

Spectral Analysis of Stochastic Processes

Theory

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E2C2 Summer School, Comorova, September 2007

Outline

Spectral Analysis of
Stochastic Processes

Hening Rust

Stochastic Processes

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Spectral Analysis

Spectral Analysis

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Some Stationary Stochastic Processes

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Parameter Estimation

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The FFF E2C2 Summer Cup

Appendix

Lecture notes in shared folder “Spectral Analysis”

Spectral Analysis

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Spectral Analysis

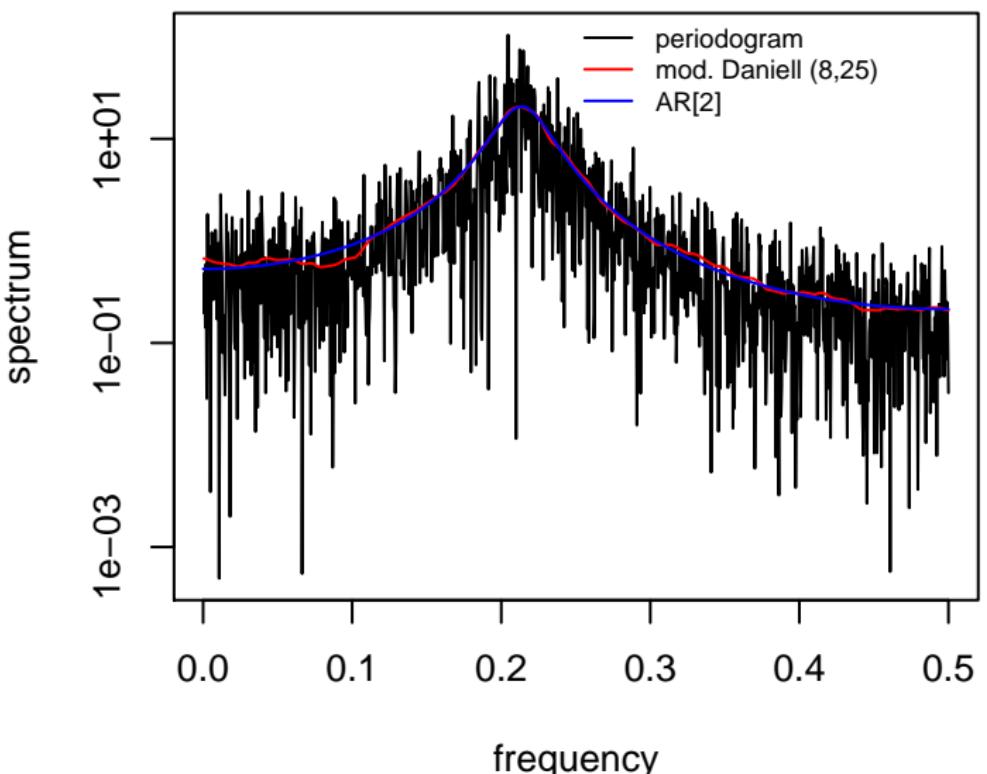
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White Noise

AR 1

AR 2 ,Oscillating

AR 2, Relaxing

Fractional Differenced

FARIMA 1,d,0

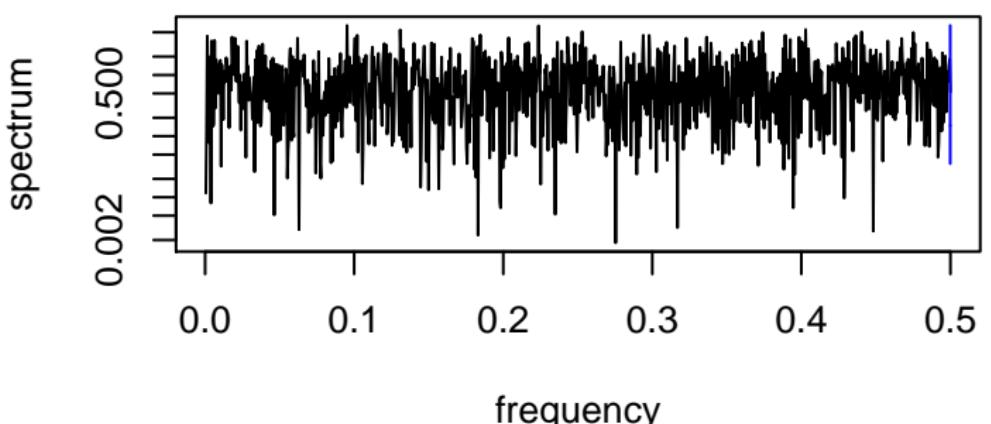
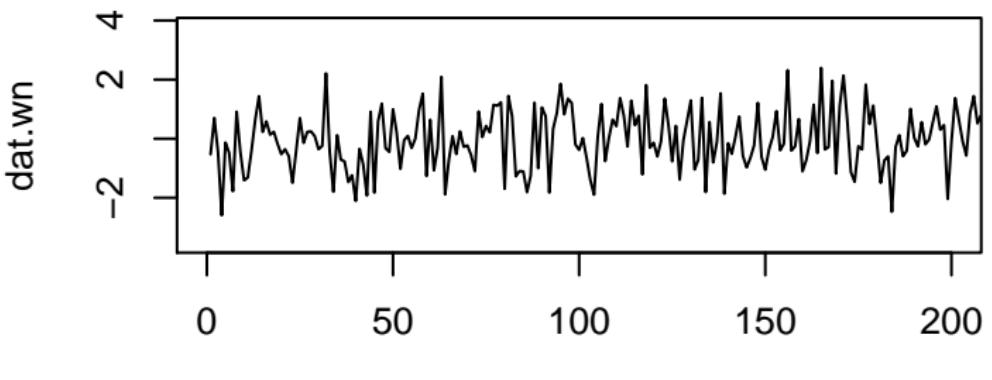
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White Noise



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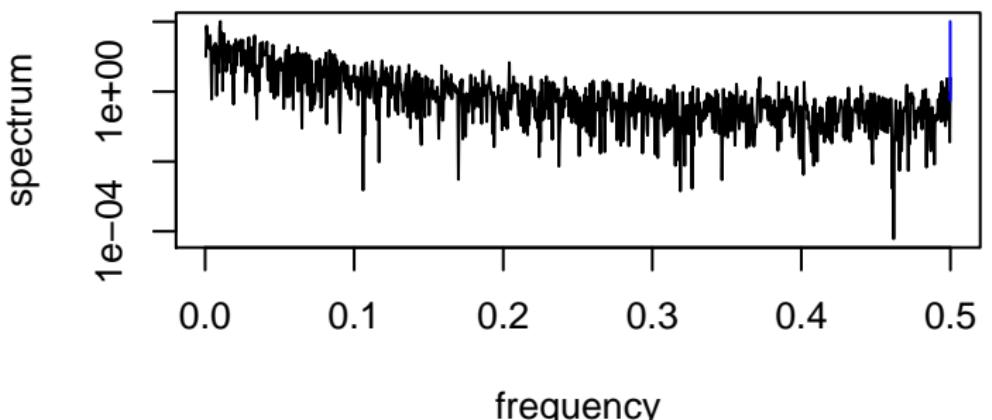
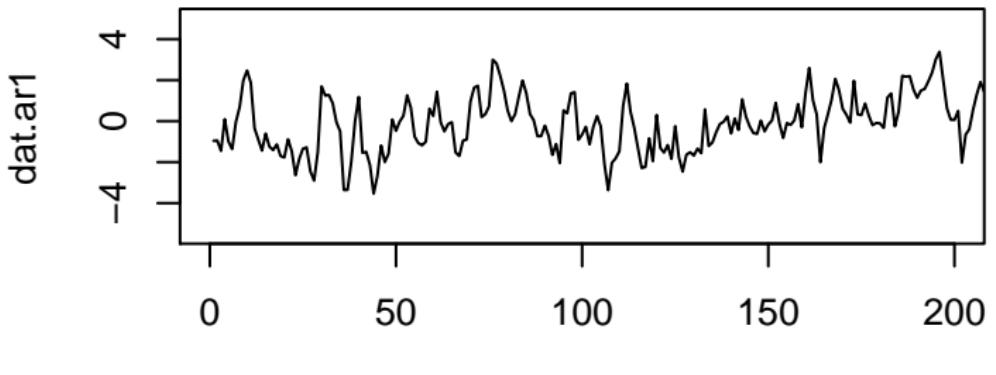
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AR[1]



AR[2], Oscillating

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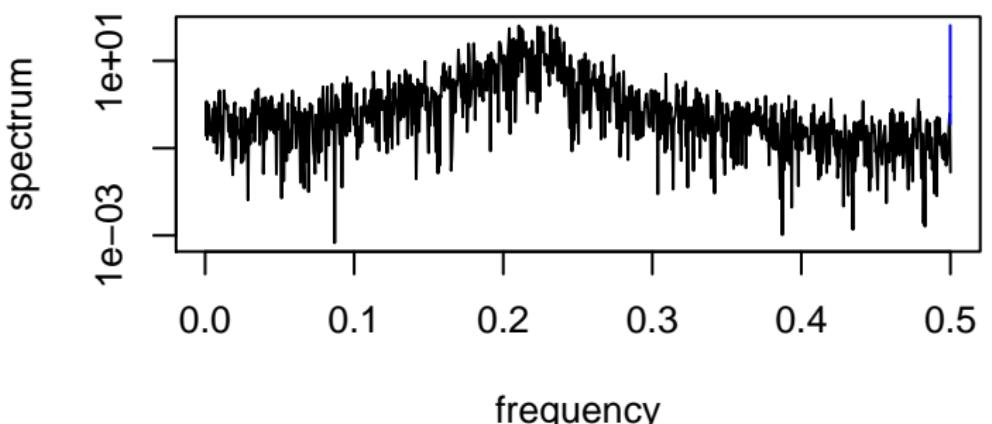
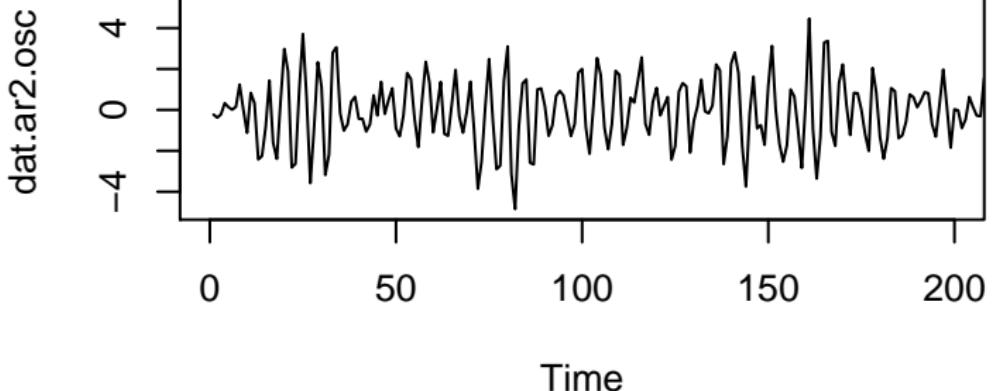
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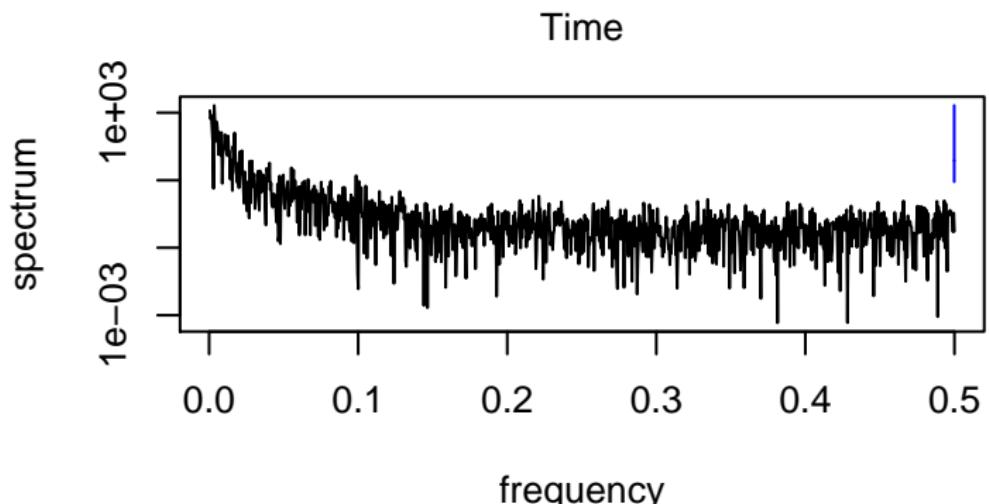
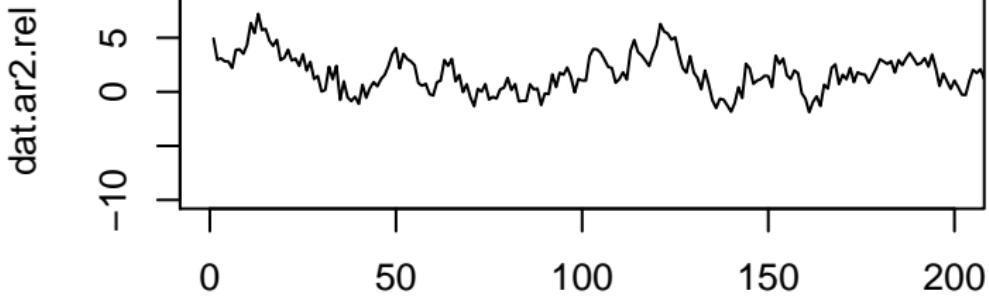
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AR[2], Relaxing



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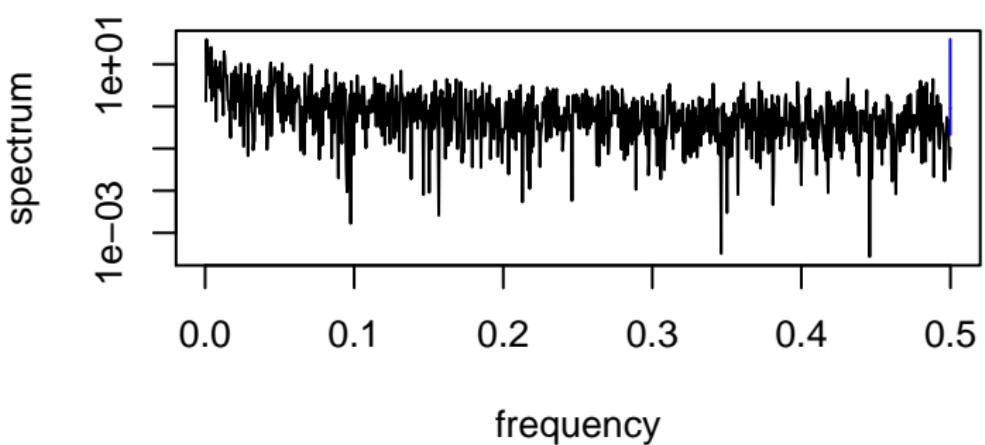
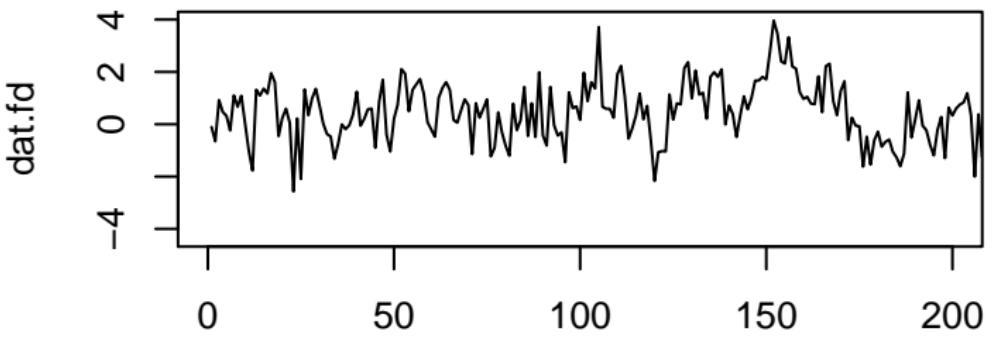
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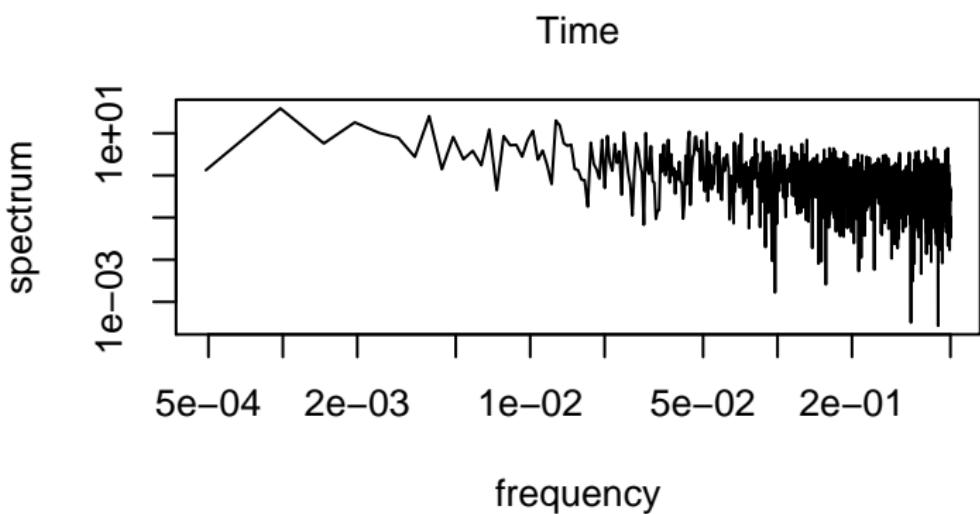
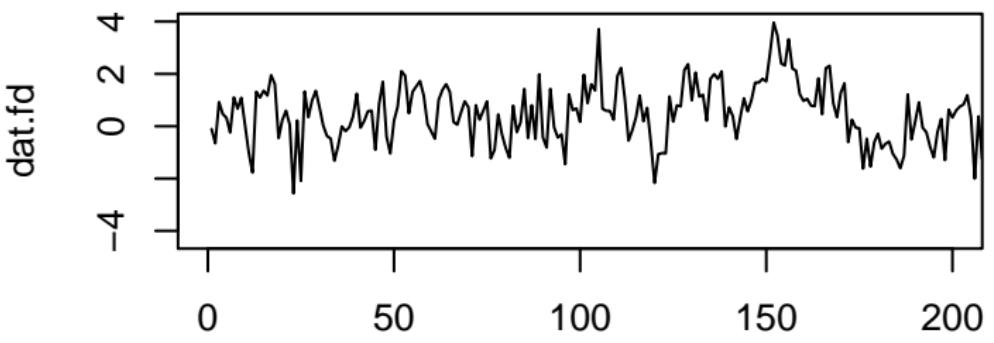
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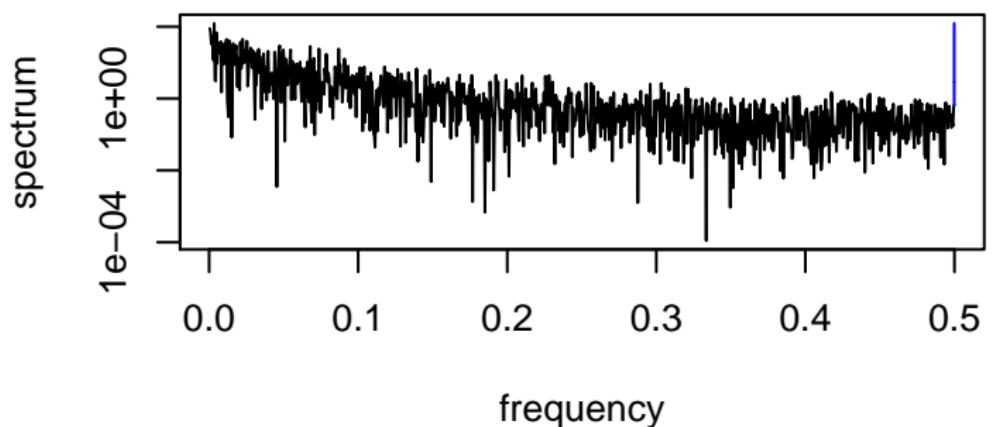
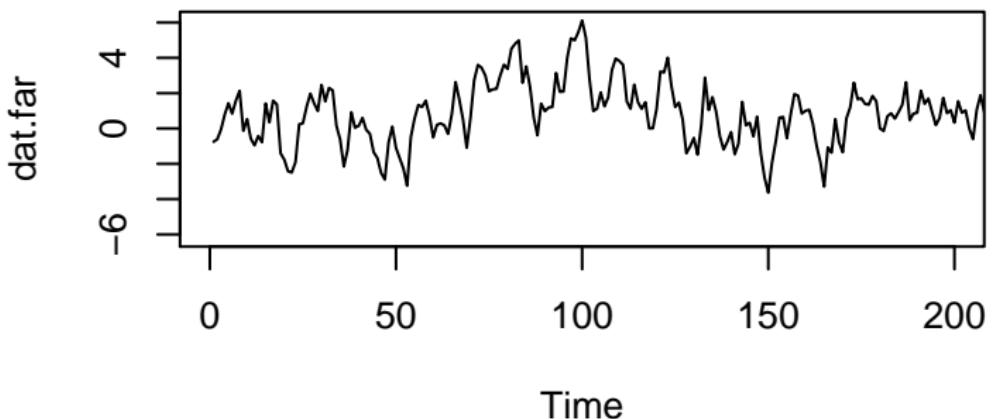
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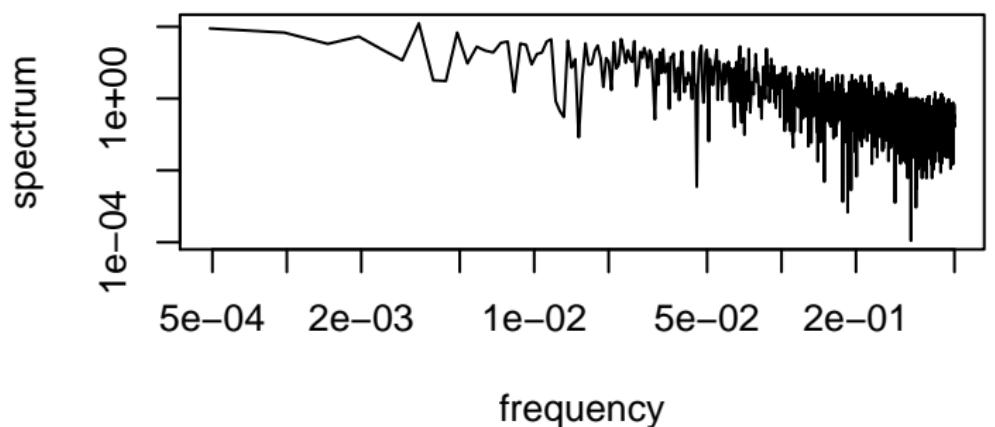
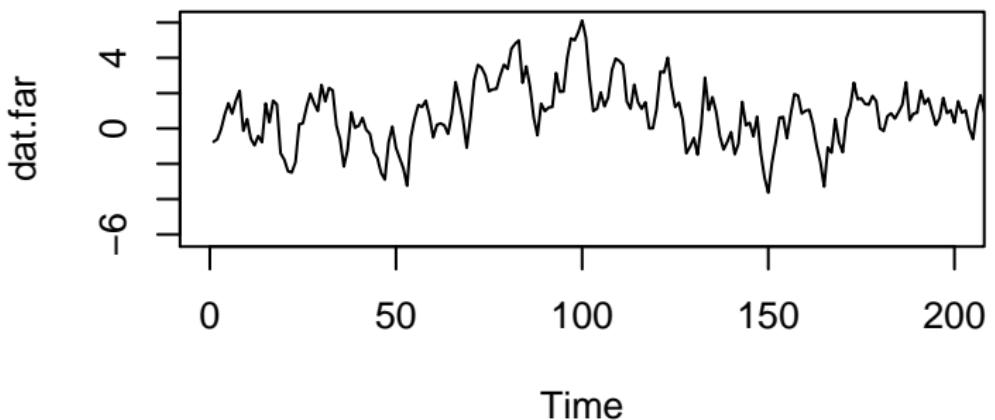
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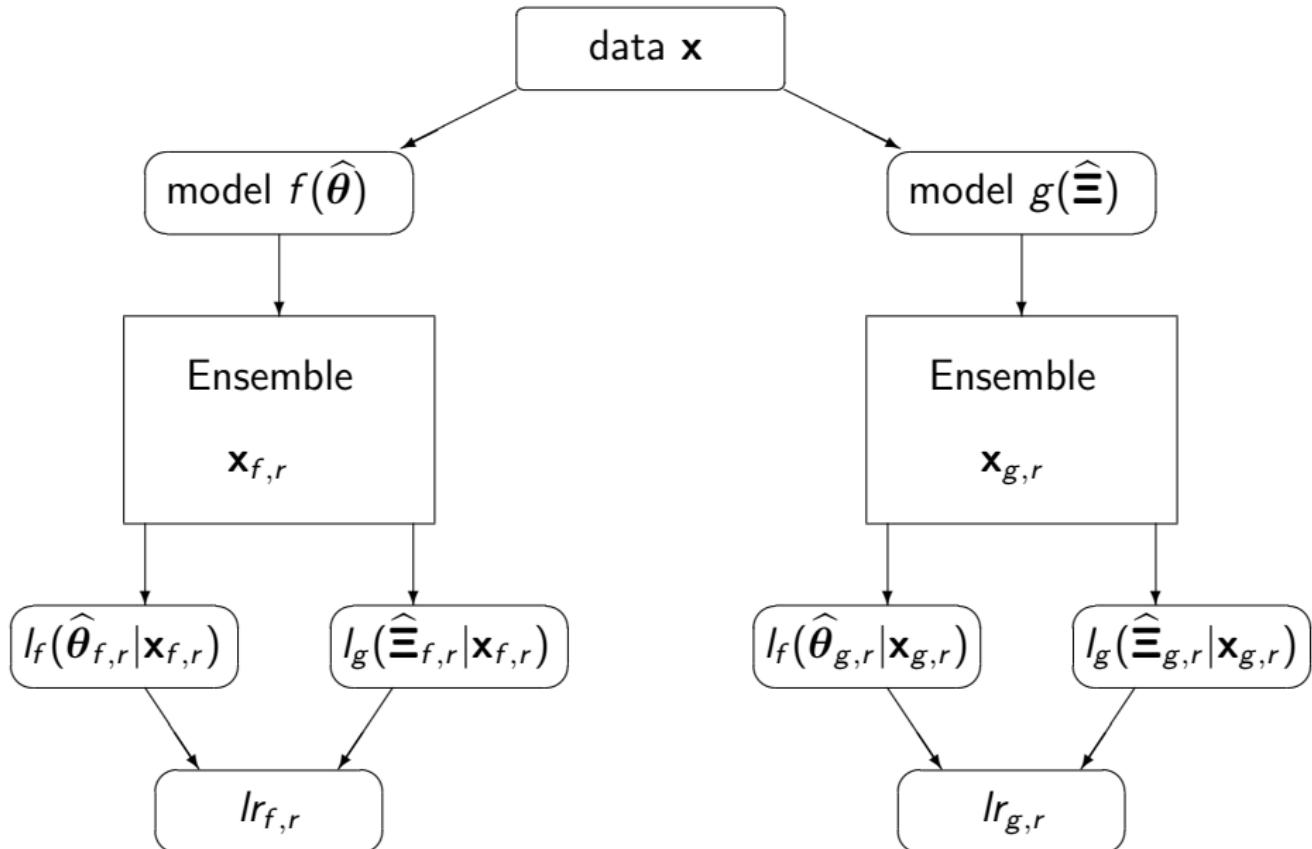
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FARIMA[1,d,0]



Non-Nested Model Selection, Scheme



Non-Nested Model Selection, Outcome

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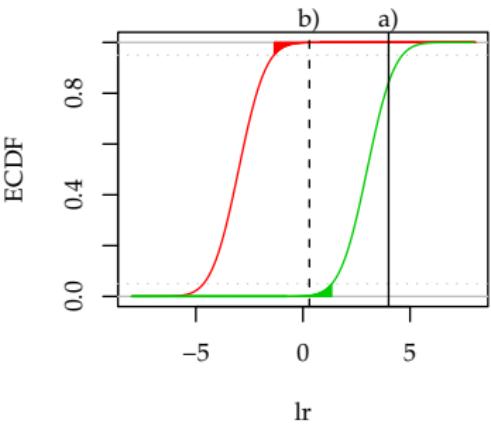
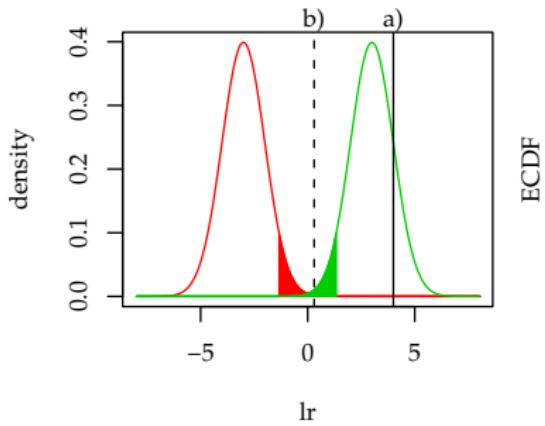
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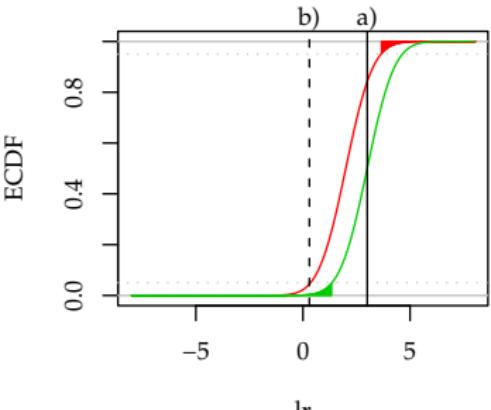
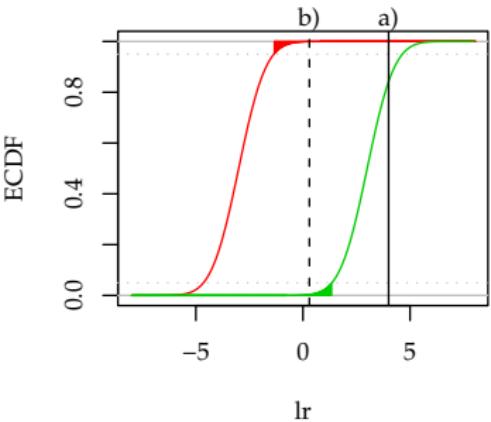
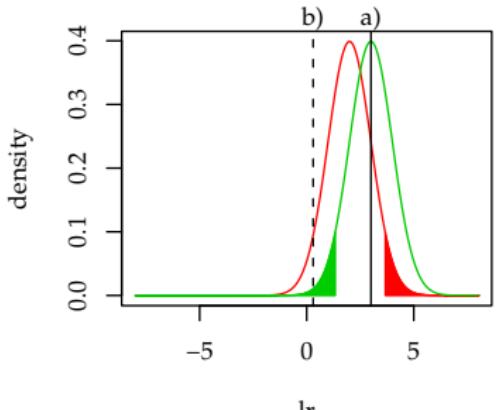
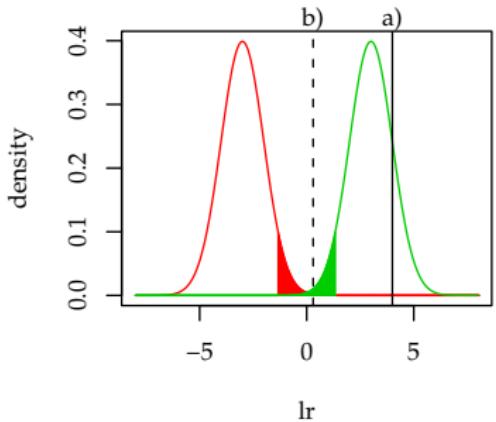
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Non-Nested Model Selection, Outcome



Prices

- ▶ one good answer – one good drink ¹

Questions

- ▶ What is the difference between the spectral density and the periodogram?
- ▶ What is the difference between SRD and LRD?
- ▶ What is the difference between AR and MA models?

¹can be substituted with organic bittersweet chocolate

Prices

- ▶ one good answer – one good drink ¹

Questions

- ▶ What is the difference between the spectral density and the periodogram?
- ▶ What is the difference between SRD and LRD?
- ▶ Which theorem links the ACF and the spectral density?

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Definition LRD
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Long-Range Dependence

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Definition

A stationary stochastic process X_t is called **long-range dependent** (LRD) if its autocorrelation function $\rho(\tau)$ is not summable, i.e.

$$\sum_{-\infty}^{\infty} \rho(\tau) = \infty.$$

Examples

• algebraically decaying ACF for large lags τ

$$\lim_{\tau \rightarrow \infty} \rho(\tau) = c_1 \tau^{-\beta}$$

• pole in the spectrum at $\omega = 0$

$$\lim_{\omega \rightarrow 0} S(\omega) = c_2 |\omega|^{-2\beta}$$

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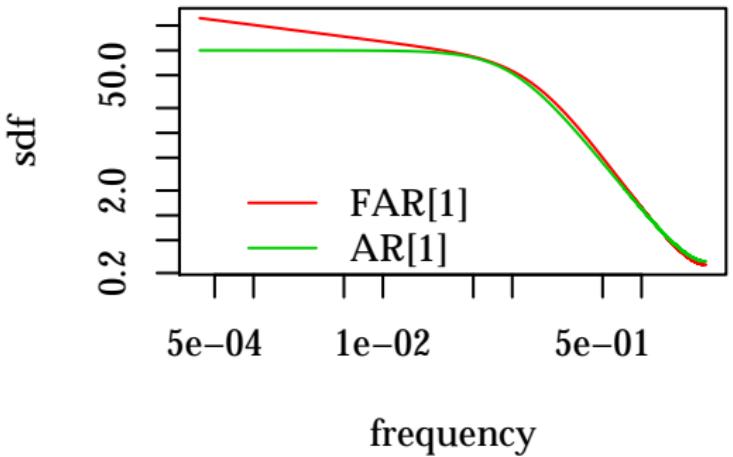
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Definition LRD

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FARIMA Processes

Example



Spectral Density

$$S(\omega) = \frac{\sigma_\eta^2}{2\pi} \frac{|\Psi(e^{i\omega})|^2}{|\Phi(e^{i\omega})|^2} |1 - e^{i\omega}|^{-2d} \xrightarrow{\omega \rightarrow 0} c_S |\omega|^{-2d}$$

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