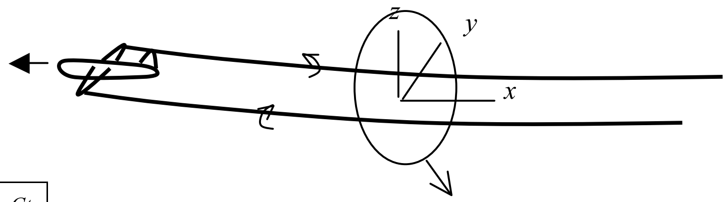
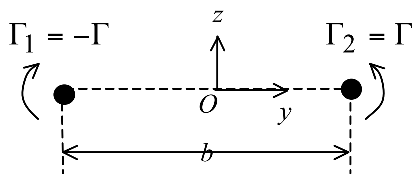
1- Interaction of two counter rotating vortices : The Crow instability (solution)



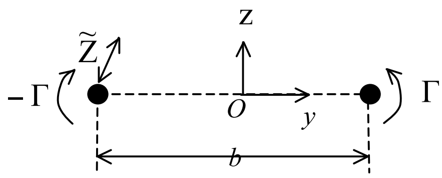
(a) Position-velocity

$$\begin{cases} \dfrac{d\bar{Z}_1}{dt} = \dfrac{\Gamma_2}{2\pi i(Z_1 - Z_2)} = \dfrac{\Gamma}{2\pi i(Z_1 - Z_2)} \\ \dfrac{d\bar{Z}_2}{dt} = \dfrac{\Gamma_1}{2\pi i(Z_2 - Z_1)} = \dfrac{-\Gamma}{2\pi i(Z_2 - Z_1)} \end{cases} \Rightarrow \boxed{\bar{Z}_2 - \bar{Z}_1 = \underbrace{(y_2 - y_1)}_{=b} - i\underbrace{(z_2 - z_1)}_{=0} = Cte}$$



$$\frac{d\bar{Z}_j}{dt} = v_j - i w_j = \frac{-\Gamma}{2\pi i b} \Rightarrow \boxed{v_1 = v_2 = 0,\ w_1 = w_2 = -\frac{\Gamma}{2\pi b}}.$$

The vortices are advected downward, and the distance between them remains constant.



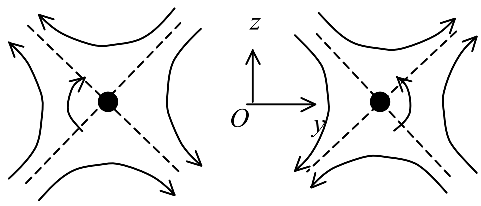
(b) At $Z_0 = Z_j + \tilde{Z}_j$, the velocity induced by the other vortex (Γ_k, Z_k) is :

$$\frac{d\bar{Z}_0}{dt} = \frac{d\bar{Z}_j}{dt} + \frac{d\bar{\tilde{Z}}_j}{dt} = \frac{\Gamma_k}{2\pi i(Z_0 - Z_k)}, \text{ with } \frac{d\bar{Z}_j}{dt} = \frac{\Gamma_k}{2\pi i b} \text{ and } Z_0 - Z_k = \pm b + \tilde{Z}_j$$

Hence : $$\frac{d\bar{\tilde{Z}}_j}{dt} = \frac{\Gamma_k}{2\pi i}\left(\frac{1}{\tilde{Z}_j \pm b} - \frac{1}{\pm b}\right) = \frac{\Gamma_k}{2\pi i(\pm b)}\left(\frac{1}{1 + \tilde{Z}_j/(\pm b)} - 1\right) = \frac{\Gamma_k}{2\pi i(\pm b)}\left(-\tilde{Z}_j/(\pm b) + O\left(\left(|\tilde{Z}_j|/b\right)^2\right)\right).$$

$$\frac{d\bar{\tilde{Z}}_j}{dt} = i\frac{\Gamma_k}{2\pi b^2}\tilde{Z}_j \text{ . So } \boxed{\frac{d\bar{\tilde{Z}}_j}{dt} = i\lambda_k \tilde{Z}_j}.$$

(c) **Solutions** $$\frac{d^2\bar{\tilde{Z}}_j}{dt^2} = i\lambda_k \frac{d\tilde{Z}_j}{dt} = (i\lambda_k)(-i\lambda_k)\bar{\tilde{Z}}_j = \lambda_k^2 \bar{\tilde{Z}}_j \Rightarrow \boxed{\tilde{Z}_j(t) \propto \tilde{Z}_j(0)e^{\pm\lambda k t}}.$$

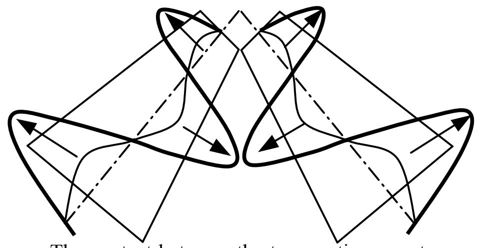


Streamlines. Streamlines are defined by $\dfrac{dy}{v} = \dfrac{dz}{w} = Cstt$.

$$\frac{d\bar{\tilde{Z}}_j}{dt} = \frac{d}{dt}(\tilde{y}_j - i\tilde{z}_j) = i\lambda_k(\tilde{y}_j + i\tilde{z}_j) \Rightarrow \boxed{\begin{cases} \dfrac{d\tilde{y}_j}{dt} = -\lambda_k \dfrac{\tilde{z}_j}{b} \\ \dfrac{d\tilde{z}_j}{dt} = -\lambda_k \dfrac{\tilde{y}_j}{b} \end{cases}}$$

Hence $\dfrac{d\tilde{y}_j}{\lambda_k \tilde{z}_j} = \dfrac{d\tilde{z}_j}{\lambda_k \tilde{y}_j} = Cstt \Rightarrow \boxed{\tilde{y}_j^{\,2} - \tilde{z}_j^{\,2} = Cstt}$.

This is a shear oriented at 45°.



(d) A sinusoidal deformation in the plane of the shear is amplified. We obtain the picture above: The contact between the two vortices creates a cut (zero total vorticity at those points), which transforms the flow into a series of vortex rings.

If the perturbation is not in the plane of the shear, one can show that each vortex rotates around its axis (self-rotation), which eventually triggers the instability described above (alignment in the plane of the shear and amplification). This instability, known as the Crow instability (Crow, *Journal of Aircraft, 1970*) can sometimes be observed behind planes.

2- Vortex stretching : tornado (Burgers vortex) : solution

Hypothesis : $\frac{\partial u}{\partial \theta} = 0$, $\underline{\omega} = \omega_z \underline{e}_z$.

(a) Vorticity : $\omega_z(r,z,t) = \frac{1}{r}\frac{\partial r u_\theta}{\partial r}$. $div(\underline{\omega}) = \frac{\partial \omega_z}{\partial z} = 0 \Rightarrow \omega_z = \omega_z(r,t) \Rightarrow \boxed{u_\theta = u_\theta(r,t)}$; $\omega_r = -\frac{1}{r}\frac{\partial u_\theta}{\partial z} = 0 \Rightarrow$ same ; $\omega_\theta = \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} = 0 \Rightarrow$

$$\frac{\partial u_r}{\partial z} = \frac{\partial u_z}{\partial r}$$

vorticity equation :

$$\underbrace{\begin{vmatrix} 0 \\ 0 \\ \partial \omega_z/\partial t \end{vmatrix}}_{\frac{\partial \underline{\omega}}{\partial t}} + \underbrace{\begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \partial \omega_z/\partial r & 0 & 0 \end{vmatrix}\begin{vmatrix} u_r \\ u_\theta \\ u_z \end{vmatrix}}_{\underline{\underline{grad}}\,\underline{\omega}.\underline{u}} = \underbrace{\begin{vmatrix} \partial u_r/\partial r & -u_\theta/r & \partial u_r/\partial z \\ \partial u_\theta/\partial r & u_r/r & 0 \\ \partial u_z/\partial r & 0 & \partial u_z/\partial z \end{vmatrix}\begin{vmatrix} 0 \\ 0 \\ \omega_z \end{vmatrix}}_{\underline{\underline{grad}}\,\underline{u}.\underline{\omega}} + \nu \Delta \omega_z$$

. The first equation leads to $0 = \omega_z \frac{\partial u_r}{\partial z} \Rightarrow u_r = u_r(r,t)$.

From $\omega_\theta = \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} = 0, \frac{\partial u_z}{\partial r} = 0 \Rightarrow u_z = u_z(z,t)$. The last one is :

$$\boxed{\frac{\partial \omega_z}{\partial t} + \underbrace{u_r \frac{\partial \omega_z}{\partial r}}_{\text{Advection}} = \underbrace{\omega_z \frac{\partial u_z}{\partial z}}_{\text{Stretching}} + \underbrace{\nu \Delta \omega_z}_{\textit{Diffusion}}}$$

(b) **Velocity**

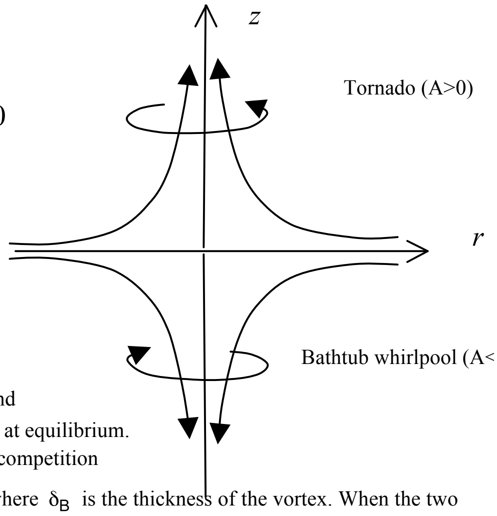
- Continuity : $\frac{1}{r}\frac{\partial r u_r}{\partial r} + \frac{\partial u_z}{\partial z} = 0 \Rightarrow \frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial r u_r}{\partial r} + \frac{\partial u_z}{\partial z}\right) = \frac{\partial^2 u_z}{\partial z^2} = 0 \Rightarrow \boxed{u_z(z,t) = S(t)z}$

$\Rightarrow \frac{1}{r}\frac{\partial r u_r}{\partial r} = -S(t) \Rightarrow \boxed{u_r(r,t) = -\frac{S(t)}{2} r}$

Note that the last equation assumes that there is no source at $r = 0$.

(c) Streamlines : $\frac{dr}{u_r} = \frac{dz}{u_z} \Rightarrow -\frac{2}{S(t)}\frac{dr}{r} = \frac{1}{S(t)}\frac{dz}{z} \Rightarrow d\ln\left(r^{-2}\right) = d\ln z \Rightarrow \boxed{z = Ar^{-2}}$. see below.

Case $S(t) > 0$



(d) The vorticity equation is therefore :

$$\boxed{\frac{\partial \omega_z}{\partial t} + \underbrace{-\frac{S(t)}{2} r \frac{\partial \omega_z}{\partial r}}_{\substack{\text{Advection} \\ \text{toward the center}}} = \underbrace{S(t)\quad \omega_z}_{\substack{\text{Amplification} \\ \text{by stretching}}} + \underbrace{\nu \Delta \omega_z}_{\substack{\textit{Viscous} \\ \textit{Spreading}}}}$$

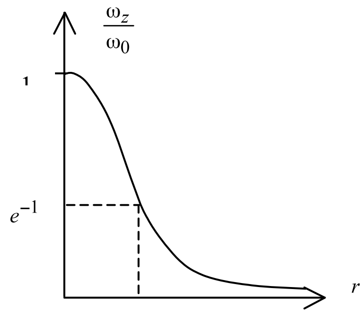
The vorticity is advected toward the center where it is amplified by stretching

(e) At the center $r = 0$, the temporal variation of ω_z results in the competition between the stretching and the diffusion which tends to spread the vortex. A steady state can be reached if those two mechanisms are at equilibrium. At $r = 0$ there is no advection term in the above equation. The evolution of $\omega(0,t)$ is determined by the competition between stretching, acting on a time scale $\tau_S = S^{-1}$ and diffusion, acting on a time scale, $\tau_\nu = \delta_B^2/\nu$ where δ_B is the thickness of the vortex. When the two opposite effects are at equilibrium, $\delta_B = \sqrt{\nu/S}$, the flow is stationary. This scale resulting from the balance stretching/spreading is the « Burgers scale ».

(f) At steady state : $0 = \frac{S}{2} r \frac{d\omega_z}{dr} + S\omega_z + \nu \frac{1}{r}\frac{d}{dr}\left(r\frac{d\omega_z}{dr}\right)$, hence :

$0 = \frac{S}{2}\frac{1}{r}\frac{dr^2\omega_z}{dr} + \nu\frac{1}{r}\frac{d}{dr}\left(r\frac{d\omega_z}{dr}\right) = \frac{1}{r}\frac{d}{dr}\left(S\frac{r^2}{2}\omega_z + \nu r\frac{d\omega_z}{dr}\right) \Rightarrow \left(Sr\frac{\omega_z}{2} + \nu\frac{d\omega_z}{dr}\right) = Cstt$. Hence :

$$\frac{d\ln\omega_z}{dr} = -\frac{Sr}{2\nu} \Rightarrow \boxed{\omega_z = \omega_0\, e^{-\frac{S r^2}{4\nu}} = \omega_0\, e^{-\left(\frac{r}{\delta_B}\right)^2}}.$$



This solution is determined by the characteristic scale (Burgers scale) :

$\delta_B = 2\sqrt{\frac{\nu}{S}}$ which results from the competition between the two mechanisms stretching/diffusion

(g) The circulation is : $\Gamma = 2\pi\int_0^\infty \omega_z r dr = 2\pi\omega_0\int_0^\infty e^{-\frac{Sr^2}{4\nu}} r dr = 2\pi\omega_0\frac{4\nu}{2S} = \omega_0\frac{4\pi\nu}{S}$. Hence : $\boxed{\omega_0 = \frac{1}{4\pi}\frac{S}{\nu}\Gamma}$ $\delta_B = 2\sqrt{\frac{\nu}{S}}$