J. Fluid Mech. (2003), vol. 481, pp. 269–290.
DOI: 10.1017/S0022112003003896 Printed in the United Kingdom

Frontal geostrophic adjustment, slow manifold and nonlinear wave phenomena in one-dimensional rotating shallow water. Part 1. Theory

By V. ZEITLIN[1], S. B. MEDVEDEV[2] AND R. PLOUGONVEN[1]

[1]LMD, Ecole Normale Supérieure, 24, rue Lhomond, 75231 Paris Cedex 05, France
zeitlin@lmd.ens.fr

[2]Institute of Computational Technologies, 6 Lavrentiev av., 630090 Novosibirsk, Russia

(Received 13 February 2002 and in revised form 29 October 2002)

The problem of nonlinear adjustment of localized front-like perturbations to a state of geostrophic equilibrium (balanced state) is studied in the framework of rotating shallow-water equations with no dependence on the along-front coordinate. We work in Lagrangian coordinates, which turns out to be conceptually and technically advantageous. First, a perturbation approach in the cross-front Rossby number is developed and splitting of the motion into slow and fast components is demonstrated for non-negative potential vorticities. We then give a non-perturbative proof of existence and uniqueness of the adjusted state, again for configurations with non-negative initial potential vorticities. We prove that wave trapping is impossible within this adjusted state and, hence, adjustment is always complete for small enough departures from balance. However, we show that retarded adjustment occurs if the adjusted state admits quasi-stationary states decaying via tunnelling across a potential barrier. A description of finite-amplitude periodic nonlinear waves known to exist in configurations with constant potential vorticity in this model is given in terms of Lagrangian variables. Finally, shock formation is analysed and semi-quantitative criteria based on the values of initial gradients and the relative vorticity of initial states are established for wave breaking showing, again, essential differences between the regions of positive and negative vorticity.

1. Introduction

The main motivation for the present study is the problem of geostrophic adjustment. What is called below the one-dimensional rotating shallow water model (1dRSW), i.e. the standard shallow water model on a rotating plane (2dRSW) where dynamical variables do not depend on one of the spatial coordinates, is the simplest possible model for studying this phenomenon. As such, it was first used in the pioneering work by Rossby (1938) (cf. also Gill 1982).†

Geostrophic adjustment is a process of relaxation of an arbitrary initial perturbation to a corresponding state of geostrophic equilibrium, i.e. the equilibrium between the pressure force and the Coriolis force. The standard scenario of adjustment established

† As usual in geophysical applications, the centrifugal force will be neglected and molecular dissipation will be absent in what follows.

in the classical works of Rossby (1938) and Obukhov (1949) (see Blumen 1972 for a review of the subsequent work) consists of an initial disturbance emitting fast inertia–gravity waves and tending to a slowly evolving geostrophically balanced flow which bears the whole of the initial potential vorticity (PV). The adjustment process, thus, provides a basis for *splitting* of the motion into fast and slow components which is essential for meteorological and oceanographic applications. For example, in weather prediction it allows one after proper initialization, to filter fast waves, insignificant for most synoptic purposes, and to follow only the slow vortical component of motion. In practice it is, of course, done in more sophisticated models than RSW. However, this latter was historically the first one used for this purpose.

The height perturbation defining completely the *geostrophic* PV, the essentially one-dimensional approach of Rossby (1938) was to use the point-wise conservation of PV in order to write a differential equation for the height distribution of the final state which is presumed to be completely adjusted, starting from the PV of the initial state. All intermediate stages of the adjustment process are, thus, omitted as only the initial and the final states are considered. Obukhov's (1949) approach was a thorough treatment of the problem in the linear approximation (valid for vanishing Rossby numbers) in two dimensions, wave emission being included.

The *nonlinear* geostrophic adjustment of single-scale vortex-like disturbances in 2dRSW was recently studied in detail by Reznik, Zeitlin & Ben Jelloul (2001) by means of the multi-time-scale perturbation theory in the Rossby number. Both slow and fast components of motion were completely resolved at the first three orders of the perturbation theory. Although the classical scenario of adjustment and fast–slow splitting were, generally, confirmed, it was also demonstrated that large-scale large-amplitude initial perturbations contain near-inertial oscillations which stay coupled to the slow vortical component of the flow for a long time and thus 'retard' the adjustment process with respect to the standard scenario of fast dispersion of the inertia–gravity waves.

Some important questions remain open regarding the fully nonlinear geostrophic adjustment. First, there is the question of existence and uniqueness of the balanced state or, more generally, the question of the (non-)existence of the slow manifold – a key problem in geophysical fluid dynamics (see e.g. Leith 1980 and, for a recent discussion, Ford, McIntyre & Norton 2000). Second, if it exists, is the balanced state always attainable? And, if yes, what is the characteristic relaxation time? (It was argued by Rossby (1938) that the adjustment process is very fast.) A related question is whether adjustment is always complete. (One may think, for instance, of possible trapped-wave modes in the interior of the balanced vortex core.) Finally, what is the influence of essentially nonlinear wave effects, such as wave breaking, on adjustment? (The PV is not conserved if wave-breaking takes place.)

In spite of its evident degeneracy, the 1dRSW model still contains all the essential dynamical ingredients of the adjustment process: fast inertia–gravity waves and 'vortices', which become transverse jets in this context. The PV is conserved, the states of geostrophic equilibrium are stationary solutions of the equations of motion and thus represent an exact slow manifold. Being, however, essentially simpler than the full 2dRSW model it allows one to go further both analytically and numerically in trying to answer the above-posed questions. Note that adjustment in the 1dRSW model is, properly speaking, semi-geostrophic (cf. e.g. Pedlosky 1984) as it is the adjustment of fronts and jets which takes place in one (cross-front) spatial direction. Recently a study of the full nonlinear adjustment problem in the 1dRSW model was undertaken numerically by Kuo & Polvani (1997) who analysed the relaxation of the stepwise perturbation of the surface elevation and demonstrated the formation of secondary shocks. Nevertheless, a good agreement of the final configuration with Rossby's scenario was found. The same authors (Kuo & Polvani 1999) considered nonlinear wave–vortex (jet) interactions, again in the framework of 1dRSW, by studying scattering of wave-trains on jets and found by numerical simulations supported by asymptotic analysis a re-adjustment of jets to their initial form, while shock formation by strong height perturbations was again observed. It should be noted in this context that the fully nonlinear geostrophic adjustment problem meets a general fluid dynamics problem of the influence of rotation on shock formation and propagation.

The aim of the present paper is to fully exploit the simplicity of the 1dRSW model by using analytic tools and to give definite answers to some of the above questions as well as to interpret the existing numerical results. The key ingredient of our approach is the use of Lagrangian variables, which proves to be crucial. The plan of the paper is as follows. After a brief discussion of the model in § 2 we introduce Lagrangian variables in § 3 and show that 1dRSW may be reduced to a single second-order PDE well-adapted for studying the initial-value (i.e. adjustment) problem. We then present in § 4 a perturbative approach to adjustment in these variables and prove the absence of trapped modes. The existence and uniqueness theorem for the non-perturbative balanced state in the case of non-negative initial PV is then proved in § 5 and the relaxation of arbitrary initial perturbations to this adjusted state is discussed. We analyse nonlinear periodic waves in 1dRSW (Shrira 1981, 1986; Grimshaw *et al.* 1998) in the Lagrangian framework (Bühler 1993) in § 6. Finally, by using the Lax method (Lax 1973; Engelberg 1996) we study the conditions of shock formation in the presence of rotation in § 7. Section 8 contains conclusions and discussion. We present a proof of existence and uniqueness of the adjusted state (i.e. of the slow manifold) in Appendix A. In Appendix B we give another one-dimensional version of the RSW equations which corresponds to axisymmetric flows and may be useful in applications.

2. General features of the model

The shallow water equations on the rotating plane with no dependence on one of the coordinates (y) are

$$\left.\begin{aligned} \partial_t u + u\partial_x u - fv + g\partial_x h = 0, \\ \partial_t v + u\partial_x v + fu = 0, \\ \partial_t h + u\partial_x h + h\partial_x u = 0. \end{aligned}\right\} \qquad (1)$$

Here u, v are the two components of the horizontal velocity, h is the total fluid depth (no topographic effects will be considered), f is the (constant) Coriolis parameter, and g is the acceleration due to gravity. Here and below ∂_t, ∂_x, etc. denote the partial derivatives with respect to the corresponding arguments and ∂^2_{xx} etc. denote the second derivatives. Physically the model represents the classical one-dimensional shallow water model (cf. e.g. Witham 1974) with a transverse flow added, all under the influence of the Coriolis force. Due to the presence of the transverse jets the dynamics is not purely one-dimensional, but rather 'one-and-a-half' dimensional. In what follows we are mainly interested in the behaviour of localized jets (localized distributions of $v(x)$) and/or fronts (localized distributions of $\partial_x h(x)$). Hence, the front or jet stretches along the y-axis and we will often call u the cross-front velocity. In general, we will call a configuration front-like if u, v, $\partial_x h$ have a common compact support in x.

The model possesses two Lagrangian invariants: the generalized momentum $M = v + fx$ and the potential vorticity (PV) $Q = (\partial_x v + f)/h$:

$$(\partial_t + u\partial_x)M = 0, \quad (\partial_t + u\partial_x)Q = 0. \tag{2}$$

Linearization around the rest state $h = H$ gives a zero-frequency mode and surface inertia–gravity waves with the standard dispersion law

$$\omega = \pm\left(c_0^2 k^2 + f^2\right)^{1/2}. \tag{3}$$

Here $c_0 = \sqrt{gH}$, ω is the wave frequency and k is the wavenumber. As usual, one may draw a parallel between the shallow-water dynamics and the gas dynamics (modified by rotation), so c_0 is analogous to the sound speed.

The steady states corresponding to the zero mode are the geostrophic equilibria:

$$fv = g\partial_x h, \quad u = 0, \tag{4}$$

which are the exact stationary solutions (note this important difference with 2dRSW, where they are not). Hence, in the state of geostrophic equilibrium the velocity is entirely determined by the height perturbation and the geostrophic PV is

$$Q^{(g)} = \left(\frac{f + (g/f)\partial_{xx}^2 h}{h}\right). \tag{5}$$

Note that a geostrophically adjusted front corresponds to a stepwise h with a jet in v given by (4). In what follows we will mostly limit ourselves to finite-energy localized front-like configurations as initial conditions for (1), thus excluding 'non-adjustable' infinite-energy (cf. (4)) configurations like, for instance, a constant shear $v \sim x$.

3. Lagrangian approach to 1dRSW

As will be seen below, it is physically more transparent and technically advantageous to use the Lagrangian version of (1). For this purpose we introduce the coordinates of Lagrangian 'quasi-particles' $X(x, t)$ via the mapping $x \to X(x, t)$, where x (Lagrangian label) is a quasi-particle's position at $t = 0$ and X its position at time t. (In fact, real Lagrangian particles are moving both in the x- and y-directions in the model; we introduce quasi-particles: strings moving only in x.) From the Eulerian point of view the X are just the Eulerian coordinates, but we use the notation x for the Lagrangian labels because the standard notation a is reserved for another purpose – see below. Hence $\dot{X} = u(X, t)$ and an over-dot will denote the (material) time-derivative from now on, while a prime (e.g. $X' = \partial_x X$) will be used for differentiation with respect to x for compactness. The two first (momentum) equations of (1) are then rewritten as

$$\left.\begin{aligned} \ddot{X} - fv + g\partial_X h &= 0, \\ \dot{v} + f\dot{X} &= 0, \end{aligned}\right\} \tag{6}$$

where v is considered as a function of x and t. By virtue of mass conservation h evolves from its initial distribution $h_I(x)$ via the Jacobi transformation

$$h(X, t) = h_I(x)\partial_X x. \tag{7}$$

By differentiating this equation (or, rather, the corresponding equation for h^{-1}) in time it is easy to check that it is equivalent to the continuity equation

$$\dot{h} + h\partial_X \dot{X} = 0. \tag{8}$$

The second of equations (6) may be immediately integrated giving

$$v(x,t) + fX(x,t) = F(x), \tag{9}$$

where $F(x)$ is an arbitrary function to be determined from the initial conditions. If $v_I(x)$ is an initial distribution of the transverse velocity then, as $X(x,0) = x$,

$$F(x) = fx + v_I(x). \tag{10}$$

By applying the chain differentiation rule to (7) we may express the partial derivative ∂_X as follows:

$$\partial_X h = \partial_X(h_I(x)\partial_x X) = h_I' \frac{1}{(X')^2} - h_I(x)\frac{X''}{(X')^3}, \tag{11}$$

and, thus, obtain a closed equation for X:

$$\ddot{X} + f^2 X + g h_I' \frac{1}{(X')^2} + \frac{g h_I}{2}\left[\frac{1}{(X')^2}\right]' = fF. \tag{12}$$

It is more convenient to rewrite this equation in terms of the deviations of quasi-particles from their initial positions $X(x,t) = x + \phi(x,t)$:

$$\ddot{\phi} + f^2\phi + g h_I' \frac{1}{(1+\phi')^2} + \frac{g h_I}{2}\left[\frac{1}{(1+\phi')^2}\right]' = f v_I. \tag{13}$$

This equation is to be solved with initial conditions $\phi(x,0) = 0$, $\dot{\phi}(x,0) = u_I(x)$, where u_I is the initial distribution of longitudinal velocity. The case of a front-like disturbance corresponds to h_I', u_I, v_I with a common compact support.

It is clear from its form that equation (13) is well-suited for studies of the Cauchy problem which is equivalent to the nonlinear geostrophic adjustment problem. An example of low-resolution calculation using the standard MATHEMATICA routine NDSolve is presented in figure 1 giving a qualitative picture of adjustment.

Note that in the absence of rotation, $f = 0$, the nonlinear equation (12) contains only derivatives of X and not X itself and may be reduced to a linear problem (i.e. completely integrated) by the hodograph method using $\dot{X}, X'$ or, equivalently, u and h as new variables and swapping dependent and independent variables: $x = x(u,h)$, $t = t(u,h)$. A linear PDE for x then results – cf. e.g. Landau & Lifshitz (1975). The presence of rotation makes the hodograph transformation ineffective.

Finally, it is worth noting that (12) may be obtained from an action principle. By multiplying the left-hand side by h_I we obtain

$$h_I(\ddot{X} + f^2 X - fF) + \left(\frac{g h_I^2}{2X'^2}\right)' = 0. \tag{14}$$

Equation (14) is an Euler–Lagrange equation following from the variational principle with the action

$$S = \int \mathrm{d}t \int \mathrm{d}x\, L \tag{15}$$

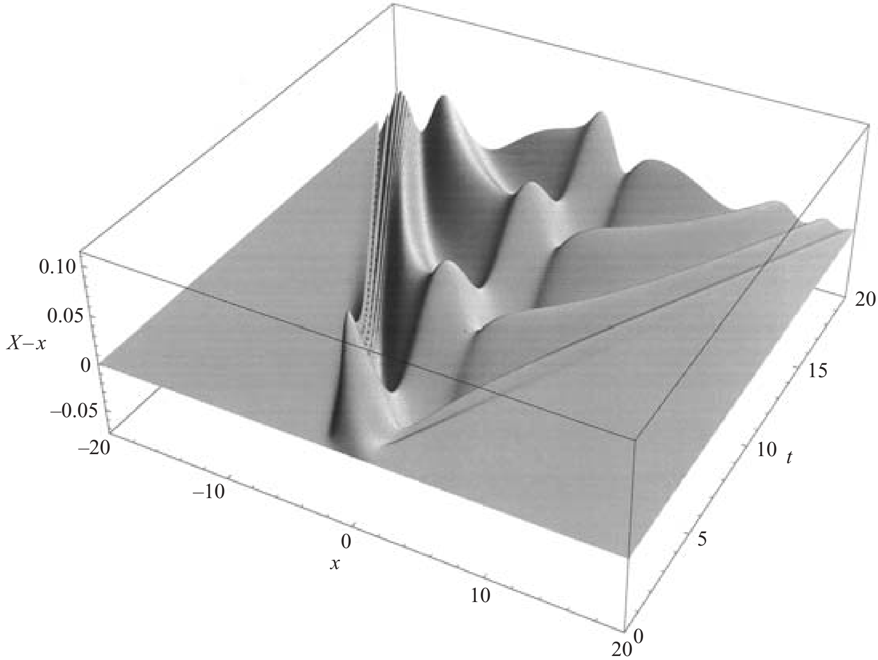

FIGURE 1. Geostrophic adjustment of the double-jet configuration (chosen for technical reasons of spatial periodicity) with a Gaussian profile of initial elevation: $h_I(x) = 1 + \exp(-x^2)$, slightly imbalanced $v_I(x) = -2(x + 0.2\sin(x))\exp(-x^2)$, and $u_I(x) = 0.1\exp(-x^2)$, as obtained by straightforward numerical integration of equation (13) using the standard MATHEMATICA routine NDSolve. Time and initial positions of Lagrangian particles are along the horizontal axes, the particle displacement $X - x$ is along the vertical axis. Emitted fast gravity waves and slowly dispersing quasi-inertial oscillations are clearly seen, as well as systematic displacements of fluid particles necessary to reach the final adjusted state. At least one shock forms and propagates toward the left, in agreement with the results of § 7, but because of low resolution the result is the fuzzy region on the left.

and the Lagrangian density

$$L = h_I\left(\frac{\dot{X}^2}{2} - f^2\frac{X^2}{2} + fFX\right) - \frac{gh_I^2}{2}\frac{1}{X'}. \tag{16}$$

Introducing canonical variables (fields) $\mathscr{P} = h_I\dot{X}, X$ we obtain a corresponding Hamiltonian:

$$H = \int \mathrm{d}x\left(\frac{\mathscr{P}^2}{2h_I} + h_I\left(f^2\frac{X^2}{2} - fFX\right) + \frac{gh_I^2}{2}\frac{1}{X'}\right). \tag{17}$$

As usual, the vanishing of the first variation of the Hamiltonian δH gives the equation for the steady state:

$$h_I(f^2X - fF) + \left(\frac{gh_I^2}{2X'^2}\right)' = 0, \quad \mathscr{P} = 0. \tag{18}$$

This is just the state of geostrophic equilibrium (4) expressed in Lagrangian coordinates. A straightforward calculation of the second variation $\delta^2 H$ shows that it is always positive-definite and, thus, the geostrophic equilibria are formally stable.

4. Perturbative semigeostrophic adjustment

We start our study of (13) with a straightforward perturbation analysis by noting that if the initial conditions v_I and h_I are in geostrophic balance

$$gh_I' = f v_I, \tag{19}$$

then $\phi = 0$ is an exact solution of (13). Therefore, for small deviations from exact geostrophy (13) may be linearized and solved perturbatively. A necessary condition for this is smallness of the initial imbalance $A_I = f v_I - gh_I'$, which will be supposed throughout this section. It is easy to show that this condition is equivalent to the smallness of the (cross-front) Rossby number

$$Ro = \frac{U}{fL}, \tag{20}$$

where U and L are the characteristic velocity and the scale of the cross-front motion. One of the advantages of the Lagrangian approach is that contrary to the Eulerian approach where it is necessary to attribute the initial imbalance either to v_I or to h_I, the imbalance A_I enters the perturbative equations as a whole.

The linearized (first-order) equation (13) is

$$\ddot{\phi} + f^2\phi - 2gh_I'\phi' - gh_I\phi'' = A_I. \tag{21}$$

We represent the solution as a combination of the slow and the fast components: $\phi = \bar{\phi} + \tilde{\phi}$ where the over-bar denotes the time mean and the tilde corresponds to fluctuations around it. For the time-mean (zero mode) one obtains

$$f^2\bar{\phi} - 2gh_I'\bar{\phi}' - gh_I\bar{\phi}'' = A_I, \tag{22}$$

a linear second-order inhomogeneous differential equation. It is useful to rewrite this equation in terms of the new variable $\bar{\Phi} = gh_I\bar{\phi}$. Dividing by gh_I yields

$$-\bar{\Phi}'' + \left(\frac{f^2 + gh_I''}{gh_I}\right)\bar{\Phi} = A_I, \tag{23}$$

where we see the geostrophic PV constructed from the initial height perturbation $Q_I^{(g)}$, cf. (5), coming into play. Equation (23) is a linear inhomogeneous ODE with variable coefficients and its solution may be obtained by the method of variation of constants once solutions of the homogeneous equation are known. We are mostly interested in the frontal case where A_I is a compact support function and h_I is monotonic and has constant asymptotics. Solutions of the homogeneous equation are then exponentially growing/decaying at both spatial infinities. However, it is not obvious that a solution of the inhomogeneous equation decaying at both spatial infinities may be found for arbitrary $Q_I^{(g)}(x)$. By the same method which is used below in the full non-perturbative proof (cf. Appendix A) we are able to show that unique solution $\phi_s(x)$ of (22) exists for non-negative geostrophic PV: $Q_I^{(g)} \geqslant 0$.

For the time varying (fast) part of the solution a homogeneous equation

$$\ddot{\tilde{\phi}} + f^2\tilde{\phi} - 2gh_I'\tilde{\phi}' - gh_I\tilde{\phi}'' = 0 \tag{24}$$

results. By introducing a new variable $\tilde{\Phi} = gh_I\tilde{\phi}$ this equation may be rewritten as

$$\ddot{\tilde{\Phi}} + (f^2 + gh_I'')\tilde{\Phi} - gh_I\tilde{\Phi}'' = 0, \tag{25}$$

where the geostrophic PV enters again.Solution of the Cauchy problem for arbitrary initial conditions $\tilde{\phi}_I$, $u_I = \dot{\tilde{\phi}}_I$ can be obtained via e.g. the Fourier transform $\Psi(x,\omega) = \int_{-\infty}^{\infty} e^{+i\omega t}\,\tilde{\Phi}(x,t)\,dt$ for which we get

$$-\Psi'' + \left[\frac{f}{g}\left(\frac{f+\dfrac{gh_I''}{f}}{h_I}\right) - \frac{\omega^2}{gh_I}\right]\Psi = 0. \tag{26}$$

Note that $Q_I^{(g)}$ plays the rôle of potential in this Schrödinger-type equation.

As $\phi_I = \tilde{\phi}_I + \bar{\phi}_I = 0$ the initial condition for $\tilde{\phi}$ and, hence, $\tilde{\Phi}$ follows once $\bar{\phi}$ is found: $\tilde{\phi}_I = -\bar{\phi}$.

This procedure may be repeated order by order in amplitude of ϕ along the lines of Reznik *et al.* (2001) with the difference that solutions of the zero-order approximation (23), (25) are not known explicitly as these are equations with non-constant coefficients. This difficulty is, however, technical, rather than fundamental. The solution of (25) represents a packet of inertia–gravity waves propagating out of the initial localized perturbation plus, possibly, some bound states (trapped modes). Hence, at least on the present perturbative level, the adjustment scenario depends on the absence (complete adjustment) or presence (incomplete adjustment) of the trapped modes. We demonstrate below that trapping by an isolated front is impossible.

A simple argument shows that the trapped modes should be sub-inertial, i.e. having a frequency below f. Consider the Fourier transform of $\tilde{\phi} : \tilde{\phi} = \int d\omega(\psi(\omega,x)e^{-i\omega t} + \text{c.c})$. Then for each Fourier component $\psi(\omega,x)$ we obtain from (24):

$$gh_I\psi'' + 2gh_I'\psi' + (\omega^2 - f^2)\psi = 0\,, \tag{27}$$

which is equivalent to (26). In the case of a front-like initial configuration the asymptotics of $\tilde{\phi}_I$ and u_I at infinity are zero and those of h_I and $Q^{(g)}$ are constant:

$$h_I|_{\pm\infty} = h_\pm, \quad Q^{(g)}\big|_{\pm\infty} = \frac{f}{h_\pm}. \tag{28}$$

Hence, at $x \to \pm\infty$ (26) becomes

$$-\Psi'' + \frac{f^2-\omega^2}{gh_\pm}\Psi = 0 \tag{29}$$

and in order to have bound states decaying at spatial infinity we should have $\omega < f$. We will show now that this is impossible.

By multiplying (27) by $gh_I\psi^*$, where the asterisk denotes complex conjugation we obtain

$$\left(g^2h_I^2\psi^*\psi'\right)' - g^2h_I^2\psi'^*\psi' + (\omega^2 - f^2)gh_I\psi^*\psi = 0 \tag{30}$$

and for states decaying at $\pm\infty$ an estimate

$$\omega^2 = f^2 + \frac{\displaystyle\int_{-\infty}^{+\infty} dx\; g^2h_I^2|\psi'|^2}{\displaystyle\int_{-\infty}^{+\infty} dx\; gh_I|\psi|^2} \geqslant f^2 \tag{31}$$

follows by integration. Thus, we arrive at a contradiction. Hence, there are no sub-inertial trapped modes in the model and the frequency spectrum is continuous. Therefore, all of the initial $\tilde{\phi}$-perturbation will be dispersed leaving only the stationary part ϕ_s in the vicinity of the initial perturbation. As shown in Reznik *et al.* (2001) the outgoing waves do not exert any drag upon the stationary state at lowest orders in Ro and, thus, slow and fast variables are split in the perturbation theory, at least for non-negative PVs. The speed of the relaxation toward the adjusted state will depend on further details of the potential $Q^{(g)}$. If quasi-stationary states, i.e. those which decay only by sub-barrier tunnelling, are present the decay rate will be exponential, as is well-known from quantum mechanics (cf. e.g. Migdal 1977). Otherwise the decay will be dispersive according to the $t^{-1/2}$ law. Here and below we mean by 'decay' a time decrease of the amplitude of a spatially localized perturbation.

5. The non-perturbative slow manifold and further discussion of the relaxation process

The X-variables we introduced before have clear physical meaning and allow one to directly incorporate the initial conditions on the free-surface elevation and along-front velocity into the evolution equation (12). However, it is technically simpler to work with partial differential equations with constant coefficients. This goal may be achieved by an additional change of variables $x = x(a)$, which 'straightens' the initial elevation profile $h_I(x)$. We obtain the following relation between J and h (recall that h is everywhere positive): $J = \partial X/\partial a = H/h(X, t)$, where the uniform mean height H is introduced. By the chain differentiation rule we obtain $g\,\partial_X h = \partial_a P$, where $P = gH/(2J^2)$ is the pressure variable. The Lagrangian equations of motion then take the form

$$\dot{u} - fv + \partial_a P = 0, \tag{32}$$

$$\dot{v} + fu = 0, \tag{33}$$

$$\dot{J} - \partial_a u = 0. \tag{34}$$

These equations may also be obtained from the original Eulerian system (1) by transforming the independent variables via solution (see for details Rozdestvenskii & Janenko 1978)

$$\mathrm{d}a = h(x, t)\,\mathrm{d}x - h(x, t)u(x, t)\,\mathrm{d}t, \quad \mathrm{d}\tau = \mathrm{d}t, \tag{35}$$

where a is called the Lagrangian mass variable. For positive h one obtains

$$\partial_x = h\,\partial_a, \quad \partial_t = \partial_\tau - hu\,\partial_a = \partial_\tau - u\,\partial_x \tag{36}$$

and the system (32)–(34) results.

System (32)–(34) is equivalent to a single equation, which may equally well be obtained by differentiation and change of variables from (12) and, thus, contains the full dynamics of the model:

$$\ddot{J} + f^2 J + \partial^2_{aa} P = fHQ\,, \tag{37}$$

where $Q(a)$ is the potential vorticity in a-coordinates:

$$Q(a) = \frac{1}{H}(\partial_a v(a, t) + fJ(a, t)) = \frac{1}{H}(\partial_a v_I(a) + fJ_I(a)), \quad \dot{Q} = 0.$$

The non-perturbative slow manifold is a stationary solution J_s of (37). For a given set of (localized) initial conditions it represents a stationary state with the same potential vorticity as the initial one. We have

$$\frac{gH}{f}\frac{\mathrm{d}^2}{\mathrm{d}a^2}\left(\frac{1}{2J_s^2(a)}\right) + fJ_s(a) = HQ(a). \tag{38}$$

This equation is nonlinear and can be rewritten as a non-autonomous ODE describing the motion of a material point in a given potential under the action of a 'time'-dependent force, where 'time' is a. Using the non-dimensional pressure variable $p = P/(gH)$ and introducing the Rossby deformation radius $R_d^2 = gH/f^2$ we obtain

$$\frac{\mathrm{d}^2 p}{\mathrm{d}a^2} + \frac{1}{R_d^2}\frac{1}{\sqrt{2p}} = \frac{f}{g}Q. \tag{39}$$

The corresponding boundary conditions are: the external force (the right-hand side) exactly equilibrates the potential force (the second term on the left-hand side) at $a = \pm\infty$. Hence, a stationary solution, if it exists, is a separatrix trajectory relating two states of unstable equilibrium for this simple one-dimensional system. Thus, the balanced solution resembles soliton or instanton solutions in a number of physical models.

It turns out, in spite of this tempting interpretation, that it is simpler to analyse the problem by re-introducing X via the change of variables $\mathrm{d}X = J\,\mathrm{d}a$ which gives

$$-\frac{g}{f}\,\frac{\mathrm{d}^2 h(X)}{\mathrm{d}X^2} + h(X)\,Q(X) = f. \tag{40}$$

Here PV is considered as a function of X via the inverse mapping $x = x(X,t)$ (or $a = a(X)$):

$$Q(X) = \frac{1}{h_I(x(X))}\left(f + \frac{\partial v_I(x(X))}{\partial x}\right).$$

The following theorem, which gives sufficient conditions for existence and uniqueness of the slow manifold may be proved then (the technicalities of the proof are standard for the ODEs theory and are given in Appendix A):

THEOREM. *For positive $Q(X)$ with compact support derivatives and arbitrary constant asymptotics (frontal case) equation* (40) *has unique bounded and everywhere positive solution $h(X)$ on $\boldsymbol{R}^1$.*

It is worth noting that like any separatrix trajectory this solution is exponentially unstable as trajectories close to it diverge exponentially, which is useful to recall while trying to find it numerically.

Another remark is that, in general, positiveness of Q is a sufficient condition for the absence of inertial instability (cf. e.g. Holton 1979; this instability as such is absent in 1dRSW due to its one-dimensionality). A slow manifold in the proper sense exists in 1dRSW with non-negative PV because the geostrophically balanced states are necessarily steady (cf. (4)). In the full 2dRSW they may be unsteady and, thus, subject to the Lighthill radiation making the slow manifold 'fuzzy' (Ford *et al.* 2000).†

Finally, the proof of the theorem is not constructive in the sense that it uses the mapping $x = x(X,t)$ which is not known explicitly. However, the Lagrangian conservation of PV guarantees that this mapping preserves the positiveness of PV and its asymptotics at infinity (provided infinity is a fixed point).

Linearization around the true adjusted state denoted by a subscript s gives

$$u = \tilde{u}, \quad v = v_s + \tilde{v}, \quad J = J_s + \tilde{J},$$

† After the present paper was submitted for publication an instructive discussion of the relevance of Lighthill radiation in the presence of rotation appeared in the literature, cf. Saujani & Shepherd (2002) and Ford, McIntyre & Norton (2002).

$$\dot{\tilde{u}} - f\tilde{v} - gH\partial_a\left(\tilde{J}/J_s^3\right) = 0, \tag{41}$$

$$\dot{\tilde{v}} + fu = 0, \tag{42}$$

$$\dot{\tilde{J}} - \partial_a u = 0. \tag{43}$$

By using

$$f\tilde{J} + \partial_a\tilde{v} = 0 \tag{44}$$

it is easy to obtain a single equation for $\tilde{J}$ and/or for $\tilde{v}$:

$$\ddot{\tilde{J}} + f^2\tilde{J} - gH\partial_{aa}^2\left(\tilde{J}/J_s^3\right) = 0, \quad \ddot{\tilde{v}} + f^2\tilde{v} - gH\partial_a\left(\partial_a\tilde{v}/J_s^3\right) = 0, \tag{45}$$

which are equivalent, as it is easy to see by applying (44).

Let us consider solutions of the following form:

$$\tilde{J} = \hat{J}(a)\,\mathrm{e}^{-\mathrm{i}\omega t} + \text{c.c.}, \quad \tilde{v} = \hat{v}(a)\,\mathrm{e}^{-\mathrm{i}\omega t} + \text{c.c.} \tag{46}$$

Then the corresponding equations for $\hat{J}$ and $\hat{v}$ are

$$\partial_{aa}^2(gH_s\hat{J}) + (\omega^2 - f^2)\hat{J} = 0, \quad \partial_a(gH_s\partial_a\hat{v}) + (\omega^2 - f^2)\hat{v} = 0, \tag{47}$$

where we denoted $H_s = H/J_s^3$. We choose the equation for $\hat{v}$ for further analysis because it is self-adjoint. Note that supra-inertiality of ω and, hence, the absence of trapped states follows trivially from (47). By using a new dependent variable ψ

$$\hat{v} = \frac{\psi}{gH_s^{1/2}} \tag{48}$$

we transform the stationary equation to the two-term canonical form

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}a^2} + \left[\frac{\omega^2 - f^2}{gH_s} - \frac{1}{4}\left(\frac{(H_s)_a}{H_s}\right)^2 - \frac{1}{2}\left(\frac{(H_s)_a}{H_s}\right)_a\right]\psi = 0. \tag{49}$$

Rewritten as

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}a^2} + k_\psi^2(a)\psi = 0, \tag{50}$$

this equation can be interpreted as the oscillator equation with variable frequency $k_\psi(a)$ (or as a Schrödinger equation with a potential V and an energy E such that $k_\psi^2 = E - V(a)$). It is clear that k_ψ^2 can be negative for $\omega > f$ and suitable H_s. This means that for certain intervals on the x-axis the wavenumber k_ψ may be imaginary (tunnelling) and quasi-stationary states may exist. For example, in the situation presented in figure 1 we have a double-jet configuration falling into this class.

Hence, for any non-negative distribution of initial PV a corresponding iso-PV stationary state exists and the linear analysis shows that small perturbations relax toward this balanced state by emission of inertia–gravity waves. The relaxation law is, in general, a combination of dispersive $\propto t^{-1/2}$ and exponential $\propto \mathrm{e}^{-\Gamma t}$ decays (cf. Migdal 1977), the latter being provided by the quasi-stationary states, if present.

The formal stability of the adjusted state discussed at the end of § 3 suggests that the adjusted state is an attractor of equation (37) with radiation boundary conditions (i.e. that localized *large-amplitude* perturbations relax toward it as well). However, this hypothesis is yet to be proved.

Note that although we concentrated above on the Cauchy problem, scattering of wave trains of sufficiently small amplitude may be also considered in the framework of (45). The absence of the bound states means that there is no resonant scattering, which corroborates the numerical results of Kuo & Polvani (1999).

6. Nonlinear waves

It is known (Shrira 1981, 1986; Grimshaw *et al.* 1998) that the 1dRSW model admits nonlinear periodic wave solutions with amplitudes bound from above by some limiting value. In this section we show that the demonstration of this fact becomes straightforward in the Lagrangian picture. This was first noticed by Bühler (1993) who discovered the nonlinear waves in the model in this way.† In the adjustment context these nonlinear waves are important in the case of periodic boundary conditions frequently used in numerical simulations. The very fact of their existence means that scenarios of adjustment are different in the closed (circle) and open (whole x-axis) domains in 1dRSW.

Consider equations (32), (33), (34) and look for stationary-wave solutions in the form $u = u(\xi)$, $v = v(\xi)$, $J = J(\xi)$, where $\xi = a - ct$. By eliminating u via $u = (c/f)v'$, where a prime denotes ξ-differentiation in this section, one finds from the equation (34) that $fJ + v' = \text{const}$. From the definition of PV it follows that this constant is equal to QH and, hence, the PV should be constant for stationary waves to exist. The value of Q is thus fixed to be $Q = f/H$. The equation for v resulting from (32) after elimination of u and J is

$$v'' + \frac{f^2}{c^2}v + \frac{gH}{2c^2}f^3\left(\frac{1}{(f-v')^2}\right)' = 0 \tag{51}$$

and may be integrated once after multiplying it by $(c^2/f^2)v'$. The following integral of motion thus results:

$$\mathscr{H} = \frac{1}{2}\left(\frac{c^2}{f^2}v'^2 + v^2 - gH\frac{v'^2}{(f-v')^2}\right) = \text{const.} \tag{52}$$

By using $v' = f(1-J)$ and $fv = c^2J' + gH(1/2J^2)'$ this expression may be rewritten as

$$\mathscr{H} = \frac{1}{2}\left[R_d^2\left[M^2J' + \left(\frac{1}{2J^2}\right)'\right]^2 + M^2(1-J)^2 - \frac{(1-J)^2}{J^2}\right], \tag{53}$$

where we used the same notation for the integral, although it is renormalized by c_0^2 with respect to (52), and introduced the Mach number $M = c/c_0$, where $c_0 = \sqrt{gH}$ and $R_d = c_0/f$. The result may be reduced to the standard 'particle-in-a-well' mechanical problem:

$$\frac{J'^2}{2} + \frac{1}{R_d^2}\frac{V(J)-\mathscr{H}}{(M^2-J^{-3})^2} = 0, \tag{54}$$

where J is the particle 'coordinate', ξ is 'time',

$$\mathscr{U}(\mathscr{J}) = \frac{1}{R_d^2}\frac{V(J)-\mathscr{H}}{(M^2-J^{-3})^2}$$

is a singular 'potential' built from the 'prepotential'

$$V(J) = \frac{(1-J)^2}{2}(M^2 - J^{-2})$$

and a constant $\mathscr{H}$, and the 'particle' always moves on the zero-energy level. The turning points of the particle trajectory are, thus, the zeros of the potential (cf. figure 2) and a stationary-wave solution exists for values of the parameters such that

† We learned about this unpublished result when the present paper was already in preparation.

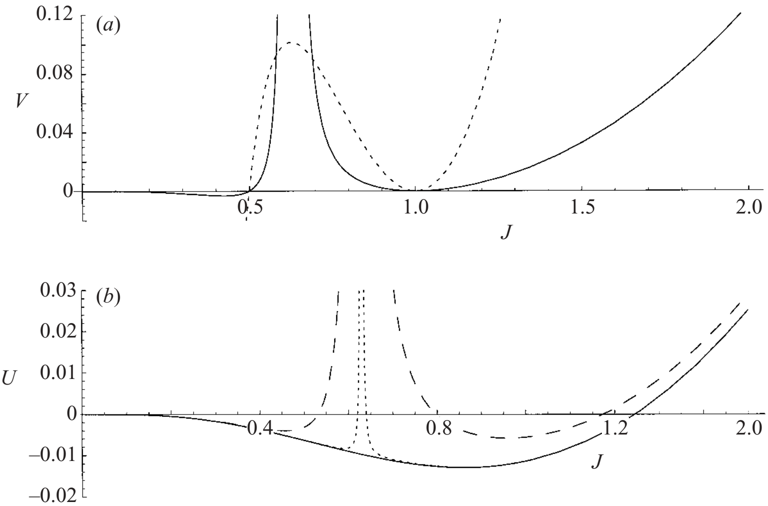

FIGURE 2. (*a*) The 'prepotential' $V(J)$ (dashed line), for $M = 2$, and the full 'potential' $\mathscr{U}(J)$ for $\mathscr{H} = 0$ and $R_d = 1$ (solid line). (*b*) 'Potential' $\mathscr{U}(J)$ for three values of the constant $\mathscr{H}$: the critical value $\mathscr{H}_c = 0.1013\ldots$ (solid), 0.101 (dotted curve), and 0.05 (dashed curve). A nonlinear wave can exist for values of $\mathscr{H}$ such that the potential has two zeros for strictly positive values of J. For $\mathscr{H} = \mathscr{H}_c$ this is no longer the case.

there is a potential well bounded by two positive zeros (J is positive by definition). For positive J the prepotential V has one double ($J = 1$) and one simple ($J = M^{-1}$) zero separated by a local extremum at $J = M^{-2/3}$ which is a maximum in the 'supersonic' ($M > 1$) and a minimum in the 'subsonic' ($M < 1$) case. The position of zeros of the whole potential $\mathscr{U}$ is easy to understand from this structure of V, which should be vertically shifted by $\mathscr{H}$ to give the numerator of $\mathscr{U}$. As $\mathscr{U}$ has a singularity at the local extremum of V there are no regular bounded solutions of the problem (54) in the 'subsonic' case. On the contrary, in the 'supersonic' case the oscillating solutions are possible. Depending on the value of $\mathscr{H}$ their amplitude varies from zero to a maximum value corresponding to the coincidence of a zero and the pole of $\mathscr{U}$. This value, which is reached at $\mathscr{H} = \mathscr{U}(M^{-2/3}) = \frac{1}{2}(M^{2/3} - 1)^3$ would give a cusp in the profile of J (reflection of the particle from the vertical wall). It is, however, unattainable due to mutual cancellation of the zero and the pole at the corresponding value of $\mathscr{H}$ and, thus, represents the asymptotic limit of the stationary-wave amplitudes. Note that for J close to one (54) becomes the harmonic oscillator equation with (spatial) frequency squared equal to

$$k^2 = \frac{1}{R_d^2(M^2 - 1)}. \tag{55}$$

Recalling the definition of M this gives $c^2 = c_0^2(1 + k^{-2}R_d^{-2})$ which is equivalent to the dispersion relation (3) for the inertia–gravity waves. Hence, the stationary nonlinear periodic waves are finite-amplitude analogues of standard infinitesimal inertia–gravity waves – cf. figure 3. However, their amplitudes cannot exceed a limiting value. The deviation of the emitted waves from linearity in the case of strong initial perturbations should manifest itself in the adjustment process, especially in the periodic geometry.

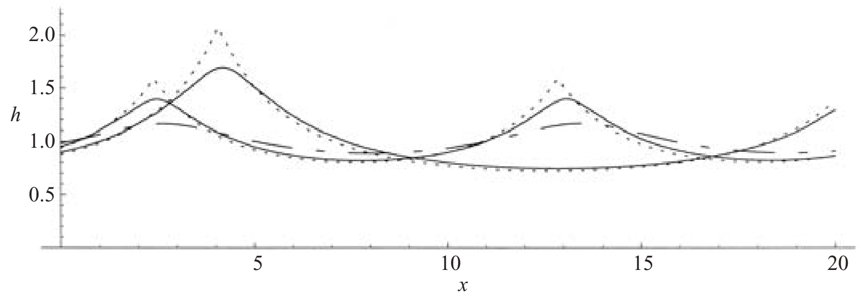

FIGURE 3. Height profiles of stationary nonlinear waves in physical space for various values of M and $\mathscr{H}$, with $R_d = 1$. The waves of shorter wavelength have Mach number $M = 2$, those with longer wavelength have $M = 3$. The two limiting asymptotics ($\mathscr{H} = \mathscr{H}_c$) are the dotted curves; the solid lines correspond to $\mathscr{H} = 0.9\mathscr{H}_c$ in both cases. Finally, for $M = 2$, the wave corresponding to $\mathscr{H} = 0.5\mathscr{H}_c$ is also shown (dash-dotted curve).

7. Wave breaking and shocks in Lagrangian variables

In order to analyse shock formation in Lagrangian variables we use Lax's method (Lax 1973), following Engelberg (1996).

Let us rewrite the Lagrangian equations of motion in the a-variables as a system of two equations (to avoid cumbersome formulae, all dimensional parameters are taken to be equal to unity in this section; correct dimensions are easy to recover)

$$\dot{u} + \partial_a p = v, \quad \dot{J} - \partial_a u = 0, \tag{56}$$

where v is not an independent variable and should be determined from the relation $\partial_a v = Q(a) - J$.

This is a quasi-linear system

$$\begin{pmatrix} \dot{u} \\ \dot{J} \end{pmatrix} + \begin{pmatrix} 0 & -J^{-3} \\ -1 & 0 \end{pmatrix} \partial_a \begin{pmatrix} u \\ J \end{pmatrix} = \begin{pmatrix} v \\ 0 \end{pmatrix}. \tag{57}$$

The eigenvalues of the matrix on the left-hand side are $\mu_\pm = \pm J^{-3/2}$ and the corresponding left eigenvectors are $(1, \pm J^{-3/2})$. Hence, the Riemann invariants are $w_\pm = u \pm 2J^{-1/2}$ and we have

$$\dot{w}_\pm + \mu_\pm \partial_a w_\pm = v. \tag{58}$$

Expressions for the original variables in terms of $w_\pm$ are

$$u = \tfrac{1}{2}(w_+ + w_-), \tag{59}$$

$$J = \frac{16}{(w_+ - w_-)^2} > 0, \tag{60}$$

$$\mu_\pm = \pm \left(\frac{w_+ - w_-}{4} \right)^3. \tag{61}$$

In terms of the derivatives of the Riemann invariants $r_\pm = \partial_a w_\pm$ we obtain

$$\dot{r}_\pm + \mu_\pm \partial_a r_\pm + \frac{\partial \mu_\pm}{\partial w_+} r_+ r_\pm + \frac{\partial \mu_\pm}{\partial w_-} r_- r_\pm = \partial_a v = Q(a) - J\,, \tag{62}$$

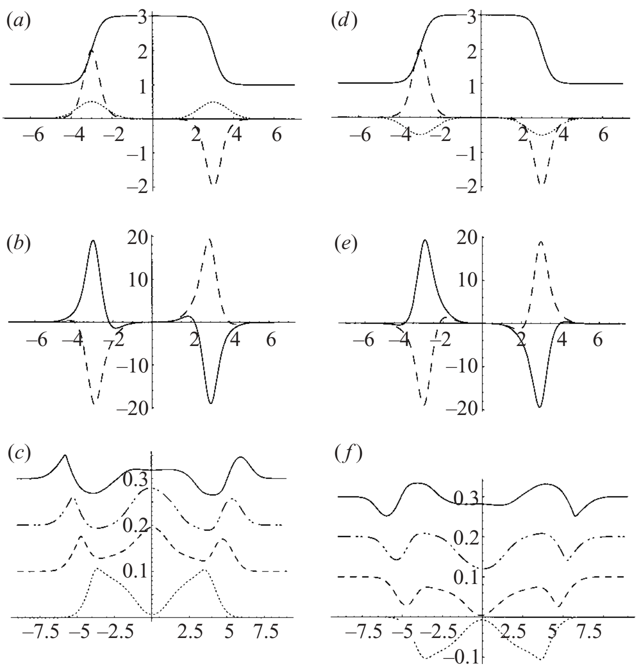

FIGURE 4. Shock formation in two reversed double-jet configurations given in the left and the right column, respectively, calculated by MATHEMATICA integration of equation (13). (*a*, *d*) Two sets of initial conditions: h_I (solid) and v_I (dashed) are always in geostrophic equilibrium, and the perturbation is entirely provided by u_I (dotted). (*b*, *e*) The corresponding initial profiles of R_+ (solid) and R_- (dashed). According to the results of § 7, shock formation, i.e. the appearance of a singularity in $R_\pm$, is favoured in the regions with sufficiently negative relative vorticity and/or sufficiently negative $R_\pm$. The singularities in R_+ and R_- propagate rightward and leftward, respectively, in x. In the case shown, shocks are expected to appear in the regions of negative shear ([−3, −2] and [2, 3]) and originate from regions with negative initial $R_\pm$ (around $x = -3$ for R_- and $x = +3$ for R_+); from the shift of the maxima of $R_\pm$ due to the contribution of u, one can see that the leftward propagating shock should be favoured in the first case and the rightward one in the second case. (*c*, *f*) The Lagrangian particle displacements $\phi(x, t)$ at successive times $t = 0.5, 1, 1.5, 2$ (dotted, dashed, dot-dashed and solid lines, respectively); for readability, each curve has been displaced by 0.1 upward relative to the previous one. A shock is a cusp in ϕ. One can see the formation of cusps appearing at the expected locations and moving in the expected direction on both figures.

which may be rewritten, using 'double Lagrangian' derivatives along the characteristics $\mathrm{d}r/\mathrm{d}t_\pm = \dot{r} + \mu_\pm \partial_a r$, as

$$\frac{\mathrm{d}r_\pm}{\mathrm{d}t_\pm} + \frac{\partial \mu_\pm}{\partial w_+} r_+ r_\pm + \frac{\partial \mu_\pm}{\partial w_-} r_- r_\pm = Q(a) - J. \tag{63}$$

Wave breaking and shock formation correspond to $r_\pm$ reaching the limit $\pm\infty$ in finite time.

Introducing the new variables $R_\pm = \mathrm{e}^{\lambda} r_\pm$, with $\lambda = (3/2)\log|w_+ - w_-|$, we rewrite equations (62) in the following form:

$$\frac{\mathrm{d}R_\pm}{\mathrm{d}t_\pm} = -\mathrm{e}^{-\lambda}\frac{\partial \mu_\pm}{\partial w_\pm} R_\pm^2 + \mathrm{e}^{\lambda}(Q(a) - J), \tag{64}$$

where $\partial \mu_\pm/\partial w_\pm = (3/64)(w_+ - w_-)^2 > 0$.

These are generalized Ricatti equations for each characteristic. The qualitative analysis of such equations (see Lemmas 1 and 2 in Engelberg 1996) gives that:

(i) if the initial relative vorticity $Q - J = \partial_a v$ is sufficiently negative breaking always takes place whatever initial conditions are;

(ii) if the relative vorticity is positive, as well as the derivatives of the Riemann invariants at the initial moment, breaking never takes place.

Hence, in the adjustment context, shocks should be more easy to produce on the anticyclonic (negative relative vorticity) side of the jet. As one of the Riemann invariants always has a negative derivative for the stepwise height profiles, shocks should always be produced by the pure height (no v_I) adjustment, as was observed in numerical simulations (Kuo & Polvani 1997; there, in addition, the initial height profile was itself discontinuous). It seems that shock production in the course of adjustment of front-like perturbations is unavoidable, because these latter never satisfy conditions of Lemma 2 in (Engelberg 1996). Test simulations done with standard MATHEMATICA routines support these conclusions – see figure 4. Although the Lagrangian variables allow one to obtain the shock-formation criteria relatively easily, it is more convenient to formulate the Rankine–Hugoniot conditions and to proceed with full-scale shock-resolving numerical simulations of adjustment in the Eulerian coordinates. This work is in progress and will be presented elsewhere (Bouchut, Le Sommer & Zeitlin 2003).

8. Summary and discussion

By using the Lagrangian coordinates in the 1dRSW equations we have

(i) demonstrated existence and uniqueness of the non-perturbative slow manifold for front-like configurations with non-negative initial PVs;

(ii) proved that trapped states are impossible in front-like disturbances, but that quasi-stationary states may exist implying, in general, more complex than simple dispersion relaxation to the adjusted state;

(iii) analysed finite-amplitude periodic nonlinear waves existing in the model;

(iv) established semi-quantitative criteria for the shock formation.

Our analysis confirms, therefore, the Rossby scenario of adjustment for localized fronts but with a restriction to non-negative initial PV distributions and small enough initial departures from the balanced state. The rate of relaxation toward the adjusted state depends on the fine structure of the latter and its capacity to support the quasi-stationary states. Shocks do appear during the adjustment process but they do not change the basic scenario as long as they are formed out of the front and are outgoing. We cannot exclude at the present stage that very strong initial imbalances may modify the classical scenario even for positive initial PVs by producing either some sort of self-sustained nonlinear oscillations or shocks at the front location.

Probably, the most interesting question is what happens when the initial PV is negative in some region and no proof of existence of the iso-PV balanced state is available. This is a *de facto* strongly nonlinear situation and, as well as the strongly nonlinear positive-PV case should be investigated numerically. It should be noted that as wave breaking and shock formation with subsequent dissipative effects is the only way for the PV-less wave component of the flow to affect the PV-bearing vortex part, shocks should be resolved with extreme care in numerical simulations. The comparison of reliability of existing Riemann solvers with proper inclusion of rotation is a subject of a related work which is in progress now and will be presented elsewhere.

V. Z. is grateful to O. Bühler for helpful discussions and for providing his unpublished results. Part of this work was done when V. Z. was visiting the Plasma Division of the Courant Institute at NYU. Numerous stimulating discussions with G. Zaslavsky and his kind hospitality at Courant as well as a partial financial support by a US Navy grant are gratefully acknowledged. S. B. M. was supported by RFBR grant No 01-01-00959. R. P. acknowledges support from ACI 'jeunes chercheurs' #0693 of CNRS.

Appendix A. Existence and uniqueness proof for the adjusted state

The dimensionless equation (40) is written as

$$\frac{\mathrm{d}^2 h}{\mathrm{d}x^2} - Q(x)h = -1. \tag{A 1}$$

We assume that

$$\left.\begin{array}{l} Q(x) > 0, \\ Q(\pm\infty) = Q_\pm = \text{const}, \\ Q'(x) \text{ has a finite support.} \end{array}\right\} \tag{A 2}$$

A general solution of (A 1) has the following form:

$$h(x) = C_1 h_1(x) + C_2 h_2(x) + h_1(x)\int_{x_0}^{x} \frac{h_2(t)}{W}\,\mathrm{d}t - h_2(x)\int_{x_0}^{x} \frac{h_1(t)}{W}\,\mathrm{d}t, \tag{A 3}$$

where $h_1(x)$, $h_2(x)$ is a fundamental set of solutions for (A 1) without the right-hand side, and the Wronskian $W = h_1 h_2' - h_1' h_2$ is constant.

We define the two fundamental sets of solutions $\psi_{1,2}(x)$ and $\varphi_{1,2}(x)$ using the asymptotic conditions at infinity. The first set is

$$\psi_1(x) = Q_+^{-1/4}\exp\left\{-Q_+^{1/2}x\right\} + o(1), \quad \psi_2(x) = Q_+^{-1/4}\exp\left\{+Q_+^{1/2}x\right\} + o(1), \quad x \to +\infty, \tag{A 4}$$

and the second set is

$$\varphi_1(x) = Q_-^{-1/4}\exp\left\{-Q_-^{1/2}x\right\} + o(1), \quad \varphi_2(x) = Q_-^{-1/4}\exp\left\{+Q_-^{1/2}x\right\} + o(1), \quad x \to -\infty. \tag{A 5}$$

Using the first set of solutions and choosing a point $x_0 = x_2$ on the right from the support we have

$$h(x) = C_1\psi_1(x) + C_2\psi_2(x) + \tfrac{1}{2}\psi_1(x)\int_{x_2}^{x} \psi_2(t)\,\mathrm{d}t - \tfrac{1}{2}\psi_2(x)\int_{x_2}^{x} \psi_1(t)\,\mathrm{d}t \tag{A 6}$$

(here $W = \psi_1\psi_2' - \psi_1'\psi_2 = 2$) and at $x \to +\infty$ we obtain

$$h(x) = \frac{1}{Q_+} + \exp\left(-xQ_+^{1/2}\right)\left(\frac{C_1}{Q_+^{1/4}} - \frac{\exp\left(+x_2 Q_+^{1/2}\right)}{2Q_+}\right) + \exp\left(+xQ_+^{1/2}\right)\left(\frac{C_2}{Q_+^{1/4}} - \frac{\exp\left(-x_2 Q_+^{1/2}\right)}{2Q_+}\right). \tag{A 7}$$

To eliminate the growing part of the solution one should put

$$C_2 = \frac{\exp\left(-x_2 Q_+^{1/2}\right)}{2Q_+^{3/4}}. \tag{A 8}$$

The first set of solutions is a linear combination of the second set:

$$\psi_1(x) = T_{11}\varphi_1(x) + T_{12}\varphi_2(x), \tag{A 9}$$

$$\psi_2(x) = T_{21}\varphi_1(x) + T_{22}\varphi_2(x), \tag{A 10}$$

where $\boldsymbol{T}$ is the transition matrix. From the conservation of the Wronskian ($\varphi_1\varphi_2' - \varphi_1'\varphi_2 = 2$) one obtains

$$T_{11}T_{22} - T_{12}T_{21} = 1. \tag{A 11}$$

Substituting (A 9) and (A 10) into (A 6) we obtain another representation of the solution:

$$\begin{aligned} h(x) = {} & C_1(T_{11}\varphi_1(x) + T_{12}\varphi_2(x)) + C_2(T_{21}\varphi_1(x) + T_{22}\varphi_2(x)) \\ & - \tfrac{1}{2}(T_{11}\varphi_1(x) + T_{12}\varphi_2(x)) \int_x^{x_2} (T_{21}\varphi_1(t) + T_{22}\varphi_2(t))\,\mathrm{d}t \\ & + \tfrac{1}{2}(T_{21}\varphi_1(x) + T_{22}\varphi_2(x)) \int_x^{x_2} (T_{11}\varphi_1(t) + T_{12}\varphi_2(t))\,\mathrm{d}t. \end{aligned} \tag{A 12}$$

For the negative infinity asymptotics we choose a limiting point $x_1 < x_2$ on the left from the support. Then we obtain for $x \leqslant x_1$

$$\begin{aligned} h(x) = {} & C_1 \left(\frac{T_{11}}{Q_-^{1/4}} \exp\left(-x Q_-^{1/2}\right) + \frac{T_{12}}{Q_-^{1/4}} \exp\left(+x Q_-^{1/2}\right) \right) \\ & + C_2 \left(\frac{T_{21}}{Q_-^{1/4}} \exp\left(-x Q_-^{1/2}\right) + \frac{T_{22}}{Q_-^{1/4}} \exp\left(+x Q_-^{1/2}\right) \right) \\ & - \frac{1}{2} \left(\frac{T_{11}}{Q_-^{1/4}} \exp\left(-x Q_-^{1/2}\right) + \frac{T_{12}}{Q_-^{1/4}} \exp\left(+x Q_-^{1/2}\right) \right) \int_{x_1}^{x_2} (T_{21}\varphi_1(t) + T_{22}\varphi_2(t))\,\mathrm{d}t \\ & - \frac{1}{2} \left(\frac{T_{11}}{Q_-^{1/4}} \exp\left(-x Q_-^{1/2}\right) + \frac{T_{12}}{Q_-^{1/4}} \exp\left(+x Q_-^{1/2}\right) \right) \\ & \times \int_x^{x_1} \left(\frac{T_{21}}{Q_-^{1/4}} \exp\left(-t Q_-^{1/2}\right) + \frac{T_{22}}{Q_-^{1/4}} \exp\left(+t Q_-^{1/2}\right) \right) \mathrm{d}t \\ & + \frac{1}{2} \left(\frac{T_{21}}{Q_-^{1/4}} \exp\left(-x Q_-^{1/2}\right) + \frac{T_{22}}{Q_-^{1/4}} \exp\left(+x Q_-^{1/2}\right) \right) \int_{x_1}^{x_2} (T_{11}\varphi_1(t) + T_{12}\varphi_2(t))\,\mathrm{d}t \\ & + \frac{1}{2} \left(\frac{T_{21}}{Q_-^{1/4}} \exp\left(-x Q_-^{1/2}\right) + \frac{T_{22}}{Q_-^{1/4}} \exp\left(+x Q_-^{1/2}\right) \right) \\ & \times \int_x^{x_1} \left(\frac{T_{11}}{Q_-^{1/4}} \exp\left(-t Q_-^{1/2}\right) + \frac{T_{12}}{Q_-^{1/4}} \exp\left(+t Q_-^{1/2}\right) \right) \mathrm{d}t. \end{aligned} \tag{A 13}$$

After a simplification we obtain

$$\begin{aligned} h(x) = {} & \frac{1}{Q_-} + \frac{2C_1 Q_-^{3/4} T_{11} + 2C_2 Q_-^{3/4} T_{21} - \exp\left(x_1 Q_-^{1/2}\right) - D_2 Q_-^{3/4}}{2Q_-} \exp\left(-x Q_-^{1/2}\right) \\ & + \frac{2C_1 Q_-^{3/4} T_{12} + 2C_2 Q_-^{3/4} T_{22} - \exp\left(-x_1 Q_-^{1/2}\right) + D_1 Q_-^{3/4}}{2Q_-} \exp\left(x Q_-^{1/2}\right), \end{aligned} \tag{A 14}$$

where

$$D_1 = \int_{x_1}^{x_2} \varphi_1(t)\,\mathrm{d}t, \quad D_2 = \int_{x_1}^{x_2} \varphi_2(t)\,\mathrm{d}t. \tag{A 15}$$

The solution has finite asymptotics if and only if

$$2C_1 Q_-^{3/4} T_{11} + 2C_2 Q_-^{3/4} T_{21} - \exp\left(x_1 Q_-^{1/2}\right) - D_2 Q_-^{3/4} = 0. \tag{A 16}$$

If $T_{11} \neq 0$ then C_1 can be found from (A 16). Now we prove that $T_{11} \neq 0$ for any solution. Assume that $T_{11} = 0$ then from (A 11) we get that $T_{12} \neq 0$. Therefore we have

$$\psi_1(x) = T_{12}\varphi_2(x). \tag{A 17}$$

From (A 5) we have that

$$\psi_1(x) = T_{12} Q_-^{-1/4} \exp\left\{+Q_-^{1/2} x\right\} + o(1), \quad x \to -\infty.$$

From (A 4) we find that $\psi_1(x)$ is an integrable function. Multiplying the homogeneous equation (A 1) by $\psi_1(x)$ for $h(x) = \psi_1(x)$ and integrating we find that

$$\int \psi_1(x) \frac{\mathrm{d}^2\psi_1(x)}{\mathrm{d}x^2}\,\mathrm{d}x - \int Q(x)\psi_1^2(x)\,\mathrm{d}x = 0. \tag{A 18}$$

By integration by parts in the first integral we obtain a negative value on the left-hand side. This value is equal to zero if and only if $\psi_1(x) \equiv 0$. But we assumed that $\psi_1(x)$ is a non-trivial solution. Hence, we proved the following:

PROPOSITION 1. *The equation* (A 1) *has a bounded solution* $h(x)$ *for positive* $Q(x)$ *which is unique for the boundary conditions* $h(x) = Q_\pm$ *as* $x \to \pm\infty$.

A.1. *Positiveness of solution*

The result that any solution of (A 1) without the right-hand side can have either no or a single zero is well-known from the Sturm theorems (cf. e.g. Hartman 1964). Now we prove this theorem for the full (A 1). Assume that $h(x_1) = h(x_2) = 0$ and $h(x) < 0$ for $x_1 < x < x_2$. Integrating (A 1) from x_1 to x_2 we obtain

$$\frac{\mathrm{d}h}{\mathrm{d}x}(x_2) - \frac{\mathrm{d}h}{\mathrm{d}x}(x_1) = \int_{x_1}^{x_2} Q(x)h(x)\,\mathrm{d}x - (x_2 - x_1). \tag{A 19}$$

Obviously, the left-hand side is positive and the right-hand side is negative. As a result we have a contradiction. Therefore there are no zeros of this kind.

PROPOSITION 2. *Any solution* $h(x)$ *of the equation* $-h''(x) + Q(x)h(x) = 1$ *has either one zero or no zeros at all for positive* $Q(x)$.

We first prove that a non-negative solution $h(x)$ cannot have a singular zero x_0: $h(x_0) = 0$, $h'(x_0) = 0$. Assume that $h(x_0) = 0$ and $h'(x_0) = 0$. If $h(x)$ is non-negative then $h''(x_0) \geqslant 0$. But from (A 1) we have that $h''(x_0) = -1$ and we arrive to a contradiction.

PROPOSITION 3. *Non-negative solutions* $h(x)$ *have no singular zeros such that* $h(x) = 0$ *and* $h'(x) = 0$.

From Propositions 1, 2, 3 we obtain the following:

PROPOSITION 4. *A solution* $h(x)$ *with asymptotics* $h(\pm\infty) = 1/Q_\pm > 0$ *is a positive function of* x $(h(x) > 0)$.

Proof. As proved, $h(x)$ can have 0 or 1 zeros. If $h(x)$ has no zeros then $h(x) > 0$ because $h(\pm\infty) > 0$. If $h(x)$ has one zero at the point x_0 then $h(x_0) = 0$ and $h'(x_0) = 0$, which contradicts Proposition 3.

Summarizing we formulate the main theorem.

THEOREM. *Under assumptions* (A 2) *equation* (A 1) *has a unique bounded and positive solution* $h(x)$.

The same proof, save positiveness which is inessential there, can be used for (23).

Appendix B. Lagrangian approach to the axisymmetric 1dRSW

Axisymmetric motion in shallow water is described by fields depending on only one space variable: r, the distance to the centre. As in the rectilinear case, it is possible to reduce the whole dynamics to a single PDE for a Lagrangian variable, $R(r,t)$, the distance to the centre of a particle initially situated at r.

We first rewrite the RSW equations in the Eulerian framework, using cylindrical coordinates (r,θ) and assuming axisymmetry ($\partial_\theta \equiv 0$):

$$(\partial_t + u_r\,\partial_r)u_r - u_\theta\left(f + \frac{u_\theta}{r}\right) + \partial_r h = 0, \tag{B 1a}$$

$$(\partial_t + u_r\,\partial_r)u_\theta + u_r\left(f + \frac{u_\theta}{r}\right) = 0, \tag{B 1b}$$

$$(\partial_t + u_r\,\partial_r)h + \frac{1}{r}\partial_r(r\,u_r\,h) = 0. \tag{B 1c}$$

Here u_r (u_θ) is the radial (azimuthal) velocity, and h the total height of the fluid. Multiplying (B 1b) by r, we recover the conservation of angular momentum:

$$(\partial_t + u_r\,\partial_r)\left(ru_\theta + f\frac{r^2}{2}\right) = 0. \tag{B 2}$$

Equations (B 1) can be rewritten using the Lagrangian coordinate for the radial position: $R(r,t)$ is the radial position at time t of the particle that was at r initially. Note that r changes its meaning, from here on, becoming a Lagrangian label instead of an Eulerian coordinate.

Equation (B 2) may be immediately integrated:

$$R(r,t)\,u_\theta(r,t) + f\frac{R^2(r,t)}{2} = G(r)\,. \tag{B 3}$$

$G(r)$ is determined from initial conditions: if $u_{\theta I}$ is the initial azimuthal velocity profile, then

$$G(r) = r\,u_{\theta I}(r) + f\frac{r^2}{2}. \tag{B 4}$$

Using the above expression, the term $u_\theta\,(f + u_\theta/r)$ in (B 1a) is expressed in terms of $R(r,t)$ and G:

$$\begin{aligned} u_\theta\left(f + \frac{u_\theta}{R}\right) &= \frac{1}{R}\left(G - f\frac{R^2}{2}\right)\left(f + \frac{G}{R^2} - \frac{f}{2}\right), \\ &= \frac{1}{R^3}\left(G^2 - \frac{f^2R^4}{4}\right). \end{aligned} \tag{B 5}$$

Mass conservation is expressed by the following relation between $h(r, t)$ and the initial height profile $h_I(r)$:

$$h(r, t)\, R(r, t)\, \mathrm{d}R = h_I(r)\, r\, \mathrm{d}r. \tag{B 6}$$

With the help of (B 5) and (B 6), the radial momentum equation becomes

$$\ddot{R} + \frac{f^2}{4} R - \frac{1}{R^3} G^2 + \frac{1}{\partial_r R} \partial_r \left(\frac{a\, h_I}{R\, \partial_r R} \right) = 0, \tag{B 7}$$

where $\dot{R}(r, t) = u_r(r, t)$. The form of this equation is similar to the one found in the rectilinear case; the advantages of the Lagrangian formulation apply here as well.

It seems odd at first glance that the second term is $f^2 R/4$, and not simply $f^2 R$. The usual rôle of the inertial frequency f is recovered when we linearize the equation for small disturbances. For instance, for small perturbations about the rest state

$$R(r, t) = r + \phi(r, t), \tag{B 8}$$

with $|\phi| \ll r$, $h_I(r) = 1$ and $u_{\theta I}(r) = 0$ the following equation is obtained after some algebra:

$$\ddot{\phi} + f^2 \phi - \frac{\partial_r \phi}{r} - \partial^2_{rr} \phi + \frac{\phi}{r^2} = 0. \tag{B 9}$$

If solutions are sought in the form $\phi(r, t) = \hat{\phi}(r)\, \mathrm{e}^{\mathrm{i}\omega t}$, equation (B 9) yields, after a change of variables, the canonical equation for the Bessel functions. The familiar axisymmetric solutions involving Bessel functions J_1 then follow (cf. e.g. Landau & Lifshitz 1975, on axisymmetric sound waves):

$$\phi(r, t) = C\, J_1(\sqrt{\omega^2 - f^2}\, r)\, \mathrm{e}^{\mathrm{i}\omega t} + \text{c.c.}, \tag{B 10}$$

where C is the wave amplitude.

REFERENCES

Blumen, W. 1972 Geostrophic adjustment. *Rev. Geophys. Space Phys.* **10**, 485–528.

Bouchut, F., Le Sommer, J. & Zeitlin, V. 2003 Frontal geostrophic adjustment, slow manifold and nonlinear wave phenomena in one-dimensional rotating shallow water. Part 2. Numerical simulations. In preparation.

Bühler, O. 1993 A nonlinear wave in rotating shallow water. GFD Summer School preprint (Woods Hole) – unpublished.

Engelberg, S. 1996 Formation of singularities in the Euler and Euler–Poisson equations. *Physica* D **98**, 67–74.

Ford, R., McIntyre, M. E. & Norton, W. A. 2000 Balance and the slow quasimanifold: some explicit results. *J. Atmos. Sci.* **57**, 1236–1254.

Ford, R., McIntyre, M. E. & Norton, W. A. 2000 Reply to Saujani and Shepherd. *J. Atmos. Sci.* **59**, 2878–2882.

Gill, A. E. 1982 *Atmosphere – Ocean Dynamics*, Chap. 7.2. Academic.

Grimshaw, R. H. G., Ostrovsky, L. A., Shrira, V. I. & Stepanyants, Yu. A. 1998 Long nonlinear surface and internal gravity waves in a rotating ocean. *Surveys in Geophys* **19**, 289–338.

Hartman, P. 1964 *Ordinary Differential Equations.* Wiley & Sons.

Holton, J. R. 1979 *An Introduction to Dynamic Meteorology.* Academic.

Kuo, A. C. & Polvani, L. M. 1997 Time-dependent fully nonlinear geostrophic adjustment. *J. Phys. Oceanogr.* **27**, 1614–1634.

Kuo, A. C. & Polvani, L. M. 1999 Wave-vortex interactions in rotating shallow water. Part 1. One space dimension. *J. Fluid Mech.* **394**, 1–27.

Landau, L. D. & Lifshitz, E. M. 1975 *Hydrodynamics.* Academic.

Lax, P. D. 1973 *Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves*. SIAM.

Leith, C. E. 1980 Nonlinear normal mode initialization and quasi-geostrophic theory. *J. Atmos. Sci.* **37**, 958–968.

Migdal, A. B. 1977 Qualitative methods in quantum theory. In *Frontiers in Physics*, vol. 48, Reading, MA.

Obukhov, A. M. 1949 On the problem of geostrophic wind. *Izv. Geog. Geophys.* **13**, 281–306 (in Russian).

Pedlosky, J. 1984 *Geophysical Fluid Dynamics*. Springer.

Reznik, G. M., Zeitlin, V. & Ben Jelloul, M. 2001 Nonlinear theory of geostrophic adjustment. Part I. Rotating shallow water. *J. Fluid Mech.* **445**, 93–120.

Rossby, C.-G. 1938 On the mutual adjustment of pressure and velocity distributions in certain simple current systems, II. *J. Mar. Res.* **1**, 239–263.

Rozdestvenskii, B. L. & Janenko, N. N. 1978 *Systems of Quasilinear Equations and their Applications to Gas Dynamics*. AMS Translation of Mathematical Monographs, vol. 55, Providence, RI.

Saujani, S. & Shepherd, T. G. 2002 Comments on "Balance and the slow quasimanifold. Some explicit results", by Ford, McIntyre and Norton. *J. Atmos. Sci.* **59**, 2874–2877.

Shrira, V. I. 1981 Propagation of long nonlinear waves in the layer of rotating fluid. *Sov. Phys. Izv. Atm. Ocean Phys.* **17**(1), 55–59.

Shrira, V. I. 1986 On the long strongly nonlinear waves in rotating ocean. *Sov. Phys. Izve. Atm. Ocean Phys.* **22**(4), 298–305.

Whitham, G. B. 1974 *Linear and Nonlinear Waves*. Wiley.