geopotential

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Geopotentiel and vertical pressure coordinates

Definitions:

- <u>geopotential altitude</u>: *z(x,y,p,t)* altitude of the pressure surface *p*
- <u>géopotential:</u> $\Phi(x,y,z,t) = g\, z(x,y,p,t)$

Pressure and temperature are easily measured by in situ instruments on board the radiosondes. Altitude is less easily measured. It can be obtained by radar measurement (reflector on the probe, radar on the ground) or with a GPS (still expensive on non-recoverable probes). It can also be obtained, in the hydrostatic hypothesis, by integration from the ground of the equation $\partial p/\partial z + \rho\, g = 0$, i.e.

$$z(x,y,p,t) = -\int_{p_{\text{sol}}(x,y,t)}^{p} \frac{R\,T'(x,y,p',t)}{g p'}\, dp'.$$

It is thus natural to show the results of meterological soundings as maps of geopotential.

There are other reasons to use the geopotential function. They are related to the use of the pressure p as vertical coordinate instead of the altitude *z*.

The use of *p* as a vertical coordinate is made possible by the hydrostatic approximation $\partial p/\partial z + \rho g = 0$. This relation implies that p is a variable decreasing vertically with altitude (the hydrostatic pressure is simply related to the weight of the column of air located above) and allows to pass from *z* to *p*.

We go from coordinates (x,y,z) to (x,y,p).

In the new coordinates, the vertical velocity is $\omega = Dp/Dt$. It is negative for upward movements and positive for downward movements. The horizontal speed is unchanged but it is defined at constant pressure rather than at constant altitude.

Several interesting properties follow from this:

- Air movements are "incompressible" in *(x,y,p)* coordinates. Indeed the mass element $\rho\, dx\, dy\, dz$ becomes $1/g\, dx\, dy\, dp$, so that the density in the new coordinates becomes $1/g$ which is a constant. Therefore, the continuity equation
$D\rho/Dt + \rho\, div_{xyz}\, u = 0$ is replaced by $div_{xyp}\, u = 0$, hence

$$\left(\frac{\partial u}{\partial x}\right|_{y,p} + \left(\frac{\partial v}{\partial y}\right|_{x,p} + \left(\frac{\partial \omega}{\partial p}\right|_{x,y} = 0$$

This relation considerably simplifies the analysis of the equations of motion as it will be seen later.

The term *-1/ρ grad p*|z appearing to the right of the horizontal motion equation can be replaced by – *grad Φ*|*p* .

First demonstration using the rules of change of variables.
Reminder: to pass variables (x,y,z) to (X,Y,Z)

$$\left(\frac{\partial F}{\partial X}\right|_{Y,Z}=\left(\frac{\partial F}{\partial x}\right|_{y,z}\left(\frac{\partial x}{\partial X}\right|_{Y,Z}+\left(\frac{\partial F}{\partial y}\right|_{x,z}\left(\frac{\partial y}{\partial X}\right|_{Y,Z}+\left(\frac{\partial F}{\partial z}\right|_{x,y}\left(\frac{\partial z}{\partial X}\right|_{Y,Z}$$

We use this relation to pass from (x,y,z) to (x,y,p).

$$\left(\frac{\partial p}{\partial x}\right|_{y,p}=0=\left(\frac{\partial p}{\partial x}\right|_{y,z}\underbrace{\left(\frac{\partial x}{\partial x}\right|_{y,p}}_{=1}+\left(\frac{\partial p}{\partial y}\right|_{x,z}\underbrace{\left(\frac{\partial y}{\partial x}\right|_{y,p}}_{=0}+\underbrace{\left(\frac{\partial p}{\partial z}\right|_{x,y}}_{=-\rho g}\left(\frac{\partial z}{\partial x}\right|_{y,p}$$

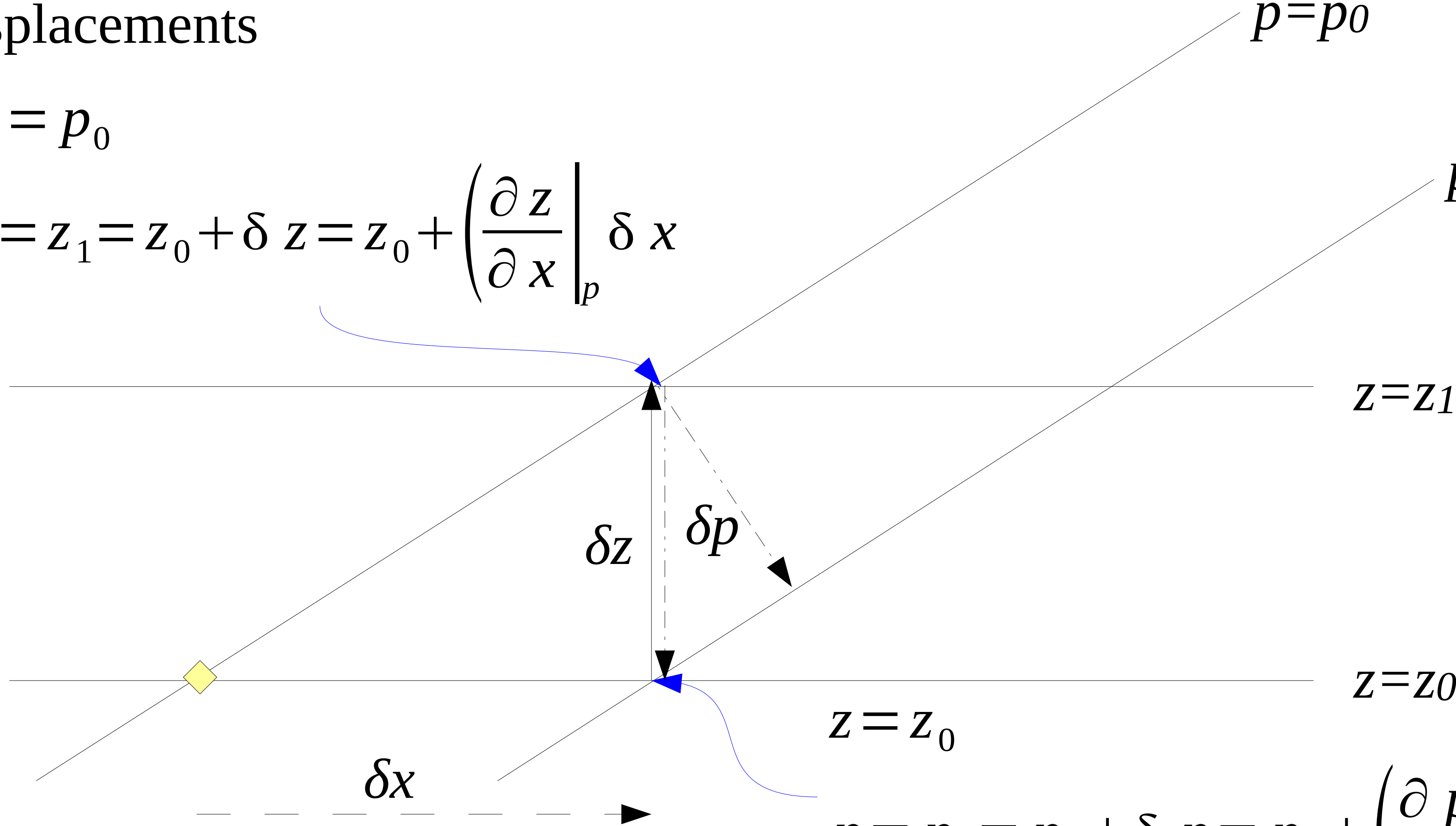
We keep finally

$$0=\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right|_{y,z}-\left(\frac{\partial \phi}{\partial x}\right|_{y,p}$$

Second demonstration using small displacements

$$p = p_0$$

$$z = z_1 = z_0 + \delta z = z_0 + \left(\frac{\partial z}{\partial x} \right|_p \delta x$$



$$z = z_0$$

$$p = p_1 = p_0 + \delta p = p_0 + \left(\frac{\partial p}{\partial x} \right|_z \delta x$$

Hence $\delta p = \left(\frac{\partial p}{\partial x} \right|_z \delta x$ et $\delta z = \left(\frac{\partial z}{\partial x} \right|_p \delta x$

According to the hydrostatic law $\frac{\delta p}{\delta z} = -\left(\frac{\partial p}{\partial z} \right|_x = \rho g$, d'où $\frac{1}{\rho} \left(\frac{\partial p}{\partial x} \right|_z = \left(\frac{\partial g z}{\partial x} \right|_p$