PC 3 - Vorticity in two dimensions - Solution

1. Biot-Savart law: complex formulation

(a) Note that: $\underline{e}_z \times (\underline{x}-\underline{x}') = \begin{pmatrix}0\\0\\1\end{pmatrix} \times \begin{pmatrix}x-x'\\y-y'\\0\end{pmatrix} = \begin{pmatrix}-(y-y')\\x-x'\\0\end{pmatrix}$. Therefore the Biot-Savart law becomes:

$$u = -\frac{1}{2\pi}\iint_A \omega(\underline{x}')\frac{y-y'}{|\underline{x}-\underline{x}'|^2}d\underline{x}' \,,\quad v = \frac{1}{2\pi}\iint_A \omega(\underline{x}')\frac{x-x'}{|\underline{x}-\underline{x}'|^2}d\underline{x}' \Rightarrow W = u - iv = -\frac{1}{2\pi}\iint_A \omega(\underline{x}')\frac{(y-y')+i(x-x')}{|\underline{x}-\underline{x}'|^2}d\underline{x}' = -\frac{1}{2\pi}\iint_A \omega(Z')\frac{i\overline{Z-Z'}}{|Z-Z'|^2}d\underline{x}' .$$

Hence: $W(Z) = \frac{1}{2\pi i}\iint_A \frac{\omega(Z')}{Z-Z'}dx'dy'$. For the impulsion, we have:

$\underline{I} = \iint (\underline{x}\times\underline{\omega})dxdy = \iint (x\underline{e}_x + y\underline{e}_y)\times \omega\underline{e}_z dxdy = \iint (-x\omega\underline{e}_y + y\omega\,\underline{e}_x)dxdy$. Therefore: $I = -I_y + iI_x = \iint \omega Z\,dxdy$.

(b) With $\omega(z) = \Sigma_j\gamma_j\delta(Z-Z_j)$, the formula $W(Z) = \frac{1}{2\pi i}\iint_A \frac{\omega(Z')}{Z-Z'}dx'dy'$ becomes $W(Z) = \frac{d\overline{Z}}{dt} = \sum_j \frac{\Gamma_j}{2\pi i(Z-Z_j)}$.

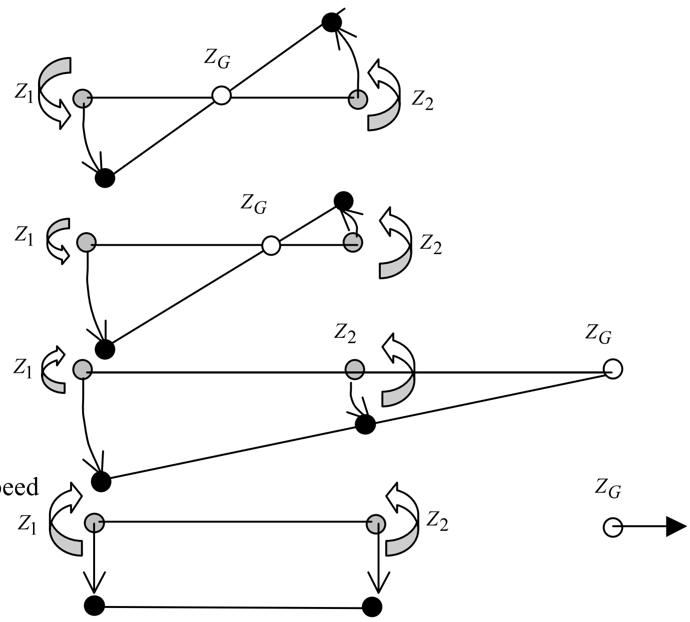
We also have: $I = -I_y + iI_x = \sum_i \Gamma_i Z_i$.

2. Case of two point vortices

(a) Zeroing the contribution of the vortex on itself, $\frac{d\overline{Z}_j}{dt} = \sum_{k\neq j}\frac{\Gamma_k}{2\pi i(Z_j - Z_k)}$.

(b) For 2 vortices, we have $\frac{d\overline{Z}_1}{dt} = \frac{\Gamma_2}{2\pi i(Z_1 - Z_2)}$ $\frac{d\overline{Z}_2}{dt} = \frac{\Gamma_1}{2\pi i(Z_2 - Z_1)}$. We want to show that $I = Z_1\Gamma_1 + Z_2\Gamma_2 = Cstt$, $Z_G = \frac{Z_1\Gamma_1 + Z_2\Gamma_2}{\Gamma_1 + \Gamma_2} = Cstt$ and $|Z_1 - Z_2| = Cstt$. But $\frac{dI}{dt} = \left(\Gamma_1\frac{dZ_1}{dt} + \Gamma_2\frac{dZ_2}{dt}\right) = \left(-\frac{\Gamma_1\Gamma_2}{2\pi i}\right) \times \left(\frac{1}{\overline{Z}_1 - \overline{Z}_2} + \frac{1}{\overline{Z}_2 - \overline{Z}_1}\right) = 0$. Hence $Z_G = Cstt$ since the total circulation $\Gamma_1 + \Gamma_2$ is constant. Also: $\frac{d(\overline{Z}_1 - \overline{Z}_2)}{dt} = \frac{1}{2\pi i}\left[\frac{\Gamma_2}{Z_1 - Z_2} - \frac{\Gamma_1}{Z_2 - Z_1}\right]$, hence $\frac{d(\overline{Z}_1 - \overline{Z}_2)}{dt} = \frac{1}{2\pi i}\frac{\Gamma_1 + \Gamma_2}{Z_1 - Z_2}$ and $\frac{d(Z_1 - Z_2)}{dt} = -\frac{1}{2\pi i}\frac{\Gamma_1 + \Gamma_2}{\overline{Z}_1 - \overline{Z}_2}$. Multiplying the first equation by $(Z_1 - Z_2)$ and the second one by $(\overline{Z}_1 - \overline{Z}_2)$, we obtain: $\frac{d|Z_1 - Z_2|^2}{dt} = 0$. The vortices therefore rotate around their center of gravity Z_G, with a constant distance between them.

(c) **Examples**



Same sign: $\Gamma_1\Gamma_2 > 0$

- $\Gamma_2 = \Gamma_1 > 0 \Rightarrow Z_G = \frac{Z_1 + Z_2}{2}$
- $\Gamma_2 = 2\Gamma_1 > 0 \Rightarrow Z_G = \frac{\Gamma_1 Z_1 + 2\Gamma_1 Z_2}{3\Gamma_1} = \frac{Z_1}{3} + \frac{2Z_2}{3}$

  The strongest vortex "embarks" the other.

Opposite signs: $\Gamma_1\Gamma_2 < 0$

- $\Gamma_2 = -2\Gamma_1 > 0 \Rightarrow Z_G = \frac{\Gamma_1 Z_1 - 2\Gamma_1 Z_2}{-\Gamma_1} = -Z_1 + 2Z_2$

  The strongest vortex again "embarks" the other
- $\Gamma_2 = -\Gamma_1 > 0 \Rightarrow Z_G \to \infty$. The vortices move downward, with speed

$$W(Z_1) = \frac{d\overline{Z}_1}{dt} = \frac{\Gamma_2}{2\pi i(Z_1 - Z_2)} = \frac{-\Gamma_1}{2\pi i(Z_1 - Z_2)} = \frac{d\overline{Z}_2}{dt} = W(Z_2) = u - iv$$

with, initially : $Z_2 - Z_1 = x_2 - x_1$. Hence : $u = 0, v = -\frac{\Gamma_2}{2\pi(x_2 - x_1)}$