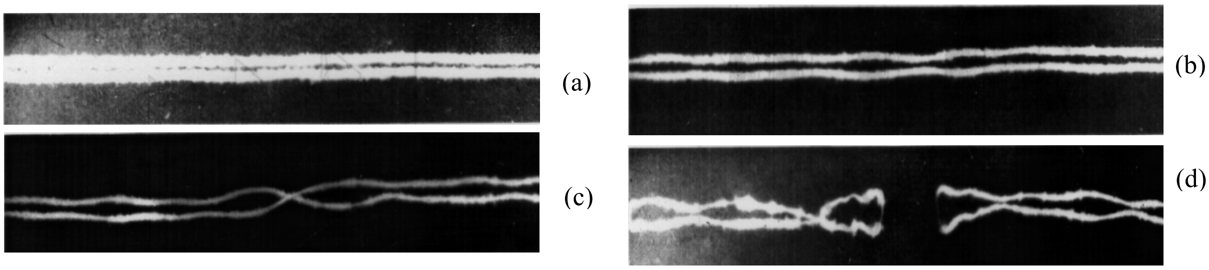
PC 4 : 3D VORTICITY

Goals
- *Investigate vortex stretching, and its competition with viscosity*
- *Applications : interaction of vortices - tornadoes, bathtub whirlpools*

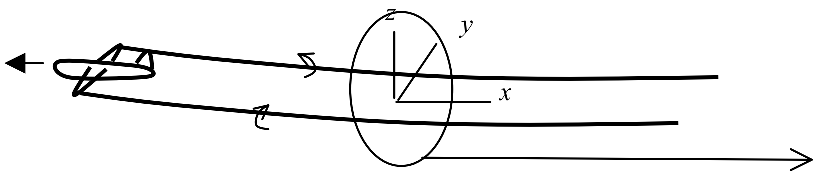
1 – Interaction of two counter rotating vortices : The Crow instability

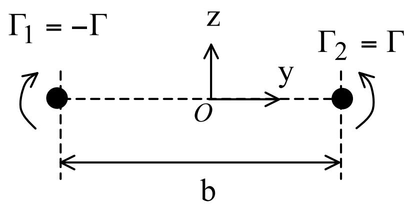
In the atmosphere, it is quite common to observe a destabilization of vortices behind planes, with a time evolution similar to the figure below: first the vortices start oscillating, at some point they meet, and finally they split. The goal of this exercise is to explain this phenomenon.



Contrails behind an aircraft,at different tiimes: (a) 15 s, (b) 30s, (c) 45s, (d) 60s (Crow, Journal of Aircraft, 1975)

Let's consider two thin parallel vortices, with strengths $\Gamma_2 = -\Gamma_1 = \Gamma\,(>0)$, far behind a plane. The two vortices can be modeled in 2D by two point vortices located at a given time at $\underline{x}_2\left(0,-b/2,0\right)$ and $\underline{x}_1\left(0,b/2,0\right)$.





(a) Determine the velocity $\underline{u}\left(v,w\right)$ of the vortices.

(b) We want to investigate the velocity of one vortex under the influence of the other vortex if the former is slightly perturbed from its equilibrium trajectory.
Compute the velocity field induced by each vortex in a neighborhood $Z = Z_j + \tilde{Z}_j$ of the other vortex, with $\left|\tilde{Z}_j\right|/b = \varepsilon << 1$. Show that: $\dfrac{d\overline{\tilde{Z}_j}}{dt} = i\lambda\tilde{Z}_j$ where $\lambda = \dfrac{\Gamma_k}{2\pi b^2}$.

(c) Let $\tilde{Z}_j = y + iz$. Find the solution $\left(y,z\right)\left(t\right)$ of this equation. Describe the streamlines of the flow.

(d) What happens if we perturb each vortex with an infinitesimal sinusoidal perturbation in the Ox direction?

2 - Vortex stretching : tornado (Burgers vortex)

Consider an incompressible homogeneous fluid, invariant by rotation around the axis 0z. The only non-zero component of vorticity is the vertical component ω_z.

(a) **Vorticity** : Starting from the definition of vorticity, show that :

$$\omega_z = \omega_z(r,t),\ u_r = u_r(r,t),\ u_\theta = u_\theta(r,t),\ u_z = u_z(z,t)$$

and that the equation for ω_z has the form : $\dfrac{\partial \omega_z}{\partial t} + u_r \dfrac{\partial \omega_z}{\partial r} = \omega_r \dfrac{\partial u_z}{\partial z} + \nu \Delta \omega_z$

(Recall that the vorticity in cylindrical coordinates is $\omega = \begin{pmatrix} \frac{1}{r}\partial_\theta u_z - \partial_z u_\theta \\ \partial_z u_r - \partial_r u_z \\ \frac{1}{r}\partial_r (r u_\theta) - \frac{1}{r}\partial_\theta u_r \end{pmatrix}$, and that $\nabla.\ \omega = \frac{1}{r}\frac{\partial (r\omega_r)}{\partial r} + \frac{1}{r}\frac{\partial (\omega_\theta)}{\partial \theta} + \frac{\partial (\omega_z)}{\partial z} = 0$.

Also the vorticity equation in that case is $\underbrace{\begin{vmatrix} 0 \\ 0 \\ \partial\omega_z/\partial t \end{vmatrix}}_{\frac{\partial \underline{\varpi}}{\partial t}} + \underbrace{\begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \partial\omega_z/\partial r & 0 & 0 \end{vmatrix} \begin{vmatrix} u_r \\ u_\theta \\ u_z \end{vmatrix}}_{\nabla \underline{\omega}\,.\,\underline{u}} = \underbrace{\begin{vmatrix} \partial u_r/\partial r & -u_\theta/r & \partial u_r/\partial z \\ \partial u_\theta/\partial r & u_r/r & 0 \\ \partial u_z/\partial r & 0 & \partial u_z/\partial z \end{vmatrix} \begin{vmatrix} 0 \\ 0 \\ \omega_z \end{vmatrix}}_{\nabla \underline{u}\,.\,\underline{\omega}} + \nu\Delta\omega_z$).

(b) **Velocity field** : From the continuity equation, determine u_r and u_z. Define $S(t) = \dfrac{du_z}{dz}$, rate of axial stretching, which we assume positive. We also suppose that $u_z(z=0) = 0$ and that there is no source for the flow.

(c) Draw the streamlines.

(d) Give the physical interpretation of each term of the vorticity equation obtained.

(e) Considering separately the stretching and the diffusion, explain what sets the value of vorticity at the center of the vortex. Determine the characteristic width δ of the vortex (Burgers scale) at steady state.

(f) Solve the steady state equation and give ω_z as a function of S and $\omega_0 = \omega_z(r=0)$. Draw the solution.

(g) What is the vorticity at the center of the vortex $\omega_0 = \omega_z(r=0)$ (as a function of the circulation Γ) ?