PC5 SOLUTION : IMPACT OF ROTATION ON THE KOLMOGOROV CASCADE

***(a)* Scales :** $\underline{x} = l\,\underline{\bar{x}}, \underline{u} = V\underline{\bar{u}}, t = \frac{l}{V}\bar{t}, \underline{\Omega} = \Omega\underline{\bar{\Omega}}, \Pi = P\bar{\Pi}$. The equations become :

$$\left(\frac{V^2}{l}\right)\frac{d\underline{\bar{u}}}{d\bar{t}} + (2\Omega V)\underline{\bar{\Omega}} \times \underline{\bar{u}} = -\left(\frac{P}{l}\right)\overline{\underline{grad}}\,\bar{\Pi} + \left(\frac{\nu V}{l^2}\right)\bar{\Delta}\underline{\bar{u}} \qquad \Rightarrow \qquad \underbrace{\left(\frac{V}{2\Omega l}\right)}_{Ro}\frac{d\underline{\bar{u}}}{d\bar{t}} + \underline{\bar{\Omega}} \times \underline{\bar{u}} = -\left(\frac{P}{2\Omega V l}\right)\overline{\underline{grad}}\,\bar{\Pi} + \underbrace{\left(\frac{\nu}{2\Omega l^2}\right)}_{E}\bar{\Delta}\underline{\bar{u}},$$ where :

$$Ro = \frac{V}{2\Omega l} = \frac{(2\Omega)^{-1}}{l/V} = \frac{\tau_\Omega}{\tau} = \frac{\text{rotation time scale}}{\text{advection time scale}}$$, or : $$Ro = \frac{V/l}{2\Omega} = \frac{\omega}{2\Omega} = \frac{\text{vorticity}}{\text{rotation rate}},$$ and $$E = \frac{\nu}{2\Omega l^2} = \frac{(2\Omega)^{-1}}{l^2/\nu} = \frac{\tau_\Omega}{\tau_\nu} = \frac{\text{rotation time scale}}{\text{diffusion time scale}}$$

<u>Remark</u> : The pressure scale is usually set by the flow, and adapts to match the leading order term(s) in the equations.

***(b)* Limit** $Ro \to 0, E \to 0$. In the limit of strong rotation, the momentum equation becomes : $2\underline{\Omega} \times \underline{u} = -\underline{grad}\,\Pi$. This is the so-called "geostrophic balance".

<u>Remark</u> : In that case, $P = 2\Omega V l$, so that the pressure term is $O(1)$, same magnitude as the Coriolis term.

Multiplying the geostrophic balance equation by $\underline{u}$ yields $0 = \underline{u}.\underline{grad}\,\Pi$. In other words, the streamlines are parallel to isobars (lines of constant Π).

Taking the rotational yields : $\underline{rot}(2\underline{\Omega} \times \underline{u}) = 2\left[\nabla\underline{\Omega}.\underline{u} - \underline{\Omega}\,div\,\underline{u} \quad -\nabla\underline{u}.\underline{\Omega} + \underline{u}\,div\,\underline{\Omega}\right] = -2\nabla\underline{u}\,.\underline{\Omega} = 0$, since $div\underline{u} = 0$ and $\underline{\Omega} = Cstt$. This shows that the velocity field is invariant in the direction of the rotation axis. Hence, the flow is two-dimensional.

Comment *: Some values of the Rossby number on Earth ($\Omega \approx 7.27 10^{-5} rad/s$)*

	l	V	Ro	E	
sink whirlpools	$10cm$	$1m/s$	$O(10^4)$	$O(1)$	No effect of the Earth's rotation in your sink!
Cyclones	$500km$	$10m/s$	$O(0.1)$	$O(10^{-11})$	geostrophic balance
Anticyclones	$1000km$	$1.5m/s$	$O(0.01)$	$O(10^{-12})$	geostrophic balance

<u>Remark</u> : On Earth Ω must be multiplied by the sine of latitude. The values given above are therefore relevant at high latitudes.

***(c)* The Kolmogorov theory does not apply when** $Ro \to 0$

- dimensional arguement : We can no longer assume that the turbulent quantities, such as the energy spectrum $E(k)$, only depend on the dissipation rate ε in the inertial regime. The rotation rate Ω must be added to the list of scales. For instance we have $E(k) = F(\varepsilon, k, \Omega)$. The $k^{-5/3}$ law is no longer verified.
- physical argument : in a two-dimensional flow, there is no vortex stretching term ($\underline{\omega}$ being perpendicular to $\nabla\underline{u}$, hence the stretching term $\nabla\underline{u}.\underline{\omega}$ vanishes). The mechanism by which turbulent energy is transfered to smaller scales (Kolmogorov-Burgers vortices that we saw last time), and which is necessary to dissipate the energy at the end of the cascade, is absent. The Kolmogorov phenomenology is no longer valid.

(d) Limiting scale

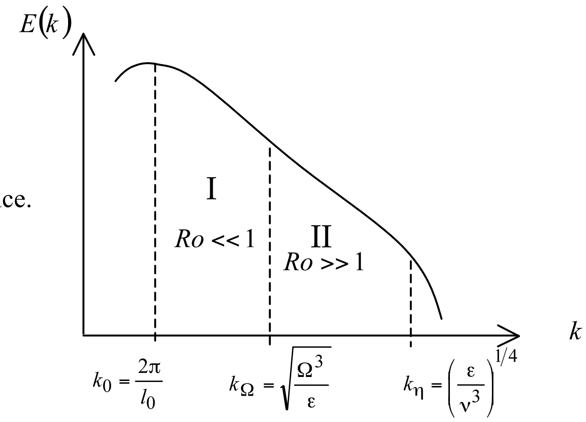
In the inertial range: scale *l*, velocity $v = (\varepsilon\, l)^{1/3}$ ($\leftarrow \varepsilon = \frac{v^2}{\tau} = \frac{v^3}{l}$), vorticity ($\omega = \frac{v}{l} = \varepsilon^{1/3} l^{-2/3}$).

One can define the Rossby number at this scale $Ro_l = \frac{\omega}{2\Omega}\left(=\frac{v}{2\Omega l}\right) = \frac{\varepsilon^{1/3}}{2\Omega l^{2/3}} \sim l^{-2/3}$. The Rossby number increases (i.e. rotation effects are smaller) as the scale decreases. The "limiting" scale corresponds to $Ro_l = \frac{\varepsilon^{1/3}}{2\Omega l_\Omega^{2/3}} = 1$. Hence $\frac{\varepsilon^{1/3} l_\Omega^{-2/3}}{2\Omega} = 1 \Rightarrow \boxed{l_\Omega = \sqrt{\frac{\varepsilon}{(2\Omega)^3}}}$. The corresponding wavenumber is : $\boxed{k_\Omega \sim \sqrt{\frac{(2\Omega)^3}{\varepsilon}}}$.

<u>Remark</u> : One can also use dimensional analysis. In the inertial range $k_\Omega = F(\varepsilon, \Omega)$, *hence:*

$$\begin{array}{cccc} & k_\Omega & \varepsilon & \Omega \\ L & -1 & 2 & 0 \\ T & 0 & -3 & 1 \end{array} \Rightarrow k_\Omega = \varepsilon^\alpha \Omega^\beta \ \textit{avec} \begin{cases} -1 = 2\alpha \\ 0 = -3\alpha - \beta \end{cases} \Rightarrow \begin{cases} \alpha = -1/2 \\ \beta = 3/2 \end{cases} \Rightarrow k_\Omega = \sqrt{\frac{\Omega^3}{\varepsilon}}$$

***(e)* Evolution of the energy spectrum with rotation**



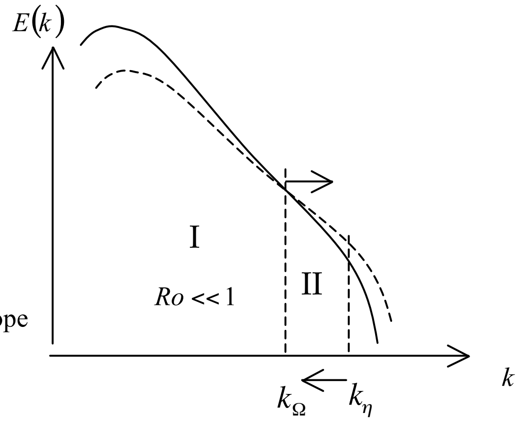
Region I : $k_0 << k << k_\Omega$, i.e. $l_0 >> l >> l_\Omega$ and $Ro \sim l^{2/3} << 1$
The flow is 2D. If we consider the fluid as an ensemble of 2D vortices, these vortices grow by vortex merging. Hence, unlike 3D turbulence, the energy is transfered toward larger and larger scales. The energy cascade is reversed compared to 3D turbulence. It is referred to as the inverse cascade.

→ 2D
→ No vortex stretching
→ No energy transfer towards small scales. Transfer towards larger scales instead

Region II : $k_\Omega << k << k_\eta$, i.e. $l_\Omega >> l >> l_\eta$ and $Ro \sim l^{2/3} >> 1$
These scales do not feel the effects of rotation. The energy spectrum is given by the Kolmogorov theory $E(k) = F(\varepsilon, k) \Rightarrow E(k) = \varepsilon^{2/3} k^{-5/3}$.

→ Usual 3D turbulence
→ $E(k) = \varepsilon^{2/3} k^{-5/3}$

Consequences :



→ The dissipative zone is no longer fed by the energy of large vortices
→ Large vortices become larger through the inverse cascade (vortex merging from 2D dynamics)
→ The energy at small scales $k > k_\Omega$ decreases by the usual forward cascade and dissipation
→ As a consequence the dissipation rate ε decreases, which increases k_Ω and decreases k_η. The inverse cascade dominates the entire spectrum. This results in an overall increase of the slope of the energy spectrum.