PC7 solution - The planar jet

1. Solution - The mean velocity satisfies the continuity equation:

$$<U_i>_{,i} = <U>_{,x} + <V>_{,y} = 0$$

Hence: $$\frac{V}{U} \sim \frac{\delta}{L} = \varepsilon << 1 \qquad \text{(S1)}$$

2. Solution - The Reynolds equation is:

$$<U_i>_{,t} + <U_j><U_i>_{,j} = -\tfrac{1}{\rho}<P>_{,i} + \left[\nu <U_i>_{,j} - <u_i u_j>\right]_{,j}$$

Hence in the Ox direction:

$$<U><U>_{,x} + <V><U>_{,y} = -\tfrac{1}{\rho}<P>_{,x} + \left[\nu <U>_{,xx} - <u^2>_{,x}\right]$$

$$+\left[\nu <U>_{,yy} - <uv>_{,y}\right] + \left[- <uw>_{,z}\right]$$

From our assumptions (no axial pressure gradient and different orders of magnitude of the various derivatives) it follows that:

$$<U><U>_{,x} + <V><U>_{,y} = \nu <U>_{,yy} - <uv>_{,y}$$

From (1), the terms on the left hand side have the same order of magnitude $O\left(U^2/L\right)$, and can not be neglected. On the right hand side, the viscous term is $O\left(\nu U/\delta^2\right)$ and the turbulent term is $O\left(u_0^2/\delta\right)$. Compared to convection, those two terms therefore yield factors $1/\left(\varepsilon^2 Re\right)$ and $u_0^2/\left(\varepsilon U^2\right)$, respectively. The viscous term is negligible from $\varepsilon^2 Re >> 1$. Hence

$$u_0 \sim \sqrt{\varepsilon} \times U \qquad \text{(S2)}$$

and:

$$<U><U>_{,x} + <V><U>_{,y} = - <uv>_{,y} \qquad \text{(S3)}$$

3. Solution (7) can be written:

$$\left(<U>^2\right)_{,x} + \left(<V><U>\right)_{,y} - <U>\left[<U>_{,x} + <V>_{,y}\right] = - <uv>_{,y}$$

where the term in square brackets vanishes from the continuity equation.

$$\left(<U>^2\right)_{,x} + \left(<V><U>\right)_{,y} = - <uv>_{,y}$$

Integrating in the y direction, we find:

$$\int_{-\infty}^{\infty}\left(<U>^2\right)_{,x} dy + \left[<V><U>\right]_{-\infty}^{\infty} = -\left[<uv>\right]_{-\infty}^{\infty}$$

Since the flow in the far field is at rest, $<U>\left(x, y=\pm\infty\right) = <uv>\left(x, y=\pm\infty, z, t\right) = 0$. Hence:

$$\int_{-\infty}^{\infty} <U>^2\, dy = fcn(x) = cstt$$

Finally $$M = \int_{-\infty}^{\infty} \rho <U>^2\, dy = Cte \qquad \text{(S4)}$$

M is the momentum flux of the jet per unit length in z.

4. Solution - We have $M/\rho = \int_{-\infty}^{\infty} <U>^2 \, dy = U_0^2(x)\delta(x)\int_{-\infty}^{\infty} f^2(\eta)d\eta = Cte$

Hence $U_0^2(x)\delta(x) = Cte$

If $\delta(x) \sim x$, then : $$U_0(x) \sim \frac{1}{\sqrt{x}} \quad \text{(S5)}$$

For the flux of kinetic energy: $$E = \rho U_0^3(x)\delta(x)\int_{-\infty}^{\infty} g^3(\eta)d\eta \sim 1/\sqrt{x} \quad \text{(S6)}$$

For the mass flux: $$m = \rho U_0(x)\delta(x)\int_{-\infty}^{\infty} g(\eta)d\eta \sim \sqrt{x} \quad \text{(S7)}$$

Comment 1: It makes sense that $dE/dx < 0$ (cf. (S6)) since the mean loses energy to the turbulent motion and dissipation transforms kinetic energy into heat.

Comment 2: Furthermore, since momentum is conserved ($M = Cte$, cf. (8)), the total mass entrained by the jet must increase (cf. (S7)): the jet absorbs fluid from the environment, and this entrainment is a direct consequence of the dissipation of energy in the shear layers of the jet.

Comment 3: In the laminar case, where viscosity enters, one can show that (looking for self-similar solutions) $\delta \sim x^{2/3}$ and $U_0 \sim x^{-1/3}$. The entrainment rate satisfies in that case: $m \sim x^{1/3}$, versus $m \sim x^{1/2}$ in the turbulent case.

5. Solution - In the case where $M = \int_{-\infty}^{\infty} \rho <U>^2 \, rdr = Cte$, then $\left(U_0(x)\delta(x)\right)^2 = Cte$

Hence, if $\delta(x) \sim x$: $$U_0(x) \sim \frac{1}{x} \quad \text{(S8)}$$

For the kinetic energy flux: $$E \sim U_0^3(x)\delta^2(x) = \left[U_0(x)\delta(x)\right]^2 U_0(x) \sim 1/x \quad \text{(S9)}$$

and for mass flux: $$m \sim U_0(x)\delta^2(x) = \left[U_0(x)\delta(x)\right]^2 / U_0(x) \sim x \quad \text{(S10)}$$

6. Solution - Because of entrainment of the surrounding air, the jet in figure (2a) is attracted towards the wall. Similarly, the two jets in figure (2b) come closer and mix. This is known as the "Coanda effect".

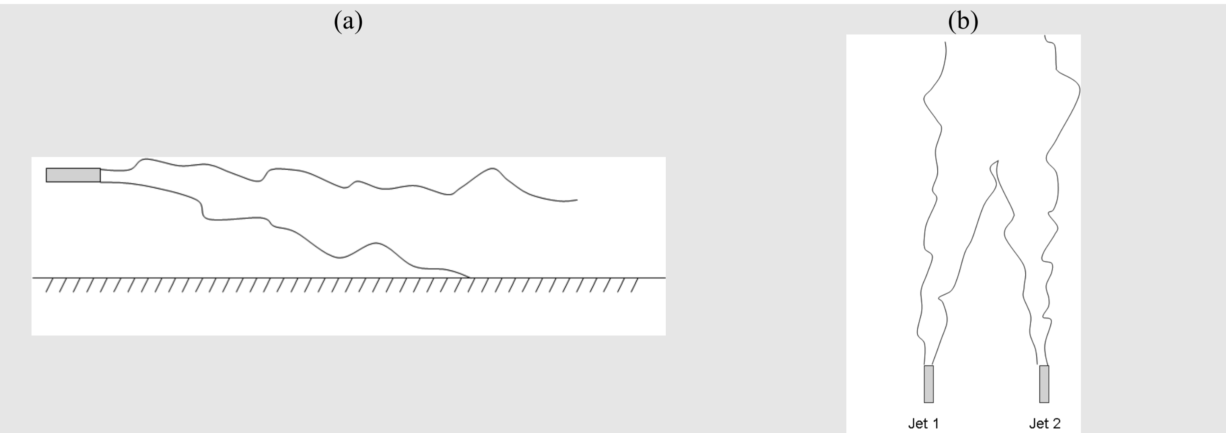

Fig. 2 – (a) : jet near a wall, (b) : parallel jets. The Coanda effect is the tendency of a fluid jet to be attracted to a nearby surface.