\tau}}^R \right\} \underline{\underline{1}} = -2\,\nu_T\, \overline{\underline{\underline{d}}}}$

Smagorinsky model (*)

• **there are many other possibilities**

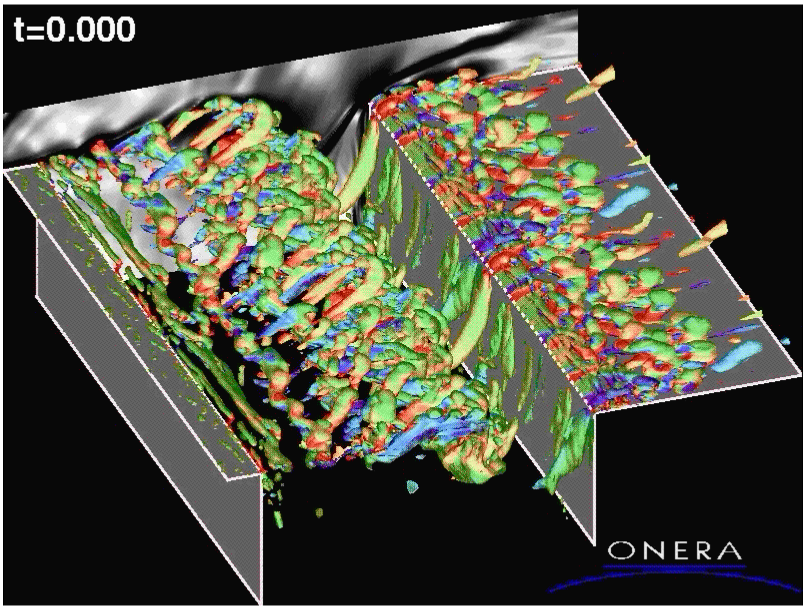
(*) 1963, researcher in meteorology which has developed firstly LES

18.2 Large Eddy Simulation (LES) (…)

- **space filtering versus ensemble average**

✓ moyenne d 'ensemble : $\langle \underline{u} \rangle (\underline{x}), \langle p \rangle (\underline{x}), \underline{\underline{R}}(\underline{x})$ • déterministic (if statistically steady)

✓ filtrage spatial : $\overline{\underline{u}}(\underline{x},t),\ \overline{p}(\underline{x},t),\ \underline{\underline{\tau}}^R(\underline{x},t)$ • random • depend on the filter parameters



Larchevèque et al. (2002)

18.2 Large Eddy Simulation (LES) (…)

- **LES : state of the art**

✓ same as DNS but at higher Reynolds

✓ flow around 2D objects (e.g. wing profiles)

✓ an efficient method for computing free turbulent shear flows (jets…)

✓ for wall flows one still needs to solve $y^{+}=1$

✓ difficult to discriminate between the respective influences of the model, the mesh size (the cutt off) and the numerics

- **perpectives**

✓ unclear (« bad » DNS...)

✓ supplanted by other cheaper methods, such as the DES which mixes the Reynodls approach in wall regions and LES in free flow regions

18.3 conclusions

- **numerical simulations**

✓ far to be operational for applications

✓ future depends on computer science

✓ meanwhile, we must keep on modelizing … and we must keep on doing experiments