PC 2 : TURBULENCE, ENERGY CASCADE AND SELF-SIMILARITY

Goals
- Concepts of self-similar cascades, fractal dimensions

1. The inertial range of the turbulent cascade Richardson (1926)- Kolmogorov (1941)

Turbulent flows arise when hydrodynamic instabilities or external forces generate motions whose scales are too large for viscosity to act. The flow "organizes" itself so that there is a transfer of energy to smaller and smaller scales, until sufficiently small scales are reached, at which viscosity can dissipate the energy.

The phenomenological picture of the inertial energy cascade, models this process of energy transfer from large to small scales. It relies on 4 points:

1- The turbulence is represented by embedded flow "structures", whose characteristic scales l and velocities u_l result from a series of events during which structures with scale l split into smaller and smaller structures.

2- Energy is transferred from the largest structures with scales l_0 and u_0 at which energy is injected, towards the smaller structures. The initial rate of energy transfer is $\varepsilon_0 \sim u_0^2/\tau_0$ where τ_0 denotes the life time of the largest structures. Letting $\tau_0 \sim l_0/u_0$, we obtain $\varepsilon_0 \sim u_0^3/l_0$.

3 - Energy is dissipated at this same rate ε_0 at the small, viscous scales. By definition, these scales (l_ν, u_ν) satisfy $u_\nu l_\nu/\nu \approx 1$ (Reynolds number unity). This is the viscous regime, where kinetic energy is lost as heat.

4 - For $l_\nu << l << l_0$, which defines the inertial range, the transfer of energy between scales occurs in a self-similar manner, that is similarly at every scale l, and in a local manner, meaning that the energy of each structure is entirely transferred to the sub-structures that it generates at the end of its life. Viscosity does not affect this energy transfer, which is purely inertial. The local rate of energy transfer at scale l, $\varepsilon(l) \sim u_l^2/\tau_l \sim u_l^3/l$, is constant and equal to ε_0 for every l. The dynamic properties of the turbulent structures in the inertial range therefore only depend on l and ε_0.

(a) Determine the relationship between u_l and *l* in the inertial range of the energy cascade. Give the life time τ_l as a function of *l*. Comment.

(b) Determine the scales l_ν and u_ν of the viscous regime.

(c) One can test the validity of this theory as follows (Kolmogorov, 1941). In a fully turbulent pipe flow, it is possible to measure the following quantity :

$$\overline{|\delta u|^p}(r) = \frac{1}{T}\int_0^T \left|u(\underline{x} + r\,\underline{e}_1, t) - u(\underline{x}, t)\right|^p dt \qquad (2.1)$$

which is equal to the p[th] moment of the differences in zonal velocities measured at two points separated by a distance *r* in the direction of the pipe. The overbar denotes an average over a time *T* sufficiently long to eliminate fluctuations.

These moments, called "structure functions of order p", are independent of the location $\underline{x}$ (as long as the side walls are not too close, otherwise the distance to the walls needs to be taken into account as an additional scale).

Show that the theory of the Richardson-Kolmogorov cascade leads to $\overline{|\delta u|^p}(r) \sim r^{\zeta_p}$ and determine the value of the exponent ζ_p.

2. An intermittent cascade : the "β-model" (Frish, Sulem and Nelkin, 1978)

The theory described previously reproduces the observations remarkably well. Nevertheless, there are some discrepancies when the exponent p in equation (2.1) is large (typically when it is larger than 6). In order to understand these departures from equation (2.1), we need to refine the idea of the cascade.

This is still an open problem, but one hypothesis is that the intermittent aspect of the cascade plays a role, as structures become more and more localized in space at smaller and smaller scales. This intermittency can be observed in all experiments on turbulence. It is particularly important for high moments (large values of p), which emphasize large velocity differences that are statistically rare.

One attempt to modify the cascade accordingly is to suppose that at each step, the volume of active structures shrinks by a certain factor. The 4 points of the phenomenological picture described in the previous exercise are modified as follows:

> *5 - At step n of the inertial cascade, each "mother" structure of size l_{n-1} generates N "children" sub-structures with scaling factor $r = l_n / l_{n-1}$, $0 < r < 1$. This cascade is assumed intermittent, that is, at each step the "children" structures occupy a fraction $\beta < 1$ of the volume of the "mother" structure. The self-similar assumption consists in supposing that this cascade is entirely characterized by two of the three parameters (r, β, N). These parameters are assumed to be constant and independent of n.*

(a) Calculate, as a function of N and r, the fraction β of the volume of the structure with size l_{n-1} occupied at step n by structures with size l_n.

<u>Application</u> : Calculate β in two dimensions, with $r = 1/2$ and $N = 3$.

(b) Calculate, in 3 dimensions, the total number N_n of structures with size l_n obtained at step *n*.

(c) Show that this three-dimensional intermittent cascade can be characterized by a fractal dimension D which you will compute as a function of N and r or as a function of β and r.

<u>Application</u> : For a three-dimensional intermittent cascade, find the values of D and β when $r = 1/2$, for various values of N.

(d) Using the same approach as in the previous exercise, determine how the velocity scale u_l changes when $\beta < 1$ ($D < 3$). Deduce the exponent ζ_p of the structure function of order p as a function of D.