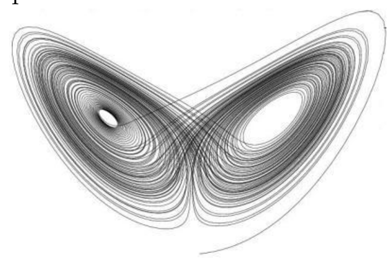
PC 3: Vorticity in two dimensions

Our aim is to introduce the notions of sensitivity to initial conditions, also known as chaos, and predictability. We will illustrate in a simple system (four point vortices) how this notion leads to different expectations from numerical weather prediction models and from climate models.



The famous "butterfly", strange attractor of Edward Lorenz

Model sensitivity is the dependence of model outputs to changes in the information which is "fed into" the model. This includes the physical content of the model itself, the computational domain and how finely it is divided into cells, but also the initial condition used to compute a forecast. There is also a sensitivity to information provided at the boundaries of the computational domain. This includes the lower boundary (exchanges of heat, water, etc. with the ground) and lateral boundaries if the computation is not performed over the whole globe, which is the case for many high-resolution applications such as air quality forecasting.

Here we focus on sensitivity to initial conditions, and address this notion with an extremely simplified model. In this model the flow is not represented by values in a set of cells, but by a small number of point vortices (like depressions or anticyclones). Each vortex possesses its own strength and moves in a plane with a velocity induced by the other vortices. This velocity is proportional to the strength of each vortex and inversely proportional to the distance between two vortices. Vortices do not attract each other but tend to turn around each other.

You are provided with a graphical interface which allows you to compute the trajectories of four interacting point vortices of adjustable strength. In addition to strength, the initial positions of the vortices need to be defined. Then the equations of motion uniquely define the trajectories of vortices. You will be able to compute several trajectories starting from almost identical initial positions. Questions that can be addressed with this setting are for instance :

- if I start two different simulations with almost identical initial conditions, will this initial discrepancy grow with time, and how fast?
- If I reduce the discrepancy between the two initial conditions, how much more time will pass before the two simulations diverge from each other?
- Can a reliable prediction be done an arbitrarily long time in advance?
- If the time scale I am interested in is much larger than the predictability time, is the numerical model useless or can it still accurately predict some interesting quantities

1 Understanding the graphical interface

The graphical interface computes the trajectories of four interacting point vortices of adjustable strengths.

1.1 Control simulation versus perturbed simulation (perturbed initial condition)

- Execute the MATLAB code predictability.m.
- In the "Simulation" panel, click on the "Simulate" button. This can take a few seconds depending on the selected length of the simulations, please leave the time step to its default value (0.02).

This will perform two simulations: a control non-perturbed simulation, and a second simulation with slightly different initial conditions. For both simulations the strengths of the vortices are identical, the two simulations only differ by their initial conditions (initial positions of vortices). The strengths of the four vortices and their initial positions are defined on the panel "Vortices". The initial position of one vortex is defined by [X Y] in brackets. For the first simulation the initial positions are exactly as defined in the boxes. For the second simulation the initial displacement defined in the third column is added to the initial positions of the vortices. When the computation of both simulations is finished, a figure appears showing the trajectory of each vortex as computed in the first (non-perturbed) simulation.

- Still in the "Simulation" panel, click on the "Animate" button. This shows how each vortex moves. You may close this figure before the end of the animation.
- Now click on the "Spread" button. A figure with three graphs appears. The first graph plots the X-position of the first vortex as a function of time. There are two curves, one for each simulation. The second graph plots the difference ("Spread") between the two simulations as a function of time. The third graph is the same as the second, with a logarithmic scale.
- Finally click on the "Density" button. This will plot two figures. The first figure has two graphs indicating how often the first vortex is inside a certain cell of a Cartesian mesh. There is one graph per simulation. The second figure compares the density obtained in the first simulation to that obtained in the second simulation.

1.2 Ensemble of simulations

You will find similar buttons in the "Ensemble" panel, with similar functionality. Now instead of two simulations, an ensemble of N simulations is performed where the value of N is defined in the box "Size". Each simulation starts with the initial positions as defined in the panel "Vortices", plus the displacements multiplied by a random number.

- In the "Ensemble" panel, click on the "Simulate" button. This computes the trajectories of the vortices in the ensemble of N simulations.
- Click "Animate".

The "Animate" button shows an animation of the position of the first vortex only, in each of the N simulations.

- Click "Spread". It shows the position X1 for each of the N simulations, and the variance of X1 across the N simulations.
- Click "Density". As before, it returns two figures. The first one shows how many times the first vortex passes inside a certain box of a mesh. The first graph is obtained from the first N/2 simulations and the second from the last N/2 simulations. The second figure compares the density obtained from the first N/2 simulations to the one obtained from the last N/2 simulations.

2 Work / Questions

First work with the following (default) values :

- strengths = 1, 1, 0.5, 0.5
- initial positions = [0 1] [1 0] [0 0] [0.5 0.5]
- initial displacements = [0 0] [0 0] [0 0] [1e-4 0]
- time step = 0.02 duration=100
- ensemble size=100

2.1 Control simulation versus perturbed simulation (perturbed initial condition)

Questions :

- After how long do the two simulations disagree with each other?
- How does the discrepancy between the two simulations evolve in time?
- Does this behavior change if a smaller initial displacement (1e-8) is used?

In dynamical systems, the rate of separation of infinitesimally close trajectories is characterized by the so-called "Lyapunov exponent". Mathematically, two trajectories with initial separation $\delta x(t=0)$ diverge at a rate given by $|\delta x(t)| \approx e^{\lambda t}|\delta x(0)|$, where λ is the Lyapunov exponent of the system.

- Try reducing/increasing the initial displacement. How much more/less time will pass before the two simulations diverge from each other? Is this what you would expect from theory, assuming that the Lyapunov exponent is a constant (i.e. assuming that it does not depend on the initial condition)?
- Can you estimate the value of the Lyapunov exponent?
- How do the two densities compare? How does this comparison evolve if you increase the length of the simulation?

2.2 Ensemble of simulations

Questions :

- Repeat the above questions (previous section 2.1) with an ensemble of simulations. How do results differ?

For stationary systems, one can either use two simulations which have run for a long time to get accurate statistics (for instance the density of the location of the first vortex), or one can use an ensemble of shorter simulations. The two approaches yield the same statistics.

2.3 The case of three vortices

- Consider again the same questions after setting the strength of the second vortex to zero, so that there are only three active vortices. In that case, how does the first vortex behave? Is its trajectory chaotic?

The system consisting of trajectories of three vortices is actually an integrable system. Therefore the trajectory of vortex one, or of any of the two other active vortices, is not chaotic.

Interestingly though, the velocity field generated by three vortices *is* chaotic. Therefore if one looks at the trajectory of the zero-strength vortex, one recover the earlier results of the chaotic system with four non-zero vortices.

- Set the strength of the second vortex back to 1, and set the strength of the *first* vortex to zero. Now the first vortex is inactive. In that case, how does the first vortex behave? Is its trajectory chaotic?

2.4 Conclusions

- Does a predictability time emerge in the previous numerical experiments? Can the numerical model reliably predict certain quantities over time scales much larger than the predictability time?

- In what you have computed, what is analog to "weather" and what is analog to "climate"?

3 Suggested further topics

3.1 Atmosphere and ocean predictions

Are the atmosphere, the ocean sensitive to initial conditions? To answer this question browse:
http://www.ecmwf.int/products/forecasts/d/charts/medium/deterministic/
http://bulletin.mercator-ocean.fr/html/produits/bestproduct/welcome_en.jsp
and compare the various forecasts made for today with different lead times: yesterday, 3 days ago, etc.

3.2 The case of three vortices: Degree of chaotization

We saw that the velocity field generated by three vortices is chaotic. It can be shown that the degree of chaotization depends on the geometry of the three vortices. More precisely, there is less chaotic behavior when the three active vortices lie on an equilateral triangle, or when two of them are very close (see for instance the article by Kuznetsov and Zaslavsky for details). The maximum chaotic behavior is somewhere in between, when two vortices are closer to each other than to the third vortex, but not too close.

Investigate this by looking at the trajectory of the passive vortex (vortex 1) for various geometries of the three active vortices. Start by setting the strength of vortex 1 to zero, and its initial position to the origin [0,0] with a small initial displacement. For the three other vortices, for simplicity set their strengths to the same value 1 and remove all initial displacements (set them to [0,0]). Now let's fix the initial position of vortex 2 to [0,1] and start with an equilateral triangle with vortex 3 at $[-1/\sqrt{3} \approx -0.577,0]$ and vortex 4 at $[1/\sqrt{3} \approx 0.577,0]$. Now slowly reduce the distance between vortices 2 and 3 by changing their initial positions to [-d,0] and [d,0], where d decreases from $1/\sqrt{3}$ to small values. What do you observe? Is this consistent with the theoretical expectations?

4 References

Kuznetsov L. and Zaslavsky G. (1998) Regular and chaotic advection in the flow field of a three-vortex system, *Physical Review E.*
Lorenz E. (1962) Deterministic nonperiodic flow, *Journal of the Atmospheric Sciences.*
Saltzman B. (1962) Finite amplitude free convection as an initial value problem–I, *Journal of the Atmospheric Sciences.*