SOLUTION PC 1 : DISSIPATION, ENERGETICS, AND TURBULENT CASCADE

1. Dissipation rate

For an incompressible fluid, the Navier-Stokes equations are :

$$div\ \underline{u} = 0, \quad \frac{d\underline{u}}{dt} = \frac{\partial \underline{u}}{\partial t} + \underbrace{\nabla \underline{u}\ .\underline{u}}_{\text{Term leading to turbulence}} = -\frac{1}{\rho}\underline{grad}\ p + \underbrace{\nu\ \Delta \underline{u}}_{\text{Viscous diffusion}}$$

The viscous term can be expressed as the divergence of the tensor of deformation rates : $div\left(2\nu\ \underline{\underline{d}}\right) = \nu\ \Delta \underline{u}$ which is related to the work of viscous constraints per unit mass $\underline{\underline{\tau}} = 2\nu \underline{\underline{d}}$ avec[1] $\underline{\underline{d}} = \frac{1}{2}\left(\nabla \underline{u} + {}^t\nabla \underline{u}\right)$. Multiplying the momentum equation with $\underline{u}$, we obtain the term $D = \nu\ \underline{u}.\Delta \underline{u}$.

(a) We want to show equation (1.1). We will use the index notations. On the one hand, we have that : $D = \nu\ u_i \frac{\partial^2 u_i}{\partial x_l \partial x_l}$. On the other hand :

$$2\nu div\left(\underline{\underline{d}}.\underline{u}\right) = 2\nu \frac{\partial}{\partial x_i}\left[d_{ij} u_j\right] = 2\nu \frac{\partial}{\partial x_i}\left[\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) u_j\right] = \nu \frac{\partial u_j}{\partial x_i}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \nu u_j \frac{\partial}{\partial x_i}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

From the continuity equation $\frac{\partial u_i}{\partial x_i} = 0$,

$$2\nu\ div\left(\underline{\underline{d}}\,\underline{u}\right) = \nu \frac{\partial u_j}{\partial x_i}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \nu u_j \frac{\partial^2 u_j}{\partial x_i \partial x_i} = \nu \frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j} + \nu\left(\frac{\partial u_j}{\partial x_i}\right)^2 + \nu u_j \frac{\partial^2 u_j}{\partial x_i \partial x_i}.$$

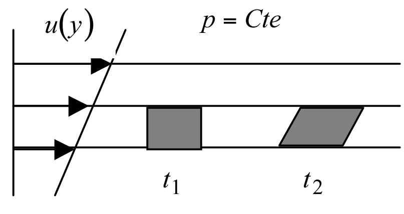
Finally : $\varepsilon = 2\nu\ \underline{\underline{d}}:\underline{\underline{d}} = 2\nu Tr\left\{\underline{\underline{d}}\ .\ \underline{\underline{d}}\right\} = 2\nu Tr\left\{\left(\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\underline{e}_i \otimes \underline{e}_j\right) \cdot \left(\frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right)\underline{e}_k \otimes \underline{e}_l\right)\right\}$

$$= 2\nu Tr\left\{\frac{1}{4}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right)\underline{e}_i \otimes \underline{e}_l\ \delta_{jk}\right\} = \frac{\nu}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right)\delta_{il}\delta_{jk} = \nu\left(\left(\frac{\partial u_i}{\partial x_j}\right)^2 + \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}\right).$$

We therefore recover equation (1.1).
Note that this implies :

$$\int_\Omega D d\Omega = \underbrace{2\nu \int_{\partial\Omega} \underline{\underline{d}}\ .\ \underline{u}\ .\ \underline{n}\ da}_{<P>=\text{power of surface friction forces}} - \underbrace{\int_\Omega \varepsilon\ d\Omega}_{<\varepsilon>=\text{energy loss due to friction, lost as heat. Irreversible energy transfer.}}$$

(b) <u>Example</u> : Simple steady shear flow : $\nabla \underline{u} = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}$, $\alpha = Cstt$



The flow being steady, the power of friction forces is balanced by the losses, and we indeed have : $D = \nu\ \underline{u}.\underbrace{\Delta \underline{u}}_{=0} = 0$.

We can compute $\underline{\underline{d}} = \frac{1}{2}\left[\begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix}\right] = \frac{1}{2}\begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}$; $2\nu div\left(\underline{\underline{d}}.\underline{u}\right) = 2\nu div\left[\frac{1}{2}\begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}\begin{pmatrix} u \\ 0 \end{pmatrix}\right] = \nu div\begin{pmatrix} 0 \\ \alpha u \end{pmatrix}$

$$= \nu\alpha \frac{du}{dy} = \nu\alpha^2 ; \quad \varepsilon = 2\nu\ \underline{\underline{d}} : \underline{\underline{d}} = 2\nu Tr\left\{\underline{\underline{d}}\ .\ \underline{\underline{d}}\right\} = 2\nu Tr\left\{\frac{1}{4}\left[\begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix} \times \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}\right]\right\} = \frac{\nu}{2} Tr\left\{\begin{pmatrix} \alpha^2 & 0 \\ 0 & \alpha^2 \end{pmatrix}\right\} = \nu\alpha^2.$$

Hence the rate of energy dissipated as heat is equal to the power of surface friction forces needed to maintain the steady shear flow.

(c) Let's look at each term of equation (1.3). $2\nu\, div(\nabla\underline{u}\,.\,\underline{u}) = 2\nu \frac{\partial}{\partial x_i}\left(u_j \frac{\partial u_i}{\partial x_j}\right)$

$= 2\nu\left[\frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j} + u_j \underbrace{\frac{\partial^2 u_i}{\partial x_i \partial x_j}}_{=0}\right] = 2\nu \frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j}$; $\nu\,\underline{\omega}^2 = \nu\left(\varepsilon_{ijk}\frac{\partial u_k}{\partial x_j}\right)\left(\varepsilon_{ilm}\frac{\partial u_m}{\partial x_l}\right) = \nu\,\varepsilon_{ijk}\varepsilon_{ilm}\frac{\partial u_j}{\partial x_k}\frac{\partial u_l}{\partial x_m}$ with $\varepsilon_{ijk}\varepsilon_{ilm} =$ $\varepsilon_{jki}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$, hence: $\nu\,\underline{\omega}^2 = \nu\left(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}\right)\frac{\partial u_j}{\partial x_k}\frac{\partial u_l}{\partial x_m} = \nu\left(\frac{\partial u_j}{\partial x_k}\frac{\partial u_j}{\partial x_k} - \frac{\partial u_j}{\partial x_k}\frac{\partial u_k}{\partial x_j}\right)$. Finally : $\nu\,\underline{\omega}^2$ $= \nu\left(\left(\frac{\partial u_i}{\partial x_j}\right)^2 - \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}\right)$. Adding these two terms indeed yields ε.

(d) The enery equation can be obtained from the momentum equation : $\underline{u}.\left(\frac{\partial \underline{u}}{\partial t} + \nabla\underline{u}.\underline{u} = -\frac{1}{\rho}\nabla p + \nu\Delta\underline{u}\right)$ $= \frac{\partial \underline{u}^2/2}{\partial t} + \underline{u}\,.\,\underline{grad}\,\frac{u^2}{2} = -\underline{u}\,.\,\underline{grad}\left(\frac{p}{\rho}\right) + \nu\,\underline{u}\,\Delta\underline{u} = 0$ where we used the vector identity $\nabla\underline{u}\,.\underline{u} = \underline{grad}\frac{u^2}{2} + \underline{\omega}\times\underline{u}$.

Hence : $\frac{\partial \underline{u}^2/2}{\partial t} = -\underline{u}\,.\,\underline{grad}\left(\frac{u^2}{2} + \frac{p}{\rho}\right) + \nu\,\underline{u}\,.\Delta\underline{u}$. Using (1.1) and the fact that $div(\phi\underline{u}) = \underline{u}.\nabla\phi + \phi\,\underbrace{div\underline{u}}_{=0}$, we find :

$$\frac{\partial \underline{u}^2/2}{\partial t} = \underbrace{div\left[\left(-\frac{p}{\rho}\underline{\underline{1}} + 2\nu\underline{\underline{d}}\right).\,\underline{u}\right]}_{\text{Power of external forces}} - \underbrace{div\left(\frac{u^2}{2}\underline{u}\right)}_{\text{Energy flux}} - \underbrace{2\nu\underline{\underline{d}}:\underline{\underline{d}}}_{\text{Dissipation}}$$

Equation (1.4) follows using $\int_\Omega div\,\underline{A}\,d\Omega = \int_{\partial\Omega}\underline{A}.\underline{n}\,da$

2. Pressure spectrum

We denote $E_\Pi(k)$ the spectrum of pressure $\Pi = p/\rho$. The dimension of $E_\Pi(k)$ is deduced from :

$<\Pi^2> = <\left(\frac{p}{\rho}\right)^2> \propto \int_0^\infty E_\Pi(k)dk$ with $\left[\frac{p}{\rho}\right] = L^2T^{-2}$, so : $[E_\Pi] = L^5T^{-4}$. The Π theorem leads to:

	E_Π	ε	k
L	5	2	−1
T	−4	−3	0

$E_\Pi = \varepsilon^\alpha k^\beta \Rightarrow \begin{cases} 5 = 2\alpha - \beta \\ -4 = -3\alpha \end{cases} \Rightarrow \begin{cases} \alpha = 4/3 \\ \beta = 2\alpha - 5 = -7/3 \end{cases} \Rightarrow \boxed{E_\Pi \sim \varepsilon^{\frac{4}{3}} k^{-\frac{7}{3}}}$

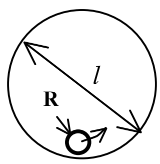
3. Size of bubbles

A population of bubbles in a liquid at rest tend to merge and form a large bubble, in order to reduce the interfacial energy.

In the presence of turbulence, the large bubbles are sheared and can split, while the small bubbles merge to form larger ones. Hence there exists an "equilibrium radius" which results from those two competing effects.
Let R denote the radius of a bubble, and let l denote the spatial scale of the turbulence. There are three possible cases:

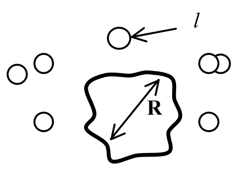
Case n°1 $l >> R$

→ Convection ⇒ shocks between bubbles ⇒ coalescence ⇒ $R\uparrow$



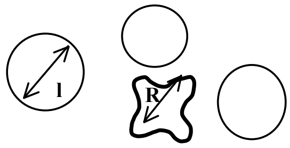
Case n°2 $l << R$

→ No effect (capillary waves)



Therefore, a bubble reacts locally to fluctuations induced by turbulent scales which have the same order of magnitude as the bubble itself $l \sim R$.

Case n°3 $l \sim R$



→ Stretching – Deformation – Fragmentation, if the energy input is sufficient

Let's look for the conditions for equilibrium at this scale. To that end, we need to look at pressure variations. For a bubble of radius R : $\Delta p = \frac{\gamma}{R}$ where $\Delta p = p_+ - p_-$ (p_+ = pressure inside the bubble, p_- = pressure outside). The turbulent motion at scale l induces pressure variations $\delta p \propto \rho_L u_l^2 \sim \rho_L (\varepsilon l)^{2/3}$ ($\varepsilon \sim u_l^3 / l$). If the scales satisfy $l \sim R$, equilibrium is possible when : $\Delta p \sim \delta p \Rightarrow \rho_L (\varepsilon R)^{2/3} \sim \frac{\gamma}{R} \Rightarrow \boxed{R \propto \left(\frac{\gamma^3}{\rho_L^3 \varepsilon^2} \right)^{1/5}}$.

A note on validity

This computation is only valid in the inertial range, $\boxed{\eta << R << l_0}$, where l_0 denotes the size of the largest structures of the turbulence. Hence : $\left(\frac{\nu^3}{\varepsilon}\right)^{1/4} << \left(\frac{\gamma^3}{\rho_L^3 \varepsilon^2}\right)^{1/5} << l_0 \Rightarrow \left(\frac{\nu^3}{\varepsilon}\right)^{5/4} << \left(\frac{\gamma^3}{\rho_L^3 \varepsilon^2}\right) << l_0^5 \Rightarrow \nu^{5/4} \varepsilon^{1/4} << \frac{\gamma}{\rho_L} << l_0^{5/3} \varepsilon^{2/3}$. Another requirement is that the upward velocity of bubbles, U_v , should be small compared to $u_R \sim (\varepsilon R)^{1/3}$. The velocity U_v is the terminal velocity of bubbles when buoyancy forces $\Delta \rho R^3 g$ ($\Delta \rho = \rho_L - \rho_G \approx \rho_L$) equal drag forces $C \frac{1}{2} \rho_L U_v^2 R^2$ where C denotes the drag coefficient (which is of order unity from the experiments of drag on a sphere). Equating these two terms : $\boxed{U_v \sim \sqrt{gR}}$. It follows that : $\sqrt{gR} << (\varepsilon R)^{1/3}$ hence $\boxed{R << \varepsilon^2 / g^3}$.

Replacing R with the expression above, we obtain the following condition: $\frac{\gamma^{3/8} g^{15/8}}{\rho_L^{3/8}} << \varepsilon$.