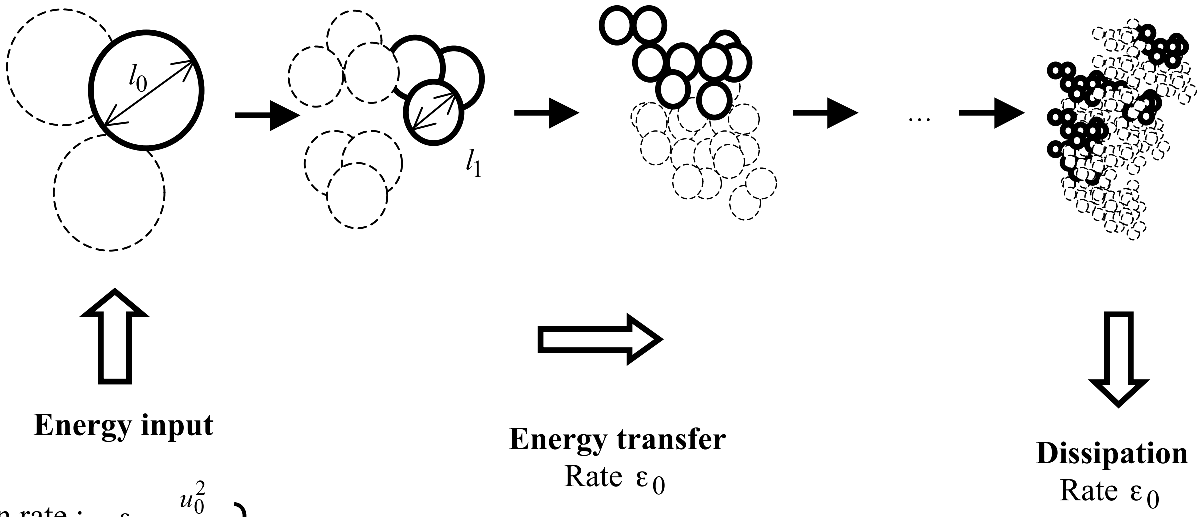
SOLUTION PC2

1. The inertial range of the turbulent cascade Richardson (1926)- Kolmogorov (1941)

The model of the energy cascade by Richardson works as follows :



Injection rate : $\varepsilon_0 \sim \dfrac{u_0^2}{\tau_0}$

Life time : $\tau_0 \sim \dfrac{l_0}{u_0}$

$$\boxed{\varepsilon_0 \sim \frac{u_0^3}{l_0}}$$

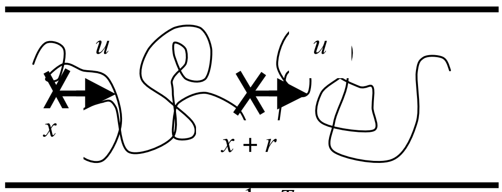
(a) Inertial range : By definition, the inertial range consists of scales l such that : $l_\nu << l << l_0$ where l_0 denotes the scale of energy input, and l_ν the scales where energy is dissipated (or viscous scales), which satisfy $\dfrac{u_\nu l_\nu}{\nu} \approx 1$. The rate of energy transfer being constant and equal to ε_0 at every scale, $\varepsilon_0 \sim \dfrac{u_l^3}{l}$, hence : $\boxed{u_l \sim (\varepsilon_0\, l)^{1/3}}$. This result can also be obtained by dimensional analysis from the relation : $u_l = F(\varepsilon_0, l)$ in the inertial range (note the absence of u_0, l_0 and ν in the inertial regime). Since $[\varepsilon_0] = \dfrac{L^2}{T^3}$ the above relationship follows. The life time is : $\boxed{\tau_l \sim \dfrac{l}{u_l} \sim \varepsilon_0^{-1/3}\, l^{2/3}}$.

Comments :

- What is conserved in the cascade is the rate of energy transfer ε_0. This means that each scale receives as much energy (from larger scales) as the energy it loses (towards smaller scales).
- For a given ε_0, the energy at equilibrium, $u_l^2 \sim (\varepsilon_0 l)^{2/3}$, decays with the scale of structures. Smaller structures contain less energy but equilibrium is maintained because the energy is transferred faster.
- This process does indeed ensure the reduction of the Reynolds number $\mathrm{Re} = \dfrac{u_l l}{\nu} \propto l^{4/3}$ until it reaches values around unity.
- The relationship $u_l \sim (\varepsilon_0\, l)^{1/3}$ implies invariance of velocity with respect to a change in scale with exponent $\dfrac{1}{3}$: $\forall \lambda \in R^+,\ \ u(\lambda\, l) = \lambda^{1/3} u(l)$. This type of scaling law is typically checked with log-log pots. Indeed: $\log[u(l_1)] - \log[u(l_2)] = 1/3 \log\lambda = 1/3(\log l_1 - \log l_2) \Rightarrow \dfrac{d \log u}{dl} = \dfrac{1}{3}$.

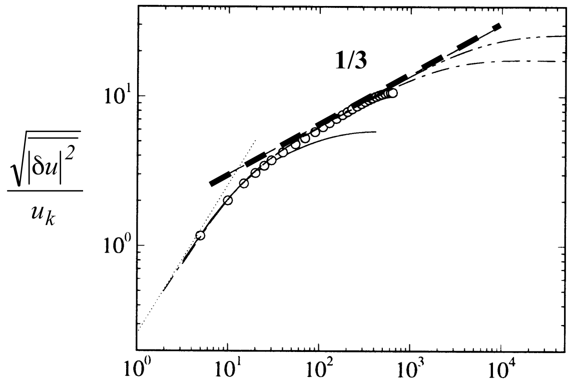
(b) Viscous regime : In order to derive the scales l_ν and u_ν, we use the relations : $\dfrac{u_\nu^3}{l_\nu} \sim \varepsilon_0$, $\dfrac{u_\nu l_\nu}{\nu} \approx 1 \Rightarrow \dfrac{u_\nu^4}{\nu} \sim \varepsilon_0$, $\dfrac{\nu^3}{l_\nu^4} = \varepsilon_0 \Rightarrow \boxed{u_\nu \sim (\nu\varepsilon_0)^{1/4} = u_k}$ et $\boxed{l_\nu \sim \left(\dfrac{\nu^3}{\varepsilon_0}\right)^{1/4} \equiv \eta}$. If we again use dimensional analysis, $F(u_\nu, l_\nu, \varepsilon_0, \nu) = 0$ where a new physical scale appears (ν) compared to the inertial range. One can rearrange the function F to account for the fact that the Reynolds number is close to unity at the viscous scales : $F'\left(\frac{u_\nu l_\nu}{\nu}=1, l_\nu, \varepsilon_0, \nu\right)=0 \;\Rightarrow\; F'(l_\nu, \varepsilon_0, \nu)=0$ or $F''\left(u_\nu, \frac{l_\nu u_\nu}{\nu}=1, \varepsilon_0, \nu\right)=0 \Rightarrow F''(u_\nu, \varepsilon_0, \nu)=0$. Dimensional analysis leads to the equations written above.

(c) Structure function

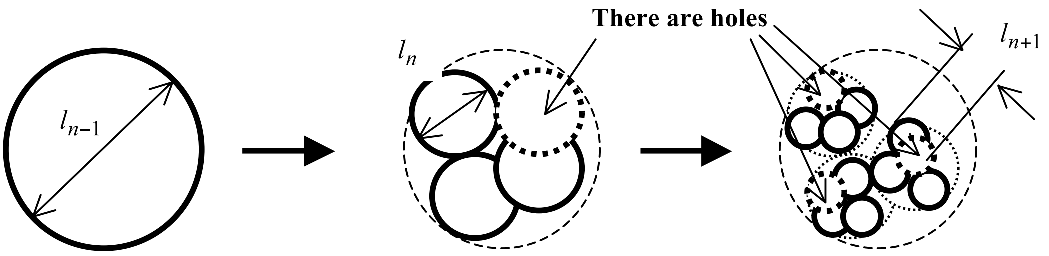


In the inertial range, $l_\nu << r << l_0$, the structure function $\overline{|\delta u|^p(r)} = \frac{1}{T}\int_0^T \left|u(\underline{x}+r\,\underline{e}_1,t) - u(\underline{x},t)\right|^p dt$ must satisfy the dimensional relation : $\overline{|\delta u|^p(r)} = F(r, \varepsilon_0)$. Hence : $\overline{|\delta u|^p(r)} \sim (\varepsilon_0\, r)^{p/3}$, and $\zeta_p = \frac{p}{3}$. The figure below shows experimental data verifying the value of the exponent with p=2.

Structure function of order 2 (square root) in various turbulent flows (in scaled coordinates).



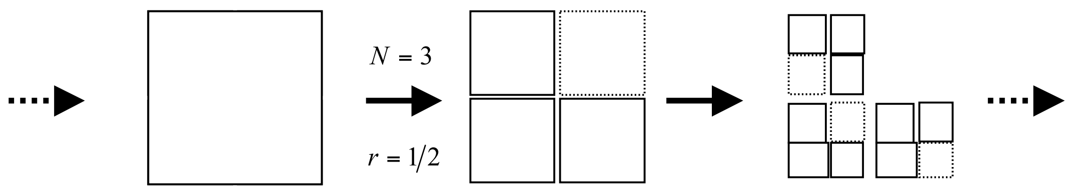
2. An intermittent cascade : the "β-model" (Frish, Sulem and Nelkin, 1978)



Step $n-1$ **Step** n **Step** $n+1$

(a) The two parameters of the self-similar cascade are: the volume fraction $\beta = N(l_n)^3/(l_{n-1})^3$ (or more generally $\beta = N(l_n)^d/(l_{n-1})^d$ in dimension d), with $\beta < 1$ in the intermittent case (the structures do not fill the whole space), and the scale ratio $r = l_n/l_{n-1} < 1$. In other words, if the structure of size l_{n-1} generates N structures of size l_n, the volume fraction is $\boxed{\beta = N\, r^3}$ (more generally $\beta = N\, r^d$ in dimension d).

2D example. We fix r and N



In this case : $\beta = \frac{3(l_n)^2}{(l_{n-1})^2} = \frac{3}{4}$.

(b) The total number of structures at the various steps is : $N_0 = 1;\ N_1 = N;\ N_2 = N^2 ...;\ N_n = N^n$. Hence the total number of structures created at step n is : $\boxed{N_n = N^n = \beta^n\ r^{-3n}}$.

(c) ***Fractal dimension*** *: The fractal dimension, or Kolmogorov capacity, is the exponent of the space in which structures with size* l_n *exactly cover the size* l_{n-1} *structure. It is there the dimension of the space such that* $Nr^D = 1,\ or\ N = \left(l_{n-1}/l_n\right)^D = r^{-D}$. It follows that $N = r^{-D} \Rightarrow \boxed{D = -\log N / \log(r)}$.

For the three-dimensional cascade, $N = \beta r^{-3} = r^{-D} \Rightarrow \log\beta - 3\log r = -D\log r \Rightarrow \boxed{D = 3 - \log\beta / \log(r)}$. (Note that $\beta = r^{3-D}$, $\boxed{\beta^n = \left(r^n\right)^{3-D} = \left(l_n/l_0\right)^{3-D}}$ where $(3-D)$ is the **co-dimension.** In the 2D example above : $N = \left(\frac{l_{n-1}}{l_n}\right)^3 = 2^D \Rightarrow$ $D = \frac{\log 3}{\log 2} = 1.58$).

Example. For $r = 1/2$, we obtain the following depending on the values of N :

Number of structures	N	8	6	4	2
Fractal dimension	$D = -\log N / \log(r)$	3	2,58	2	1
Volume fraction	$\beta = N\, r^3$	1	3/4	1/2	1/4

Volume fraction and fractal dimension for a scale ratio $r = 1/2$

(d) The consequences of the intermittency of the cascade are :

- There are less and less structures for the same energy flux. The energy at each step of the cascade is now : $e_n \sim \frac{N_n\left(l_n\right)^3 u_n^2}{l_0^3} = \beta^n u_n^2$

- The energy flux becomes $\varepsilon_n \sim \frac{e_n}{\tau_n} \sim \frac{\beta^n u_n^3}{l_n} = \varepsilon_0$, hence the velocity is $u_n \sim \left(\frac{\varepsilon_0 l_n}{\beta^n}\right)^{1/3} = \left(\varepsilon_0 l_n\right)^{1/3}\beta^{-n/3}$ and the life time $\tau_n \sim \frac{l_n}{u_n} \sim \varepsilon_0^{-1/3} l_n^{2/3}\beta^{n/3}$. Therefore, as the intermittency increases ($\beta < 1$ decreases), the characteristic velocity of the structures increases and their life time decreases. (In terms of the co-dimension, $\beta^n = \left(l_n/l_0\right)^{3-D}$, we obtain $u_n \sim l_n^{1/3}\left(l_n/l_0\right)^{-(3-D)/3}$, $\tau_n \sim \frac{l_n}{u_n} \sim l_n^{2/3}\left(l_n/l_0\right)^{(3-D)/3}$).

- For the structure function $\overline{|\delta u|^p}(l)$, using the same argument than the one we used for energy, we obtain $\overline{|\delta u|^p}(l) \sim \beta^n u_n^p \sim \beta^n\left(\varepsilon_0\ l\right)^{p/3}\beta^{-pn/3} = \beta^{n(1-p/3)}\left(\varepsilon_0\ l\right)^{p/3} \sim l^{p/3}\left(l/l_0\right)^{(3-D)(1-p/3)}$. We recover the earlier result when D=3 (no intermittency). The exponent is now $\zeta_p = \frac{p}{3} + (3-D)(1-p/3)$. It is reduced (smaller than p/3) when $p > 3$, which matches observations. Nevertheless the exponent is increased (larger than p/3) when $p < 3$. For instance with p=2 the structure function has exponent $\zeta_2 = \frac{2}{3} + (3-D)(1-2/3) = \frac{2}{3} + \frac{3-D}{3}$ whereas $\zeta_2 = \frac{2}{3}$ agrees very well with observations as shown in the figure of the previous exercise. This problem is still the subject of active research.