VADE-MECUM OF THE POTENTIAL VORTICITY

Definition (PV) $P = \frac{(\vec{rot}\,\vec{u} + 2\vec{\Omega}) \cdot \vec{\nabla}\theta}{\rho}$

Unit 1PVU = 10^{-6} K $kg^{-1}m^{2}s^{-1}$

PV is exactly conserved the flow under adiabatic , inviscid and frictionless conditions.

Simplified form $P \approx g(\zeta + f)(\frac{-\partial\theta}{\partial p})$

- ζ is the vertical component of relative vorticity
- justification: the vertical gradient of θ (4K/km in the tropospherere, 25K/km in the lower stratosphere) is from 1000 to 5000 larger than the horizontal gradient (50K/10000km); the vertical component of vorticity is dominated by f of the order of 10^{-4} s^{-1}; the horizontal component is dominated by the shear of order 10^{-2} s^{-1}.

The PV increases in magnitude towards the poles.

This is due to the increase of $|f|$ towards the poles.

The PV increases with altitude in the stratosphere.

The PV is small in the troposphere (< 2 PVU) and increases abruptly at the tropopause due to the increase in te stratification. It grows above in the stratosphere due to the density decrease with altitude.

The PV does not vanish in the absence of motion relative to the Earth

It remains $P = \frac{2\vec{\Omega} \cdot \vec{\nabla}\theta}{\rho}$

Latitude motion at tropopause level induces PV anomalies

The tropopause varies in altitude from equator (18km) to pole (6km), with jumps corresponding to the subtropical and polar jets. The potential temperature surfaces are on the contrary sloping upward towards the pole and intersect the tropopause for levels between 351K and 380K. Large horizontal variations of PV occurs where a potential temperature surface crosses the tropopause. Quasi-adiabatic northward motion on potential temperature surface is able to carry low PV air from near the surface in the subtropics to layers at or above the tropopause high latitudes among high PV air. Inversely, southward quasi-adiabatic motion brings poaches of high PV air in altidue towards the surface in the subtropics.

Effect of a positive PV anomaly at the tropopause.

Deep cyclonic motion around the anomaly (down to the ground and in the stratosphere above).

Increase of stratification inside the anomaly.

Decrease of the stratification above and below the anomaly (compensation of the cyclonic vorticity to preserve zero anomaly).

Cooling below the anomaly (raising potential temperature surfaces) and warming above (descending potential temperature surfaces).

Descent of the tropopause .

Effect of a negative PV anomaly at the tropopause.

Deep anticyclonic motion around the anomaly (down to the ground and in the stratosphere above).
Decrease of stratification inside the anomaly.
Inecrease of the stratification above and below the anomaly (compensation of the anticyclonic vorticity to preserve zero anomaly).
Warming below the anomaly (descending potential temperature surfaces) and cooling above (raising potential temperature surfaces).
Ascent of the tropopause .

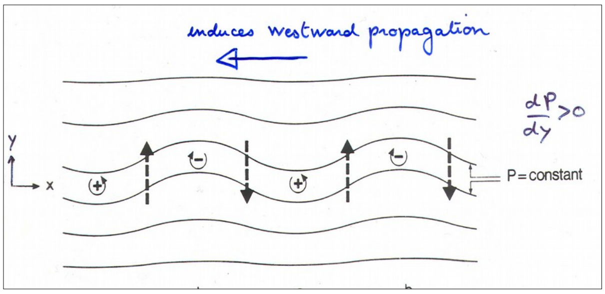
At the ground level: positive PV anomaly <=> positive T anomaly, negative PV anomaly <=> negative T anomaly

(<=> means « induces the same effects »)

Attention! The relation between the sign of PV anomaly and that of the the temperature anomaly is reversed at a top boundary (e.g. the tropopause in the Eady model, in this case cold <=> PV anomaly >0, warm <=> PV anomaly <0)

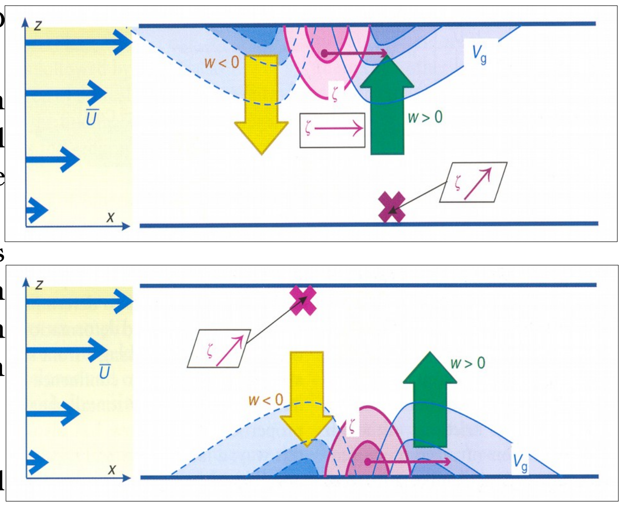
Rossby wave

Ondulation of the PV contours on a potential temperature surface (with PV gradient to the north) => creation of a chain of positive and negative PV anomalies => induced motion in quadrature with the wave => westward propagation (no change of amplitude since the motion vanishes on crests).
The direction of propagation is reversed if the PV gradient is reversed.
The mean wind must be added to get the propagation with respect to a ground observer.



Action of PV anomalies superimposed to a temperature gradient

The anomaly is located within a westerly jet in wind-thermal balance with potential temperature surfaces sloping towards the pole and a northward PV gradient.
Positive anomaly at the tropopause induces northward and upward motion on its eastern flank. Motion is southward and downward on the western flank. This implies warm advection on the east and cold advection on the west.



The warm or cyclonic anomaly at the ground level generates high PV advection on the west and low PV advection on the east.

Role of vertical advection

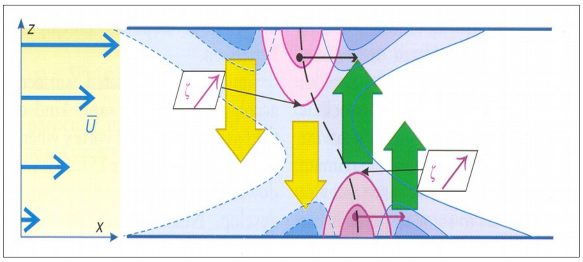
The relative vorticity is advected by the mean wind and modified by vertical stretching according to $\frac{\partial \zeta}{\partial t}+u\frac{\partial \zeta}{\partial x}=f\frac{\partial w}{\partial z}$

The ground at the bottom boundary and the strong stratospheric stratification above the tropopause (with much less sloping potential temperature surfaces) are blocking the vertical advection.

Hence, in the case of a positive PV anomaly at the tropopause, we find just below the tropopause $\frac{\partial w}{\partial z}<0$ on the eastern flank and $\frac{\partial w}{\partial z}>0$ on the western flank. These contributions create negative vorticity on the east and positive vorticity on the west, thus tending to propagate the PV anomaly to the west, opposing the advection by the westerly wind. This is the thermal version of the Rossby waves which complement the mechanism described above which works in the pure 2D approximation with f varying in latitude.

In the vicinity of the ground, we have on the opposite $\frac{\partial w}{\partial z}>0$ on the east side and $\frac{\partial w}{\partial z}<0$ on the west side, leading to reinforce the eastward advection. This is again consistent with the Rossby mechanism since the ground temperature gradient is southward oriented.

Destabilization by coupling



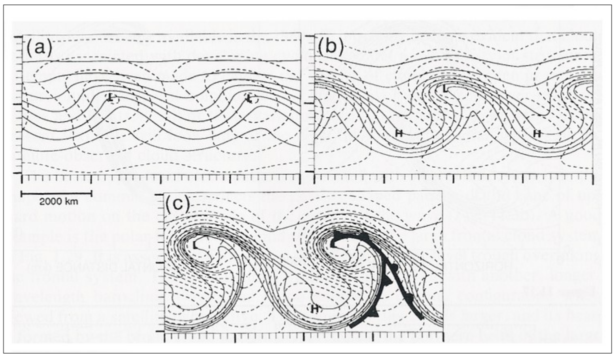
When the PV anomaly at the tropopause and the cyclonic warm anomaly at the ground get coupled, an unstable situation may result due to mutual amplification. The favorable situation is with the upper anomaly on the west of the lower anomaly. The upper anomaly is then reinforced by the southward advection of high PV air due to the surface cyclone while the warm surface anomaly is reinforced by the warm northward advection induced by the upper layer cyclone.

To maintain this situation, the requiremenst are :

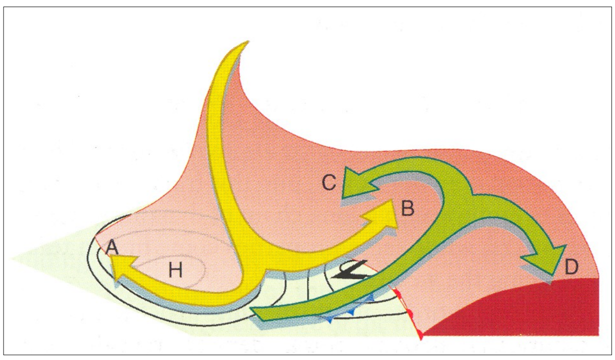
- that the mutual influence extends over the depth H of the troposphere, therefore that the minimal horizontal size is of the order of $H\,N/f$ that is about 500 km
- that the system remains locked against the advection by the mean wind which tends to propagate rapidly the upper layer anomaly to the east. This is obtained by a combination of self and mutual advection.

Instability development

The subsequent development of the instability leads the ascending warm current to roll up at the tropopause and the cyclonic core to move northward while the spread of the descending cold current against the warm current generates the cold front.



Maps of the ground temperature (solid) and pressure (dashed) during the development of the instability



ANNEXE : Exact derivation of the Ertel potential vorticity

We begin with the full vorticity equation in 3D

$$\frac{\partial}{\partial t}\vec{\zeta}_a+\vec{\nabla}\times(\vec{\zeta}_a\times\vec{u})=\frac{1}{\rho^2}\vec{\nabla}\rho\times\vec{\nabla}p+\vec{\nabla}\times\vec{F} \qquad (1)$$

where $\vec{\zeta}_a=\vec{\nabla}\times\vec{u}+2\vec{\Omega}$ is the absolute vorticity.
The second term on the left contains advection and vorticity stretching.
The first term on the right is the baroclinic term and the second one arises from the internal forces; it includes viscosity and friction.
We consider now θ, a thermodynamical function depending only of ρ and p, thus implying $\vec{\nabla}\theta\cdot(\vec{\nabla}\rho\times\vec{\nabla}p)=0$. In other words, the scalar product of the vorticity equation by $\vec{\nabla}\theta$ eliminates the baroclinic term.
From the definition ofe $D\theta/Dt$, we have

$$\frac{\partial}{\partial t}\vec{\nabla}\theta\equiv\vec{\nabla}\frac{D\theta}{Dt}-\vec{\nabla}(\vec{u}\cdot\vec{\nabla}\theta) \qquad (2)$$

By comining equations (1) and (2), we obtain

$$\frac{\partial}{\partial t}(\vec{\zeta}_a\cdot\vec{\nabla}\theta)+\vec{\nabla}\theta\cdot\vec{\nabla}\times(\vec{\zeta}_a\times\vec{u})+\vec{\zeta}_a\cdot\vec{\nabla}(\vec{u}\vec{\nabla}\theta)-\vec{\zeta}_a\cdot\vec{\nabla}\frac{D\theta}{Dt}-\vec{\nabla}\theta\cdot\vec{\nabla}\times\vec{F}=0 \qquad (3)$$

We use now the vectorial relation $\vec{\nabla}\cdot(\vec{A}\times\vec{B})\equiv\vec{\nabla}\times\vec{A}\cdot\vec{B}-\vec{\nabla}\times\vec{B}\cdot\vec{A}$,
with $\vec{A}=\vec{\nabla}\theta$ and $\vec{B}=\vec{\zeta}_a\times\vec{u}$, hence

$$\vec{\nabla}\theta\cdot\vec{\nabla}\times(\vec{\zeta}_a\times\vec{u})=\vec{\nabla}\cdot(\vec{\nabla}\theta\times(\vec{u}\times\vec{\zeta}_a))=\vec{\nabla}\cdot((\vec{\nabla}\theta\cdot\vec{\zeta}_a)\vec{u})-\vec{\zeta}_a\cdot\vec{\nabla}(\vec{\nabla}\theta\cdot\vec{u})$$

By replacing in (3), we obtain

$$\frac{\partial}{\partial t}(\vec{\zeta}_a\cdot\vec{\nabla}\theta)+\vec{\nabla}\cdot((\vec{\zeta}_a\cdot\nabla\theta)\vec{u})-\vec{\zeta}_a\cdot\vec{\nabla}\frac{D\theta}{Dt}-\vec{\nabla}\theta\cdot\vec{\nabla}\times\vec{F}=0$$

that is $\frac{D}{Dt}(\vec{\zeta}\cdot\vec{\nabla}\theta)+(\vec{\zeta}_a\cdot\vec{\nabla}\theta)\vec{\nabla}\cdot\vec{u}=\vec{\zeta}_a\cdot\vec{\nabla}\frac{D\theta}{Dt}+\vec{\nabla}\theta\cdot\vec{\nabla}\times\vec{F}$

By now combining (4) with the continuity equation $\frac{D\rho}{Dt}+\rho\vec{\nabla}\cdot\vec{u}=0$, we obtain finally

$$\boldsymbol{\frac{D}{Dt}\left(\frac{\vec{\zeta}_a\cdot\vec{\nabla}\theta}{\rho}\right)=\frac{1}{\rho}\vec{\zeta}_a\cdot\vec{\nabla}\frac{D\theta}{Dt}+\frac{1}{\rho}\vec{\nabla}\theta\cdot\vec{\nabla}\times\vec{F}}$$

The quantity $P=\frac{\vec{\zeta}_a\cdot\vec{\nabla}\theta}{\rho}$ is the Ertel potential vorticity
It is conserved for each parcel of the fluid under adiabatic ($D\theta/Dt=0$) and inviscid ($\vec{F}=0$) conditions.