PC 3: Vorticity in two dimensions

Goals
- *Biot-Savart law: formulation in complex coordinates*
- *Illustration with two point vortices*

1. Biot-Savart law: complex formulation

Let's consider an incompressible, homogeneous fluid, in a finite size domain A of the 2D plane. Its vorticity is denoted $\underline{\omega} = \omega(\underline{x},t)\underline{e}_z$ and its velocity $\underline{u}(x,y) = (u,v,0)$.

(a) The location of a point in space is determined in complex coordinates $Z = x + iy$. We define the complex velocity

$$W(Z) = u - iv = \frac{d\overline{Z}}{dt} \tag{3.1}$$

- Show that the Biot-Savart law
$$\underline{u}(\underline{x}) = \frac{1}{2\pi}\iint_A \frac{\underline{\omega}(\underline{x}') \times (\underline{x} - \underline{x}')}{|\underline{x} - \underline{x}'|^2} d\underline{x}'$$
can be written in the following form:

$$W(Z) = \frac{1}{2\pi i}\iint_A \frac{\omega(Z')}{Z - Z'} dx'\, dy' \tag{3;2}$$

- Show that the following quantity, called the impulsion $\underline{\mathrm{I}} = \iint (\underline{\mathrm{x}} \times \underline{\omega})\mathrm{dxdy} = (I_x, I_y)$ (this is a conserved quantity, which characterizes the momentum induced by the vorticity distribution) can be written
$$I(Z) = -I_y + i\,I_x = \iint_A \omega\, Z'\, dx'\, dy' \tag{3.3}$$

(b) What do those expressions become in the case of N point vortices at $Z = Z_j$ with circulations Γ_j ? In that case the vorticity is simply $\omega(Z) = \Sigma_j \Gamma_j\, \delta(Z - Z_j)$, $j = 1, N$

2. Case of 2 point vortices

(a) From (3.1) and the Biot-Savart law for N point vortices derived in 1b, compute the velocity of a vortex at Z_j with circulation Γ_j admitting (without proof) that the velocity induced by the vortex on itself is zero.

(b) Consider the case of 2 point vortices (Γ_1, Z_1) and (Γ_2, Z_2). Show that the center of gravity $Z_G = I/\Gamma$ is fixed and that the distance $|Z_1 - Z_2|$ is constant. ($\Gamma = \Gamma_1 + \Gamma_2$ denotes the total circulation).

(c) For each of the following 4 cases, determine the center of gravity of the system and describe qualitatively the movement of the vortices:

case 1 : $\Gamma_2 = \Gamma_1 > 0$; **case 2** : $\Gamma_2 = 2\Gamma_1 > 0$; **case 3** : $\Gamma_2 = -2\Gamma_1 > 0$; **case 4** : $\Gamma_2 = -\Gamma_1 > 0$.