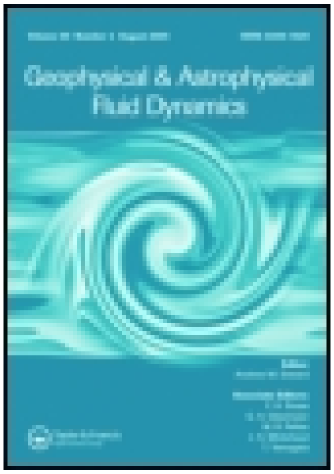

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Turbulent phase shift of rossby waves

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Turbulent Phase Shift of Rossby Waves

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The properties of turbulent flows are strongly modified by the presence of dispersive waves the period of which can be less than the eddy-turnover time of turbulence for sufficiently large scales. Using a systematic method due to André, we show that the presence of such waves induces complex damping rates in stochastic models like for instance the Eddy-Damped Quasi-Normal Markovian model: in other terms, turbulence acts to renormalise both viscosity and frequency. This method is applied to spherical Rossby waves; it is found that the relative correction to Rossby frequencies, negligible at planetary scales, might become important around 1000 km and tends to a limit close to 0.3 for sufficiently small scales.

1. INTRODUCTION

Geophysical fluid flows are greatly affected by turbulent processes whose predominance in the smaller scales is responsible for the unpredictability of the larger ones. It seems therefore attractive to apply the stochastic tools of turbulence theory to the problem of simulating the general circulation statistics of the atmosphere. The problem of large scale geophysical turbulence however exhibits several specific features usually absent in turbulence theory. Baroclinicity is one of them: the way in which it modifies the usual concept of two-dimensional turbulence has been studied in particular by Charney (1971), Salmon (1978) and others. Even for barotropic flow specific features arise due to the presence of linear terms in the basic equation, related to either orography (Herring, 1977) or variation of Coriolis parameter with latitude (Holloway and Hendershott, 1977). We shall be interested here in the latter aspect of the problem, i.e. the way in which two-dimensional turbulence becomes modified in the presence of Rossby wave propagation. We shall use the formulation of the spherical vorticity equation

$$\frac{\partial \zeta}{\partial t}+\frac{\partial(\Psi, \zeta+f)}{\partial(\lambda, \mu)}=\nu \nabla^2 \zeta+F, \tag{1}$$

where Ψ is the stream-function such that the velocity is $\mathbf{u}=\mathbf{k}\times\nabla\Psi$ with $\mathbf{k}$ the unit normal vector; ζ is the vorticity $\mathbf{k}.\mathrm{curl}\,\mathbf{u}$; λ and μ are respectively the longitude and the sine of latitude, ν is a dissipation coefficient, F is a forcing term and f is the Coriolis parameter $2\Omega\mu$; we suppose the radius of the sphere is unity. The natural geometry of the problem is spherical; however, if one is only interested in intermediate scales, a usual approximation to (1) is the well known β-plane approximation which is especially attractive for oceanic problems:

$$\frac{\partial \zeta}{\partial t}+\frac{\partial(\Psi, \zeta)}{\partial(x, y)}+\beta\frac{\partial \Psi}{\partial x}=\nu \nabla^2 \zeta+F, \tag{2}$$

where the x direction is oriented westward. Various types of boundary conditions can be associated with (2): free-slip closed basin, x-periodic free-ship channel, doubly periodic domain, or vanishing at infinity.

In the following, we shall consider mainly spherical geometry, while keeping an eye on the side on the planar analogues of our results in order to emphasize resemblances and differences. Rossby waves appear when one linearises the left-hand side of (1) or (2). In the space of spherical harmonics (cf. Section 2), the dispersion relation reads

$$\omega_l^m=-2\Omega m/l(l+1), \tag{3}$$

instead of

$$\omega_{\mathbf{k}}=-\beta k_x/k^2$$

in Fourier space. Rossby waves are therefore dispersive and strongly anisotropic: the phase velocity is always directed westward and no propagation occurs in the purely meridional direction. The interested reader will find in Longuet-Higgins (1964, 1965) a detailed discussion of the geometrical properties of Rossby waves.

From the standpoint of turbulence, one of the most important features of Rossby waves is that frequency grows with wavelength. If the relative enstrophy Z of the flow is less than Ω^2, the period of the larger scale waves is smaller than the eddy-turnover time at the same scales (typically greater than $Z^{-1/2}$): one thus expects in that case a strong modification of turbulence by the waves. In physical space, where nonlinear interactions are essentially local, waves spread the energy of the large eddies before they interact efficiently; one can also say that waves tend to organise the flow in competition with nonlinear instabilities which tend to disorganise it. Rhines (1975) distinguishes two regimes in the flow: a turbulent regime towards the smaller scales where waves are negligible and a regime dominated by wave propagation towards the larger scales. The separation is characterised by a transition wave number $k_\beta=(\beta/2U)^{1/2}$ estimated by equating the r.m.s. speed U of the flow to the phase speed of the Rossby waves. Rhines shows how the inhibition of nonlinear transfer within the larger scales yields, in the vicinity of k_β, a strong weakening of the reverse energy cascade usually observed in two-dimensional turbulence, associated with a build up of anisotropy in the larger scales from a supposedly isotropic initial state.

Holloway and Hendershott (1977) have applied the technique of stochastic modelling to the β-plane problem. They use the Eddy-Damped Quasi-Normal Markovian approximation (EDQNM) and give an estimate of the characteristic time θ_{kpq} for the relaxation of triple correlations in the presence of Rossby waves. What we propose here is a finer estimation of this characteristic time, taking into account the dephasing effects induced by waves at all orders of the hierarchy of the cumulant equations.

2. SPECTRAL INTERACTIONS

The spectral form of problem (1) obtains by expanding all scalar functions, on the basis of the spherical harmonics $Y_\alpha(\lambda,\mu)$.† Vorticity expands in the form

$$\zeta(\lambda,\mu,t)=\sum_\alpha \zeta_\alpha(t)Y_\alpha(\lambda,\mu),$$

with

$$\zeta_{\bar\alpha}=(-1)^{m_\alpha}\bar\zeta_\alpha$$

ensuring that ζ is a real field.

†The spherical harmonics are eigen functions of the laplacian operator

$$\nabla^2 Y_\alpha(\lambda,\mu)=-l_\alpha(l_\alpha+1)Y_\alpha(\lambda,\mu).$$

We shall use the complex subscript $\alpha=l_\alpha+im_\alpha$ where the degree l_α and the order m_α are integers with $l_\alpha\geqq|m_\alpha|$. We thus have

$$Y_\alpha(\lambda,\mu)=\left[(2l_\alpha+1)\frac{(l_\alpha-m_\alpha)!}{(l_\alpha+m_\alpha)!}\right]^{1/2}(1-\mu^2)^{m_\alpha/2}\frac{d^{l_\alpha+m_\alpha}}{d\mu^{l_\alpha+m_\alpha}}(1-\mu^2)^{l_\alpha}e^{im_\alpha\lambda}.$$

Conjugation and orthogonality rules for the spherical harmonics read

$$Y_{\bar\alpha}(\lambda,\mu)=(-1)^{m_\alpha}\overline{Y_\alpha(\lambda,\mu)},$$

$$\frac{1}{4\pi}\int_{-1}^{+1}d\mu\int_0^{2\pi}d\lambda\, Y_\beta(\lambda,\mu)\overline{Y_\gamma(\lambda,\mu)}=\delta_{l_\beta,l_\gamma}\delta_{m_\beta,m_\gamma}.$$

The spectral vorticity-equation is

$$(d/dt - i\omega_\alpha + \nu j_\alpha^2)\zeta_\alpha = i \sum_{\beta,\gamma} A_{\alpha\beta\gamma}\zeta_\beta\zeta_\gamma, \tag{4}$$

with

$$j_\alpha = [l_\alpha(l_\alpha + 1)]^{1/2}.$$

The interaction coefficients $A_{\alpha\beta\gamma}$ are real and defined as

$$A_{\alpha\beta\gamma} = \frac{i}{2}[j_\gamma^{-2} - j_\beta^{-2}]\frac{1}{4\pi}\int_{-1}^{+1} d\mu \int_0^{2\pi} d\lambda\, Y_\alpha \frac{\partial(Y_\beta, Y_\gamma)}{\partial(\lambda, \mu)}. \tag{5}$$

An expression of $A_{\alpha\beta\gamma}$ with Wigner's 3-l symbols has been given by Thiébaux (1971):

$$\begin{aligned} A_{\alpha\beta\gamma} = \tfrac{1}{4}(j_\gamma^{-2} - j_\beta^{-2})&[(2l_\alpha + 1)(2l_\beta + 1)(2l_\gamma + 1)]^{1/2} \\ &\times[(1+L)(1+L-2l_\alpha)(L-2l_\beta)(L-2l_\gamma)]^{1/2} \\ &\times\begin{pmatrix} l_\alpha - 1 & l_\beta & l_\gamma \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} l_\alpha & l_\beta & l_\gamma \\ m_\alpha & m_\beta & m_\gamma \end{pmatrix}, \end{aligned} \tag{6}$$

with $L = l_\alpha + l_\beta + l_\gamma$. A coefficient $A_{\alpha\beta\gamma}$ is non-zero if and only if the triad (α, β, γ) satisfies the following selection rules:

$$\begin{cases} m_\alpha + m_\beta + m_\gamma = 0 \quad \text{with } m_\alpha, m_\beta, m_\gamma \text{ not all zero,} & (7a) \\ l_\alpha, l_\beta, l_\gamma \quad \text{are the sides of a true triangle,} & (7b) \\ l_\alpha + l_\beta + l_\gamma \quad \text{odd,} & (7c) \\ l_\beta \neq l_\gamma, & (7d) \\ \alpha \neq \beta \quad \text{and} \quad \alpha \neq \gamma. & (7e) \end{cases}$$

The symmetry properties of $A_{\alpha\beta\gamma}$ are easily deducible from (5) or (6). One obtains

$$\begin{cases} A_{\alpha\beta\gamma} = -A_{\bar{\alpha}\bar{\beta}\bar{\gamma}}, & (8a) \\ A_{\alpha\beta\gamma} = A_{\alpha\gamma\beta}, & (8b) \\ A_{\alpha\beta\gamma} + A_{\beta\gamma\alpha} + A_{\gamma\alpha\beta} = 0, & (8c) \\ j_\alpha^{-2}A_{\alpha\beta\gamma} + j_\beta^{-2}A_{\beta\gamma\alpha} + j_\gamma^{-2}A_{\gamma\alpha\beta} = 0. & (8d) \end{cases}$$

Relations (8c) and (8d) express the detailed conservation of enstrophy and energy respectively in each triad interaction.

It may be interesting to remember here the analogous formulation in planar geometry when problem (2) is formulated on the basis of Fourier harmonics, namely

$$\zeta(\mathbf{x},t)=\sum_{\mathbf{k}}\zeta_{\mathbf{k}}(t)e^{i\mathbf{k}\cdot\mathbf{x}},$$

We have thus

$$(d/dt+i\omega_{-\mathbf{k}}+\nu k^2)\zeta_{-\mathbf{k}}(t)=\sum_{\mathbf{p},\mathbf{q}}A_{\mathbf{kpq}}\zeta_{\mathbf{p}}(t)\zeta_{\mathbf{q}}(t), \tag{9}$$

with

$$A_{\mathbf{kpq}}=\tfrac{1}{2}\mathbf{k}\times\left(\frac{\mathbf{p}}{p^2}+\frac{\mathbf{q}}{q^2}\right).$$

The selection rule reads

$$\mathbf{k}+\mathbf{p}+\mathbf{q}=0.$$

The resemblance between (4) and (9) is of course not surprising. One difference between the two forms is the appearance of a factor i on the right-hand side of Eq. (4), arising from the fact that each complex exponential is derived only once in the jacobian of two spherical harmonics. The symmetry properties of $A_{\mathbf{kpq}}$ are strictly analogous to relations (8), save on one point: the equivalent of (8a) in the plane case reads

$$A_{\mathbf{k},\mathbf{p},\mathbf{q}}=A_{-\mathbf{k},-\mathbf{p},-\mathbf{q}},$$

because of the absence of the factor i in this case. Note that the scale of motion associated with a given eigenmode is $|\mathbf{k}|$ on the plane and the degree l of the harmonic on the sphere. These formal identities allow straightforward extensions of a number of results from planar to spherical geometry. For inviscid truncated systems ($l\leq l_{\max}<\infty$) equipartition equilibria obtain, of the following form:

$$Z_\alpha=\tfrac{1}{2}\langle\zeta_\alpha\bar{\zeta}_\alpha\rangle=j_\alpha^2/(a+bj_\alpha^2),$$

where a and b are determined from the values of total energy and enstrophy (e.g. Kraichnan, 1967).

Some differences however between the two cases are worth noticing. In planar geometry, the only modes $\pm\mathbf{k}$ with which two Fourier modes $\pm\mathbf{p}$ and $\pm\mathbf{q}$ directly interact are $\mathbf{k}_1=\mathbf{p}+\mathbf{q}$ and $\mathbf{k}_2=\mathbf{p}-\mathbf{q}$; whereas in spherical geometry, relation (7b) allows two harmonics to interact directly with more than two others: for examples $5+3i$ and $3-2i$ interact directly with

$7+i$, $5+i$, $3+i$, $7+5i$, $5+5i$ and conjugate modes.

Therefore, one can point out that the processes which distribute energy throughout the spectrum are more efficient on the plane than on the sphere (Tang and Orszag, 1977). Another peculiarity of spherical geometry is that the concepts of homogeneity and isotropy are indistinguishable, since the displacement group contains rotations only. This remark has no consequence of interest for a problem formulated within the framework of homogeneity and isotropy. This is not the case here: Eq. (2) remains invariant within the group of translations, but not within the group of rotations, so that the only proper assumption to make is that of homogeneity; while Eq. (1) on the sphere, is invariant only within the group of rotations around the polar axis, the only proper assumption there being that of zonal homogeneity. In the former case, homogeneity warrants that statistical moments $\langle\zeta_{\mathbf{k}_1},\ldots,\zeta_{\mathbf{k}_N}\rangle$, defined as averages over an infinite set of realisations of the flow, are non-zero only if $\mathbf{k}_1+\ldots+\mathbf{k}_N=0$; in particular, the averages $\langle\zeta_{\mathbf{k}}\rangle$ are identically zero and the only non-zero moments of order two are the quadratic terms $\langle\zeta_{\mathbf{k}}\zeta_{-\mathbf{k}}\rangle$. In the latter case, the moments read $\langle\zeta_{l_1}^{m_1}\ldots\zeta_{l_N}^{m_N}\rangle$ and the zonal homogeneity implies only $m_1+\ldots+m_N=0$; thus, zonal averages $\langle\zeta_l^0\rangle$ and double correlations $\langle\zeta_{l'}^{m}\zeta_{l''}^{-m}\rangle$ are allowed to be non-zero...etc. In this manner, spherical geometry implies the existence of a mean zonal flow and interactions with this flow. Even though such interactions are only second order in time if one considers the initial departure of the flow from a gaussian spectrum, they can be expected to play a noticeable part in fully developed turbulence. Nevertheless, since the proper processing of such terms in the equations can be achieved only at the price of a considerable increase in analytic complexity we have chosen in a first approach to neglect such interactions. This simplification allows a simpler analysis but without any doubt obliterates a part of the specific features of the spherical problem.†

Boundary conditions may cause difficulties in the β-plane: with infinite boundaries, $\mathbf{k}$ varies continuously over spectral space, so that Rossby frequencies diverge as $|\mathbf{k}|\to 0$, although they are bounded in the natural problem.‡

---

†Existence of non-zero zonal averages $\langle\zeta_l^0\rangle$ implies that terms like $i\langle\zeta\rangle\langle\zeta\zeta\zeta\rangle$ are included on the right-hand side of the triple correlation equation. Since $\langle\zeta_l^0\rangle$ is real and varies over a larger time scale than $\langle\zeta\zeta\zeta\rangle$, such terms are essentially dispersive for triple correlations: they add up to the Rossby wave term and modify accordingly the wave frequencies. Note however that the operator is not diagonal if one uses spherical harmonics as basis functions. A formulation easier to work with may be obtained with normal modes for the average instantaneous zonal flow but at the expense of greater complexity in practical calculation.

‡This difficulty disappears if one takes into account the upper free surface condition in a more realistic model. The Laplacian operator ∇^2 is then replaced by $\nabla^2-\lambda^{-2}$ where λ is the internal deformation radius; Rossby frequencies are then given by $\omega_{\mathbf{k}}^{-1}=-\beta k_x/(k^2+\lambda^{-2})$ and no longer diverge as $|\mathbf{k}|\to 0$.

3. EQUATIONS OF MOMENTS

The evolution equations for the statistical averages of the flow are easily deductible from (4). Defining the modal enstrophy $Z_\alpha=\frac{1}{2}\langle\zeta_\alpha\bar{\zeta}_\alpha\rangle$ one obtains

$$(d/dt+2\nu_\alpha)Z_\alpha=\sum_{\beta,\gamma}A_{\alpha\beta\gamma}\,Im\langle\zeta_\alpha\zeta_\beta\zeta_\gamma\rangle. \tag{10}$$

Triple order moments are governed by

$$(d/dt+i\omega_{\alpha\beta\gamma}+\nu_{\alpha\beta\gamma})\langle\zeta_\alpha\zeta_\beta\zeta_\gamma\rangle=i\sum_{\delta\xi}[(-1)^{m_\alpha}A_{\bar{\alpha}\delta\xi}\langle\zeta_\beta\zeta_\gamma\zeta_\delta\zeta_\xi\rangle$$
$$+(-1)^{m_\beta}A_{\bar{\beta}\delta\xi}\langle\zeta_\gamma\zeta_\alpha\zeta_\delta\zeta_\xi\rangle+(-1)^{m_\gamma}A_{\bar{\gamma}\delta\xi}\langle\zeta_\alpha\zeta_\beta\zeta_\delta\zeta_\xi\rangle]. \tag{11}$$

We use here the contracted notations

$$\omega_{\alpha_1}+\omega_{\alpha_2}+\ldots+\omega_{\alpha_p}=\omega_{\alpha_1\ldots\alpha_p},\qquad \nu(j^2_{\alpha_1}+j^2_{\alpha_2}+\ldots+j^2_{\alpha_p})=\nu_{\alpha_1\ldots\alpha_p}.$$

The equation for the nth order moment always includes a $n+1$st order term. The whole set of equations thus reads as an infinite hierarchy of independent equations the closure of which, in the present state, requires semi-phenomenological approximations such as the EDQNM assumption (Section 4). Rotation does not appear explicitly in the enstrophy equation (10). However, a linear dispersive term is introduced in all moment equations of order larger than 2 (with the exception of the resonant interactions), and in particular in the equation for triple correlations (11). Rotation thus acts on the energy spectrum by means of a modification of the dynamic transfer. Moreover, this modification is itself not confined to the propagative effect $i\omega_{\alpha\beta\gamma}$ which appears explicitly in (11). It also arises by mean of moments $\langle\zeta_\beta\zeta_\gamma\zeta_\delta\zeta_\xi\rangle$ from all kinds of changes induced by rotation on the dynamics of the whole set of higher order moments.

4. EDDY-DAMPED QUASI-NORMAL MARKOVIAN APPROXIMATION (EDQNM)

The assumption of zonal homogeneity allows a simple formulation for (11)

$$(d/dt+i\omega_{\alpha\beta\gamma}+\nu_{\alpha\beta\gamma})\langle\zeta_\alpha\zeta_\beta\zeta_\gamma\rangle+iR_{\alpha\beta\gamma}=Q_{\alpha\beta\gamma}, \tag{12}$$

where

$$R_{\alpha\beta\gamma}=8[A_{\alpha\beta\gamma}Z_\beta Z_\gamma+A_{\beta\gamma\alpha}Z_\gamma Z_\alpha+A_{\gamma\alpha\beta}Z_\alpha Z_\beta];$$

$Q_{\alpha\beta\gamma}$ is a residual sum of fourth-order cumulants.† In the classical quasi-normal approximation, one neglects $Q_{\alpha\beta\gamma}$ in Eq. (12) which amounts to a particular closure of the hierarchy of moments equations. Such a model, however, yields an unrealistic evolution of the energy spectrum with the occurrence of strong negative values (Ogura, 1963); it is also well known that a similar closure produces the same result when applied at any higher order in the hierarchy. This deficiency is mainly due to the lack of irreversibility (independently from viscous dissipation) in the moment dynamics of the model (Orszag, 1970). Indeed, one can show (André, 1975) in the case of a non-gaussian flow that a typical effect of the nth-order cumulants is to relax the cumulants of lower order; this relaxation allows a convergence toward statistical equilibrium for an inviscid system with a finite number of modes.

Leith (1971) and Orszag (1970) proposed to parameterise the relaxation effect of the fourth-order cumulants in the equation for triple correlations by introducing a turbulent damping coefficient $\mu_{\alpha\beta\gamma}(t)$:

$$[d/dt+\mu_{\alpha\beta\gamma}(t)+i\omega_{\alpha\beta\gamma}+v_{\alpha\beta\gamma}]\langle\zeta_\alpha\zeta_\beta\zeta_\gamma\rangle+iR_{\alpha\beta\gamma}(t)=0. \tag{13}$$

By integration of (13) and replacement in (10) we obtain

$$(d/dt+2v_\alpha)Z_\alpha(t)=\int_0^t ds\sum_{\beta,\gamma}Re\left(\exp\left\{-(v_{\alpha\beta\gamma}+i\omega_{\alpha\beta\gamma})(t-s)-\int_0^t \mu_{\alpha\beta\gamma}(n)\,dn\right\}\right)A_{\alpha\beta\gamma}R_{\alpha\beta\gamma}(s). \tag{14}$$

The markovianisation method consists in replacing the historical integral on the right-hand side of (14) by a characteristic time $\theta_{\alpha\beta\gamma}$ (necessarily positive).

The markovianised equation reads

$$[d/dt+2\eta_\alpha(t)+2v_\alpha]Z_\alpha(t)=\sum_{\beta,\gamma}\theta_{\alpha\beta\gamma}(t)a_{\alpha\beta\gamma}Z_\beta(t)Z_\gamma(t), \tag{15}$$

$$\eta_\alpha(t)=\tfrac{1}{2}\sum_{\beta,\gamma}\theta_{\alpha\beta\gamma}(t)b_{\alpha\beta\gamma}Z_\beta(t), \tag{16}$$

where

$$a_{\alpha\beta\gamma}=8(A_{\alpha\beta\gamma})^2,\quad b_{\alpha\beta\gamma}=16A_{\alpha\beta\gamma}A_{\gamma\alpha\beta}.$$

†See the definition of the cumulants given in Section 5 on p. 264.

The EDQMN equation (15) ensures that energy will remain positive at all times. This property corresponds to the fact that (15)–(16) can be obtained as the exact closure of a Langevin equation for a random vorticity field (Leith, 1971; Holloway, 1976)

$$
\begin{gathered}
[d/dt+\eta_\alpha(t)+i\omega_\alpha+\nu_\alpha]\zeta_\alpha(t)=f_\alpha(t),\\
\langle f_\alpha(t)f_\alpha(t')\rangle=\delta(t-t')\sum_{\beta,\gamma}\theta_{\alpha\beta\gamma}(t)a_{\alpha\beta\gamma}Z_\beta(t)Z_\gamma(t).
\end{gathered}
\tag{17}
$$

We still have to define $\theta_{\alpha\beta\gamma}(t)$ in order to completely specify the model. A first estimate can be

$$\theta_{\alpha\beta\gamma}(t)=Re\int_0^t G_{\alpha\beta\gamma}(t,s)\,ds,$$

where

$$G_{\alpha\beta\gamma}(t,s)=\exp\left\{-(\nu_{\alpha\beta\gamma}+i\omega_{\alpha\beta\gamma})(t-s)-\int_s^t \mu_{\alpha\beta\gamma}(n)\,dn\right\}$$

is the Green function associated to (13). Since $\theta_{\alpha\beta\gamma}$ is an integral time-scale for the triple correlation $\langle\zeta_\alpha\zeta_\beta\zeta_\gamma\rangle$, it can also be derived from (17), as

$$\theta_{\alpha\beta\gamma}(t)=Re\int_0^t G_\alpha(t,s)G_\beta(t,s)G_\gamma(t,s)\,ds \tag{18}$$

where $G_\alpha(t,s)$ is the Green function associated with (17). This compels us to take

$$\tilde{\mu}_{\alpha\beta\gamma}(t)=\eta_\alpha(t)+\eta_\beta(t)+\eta_\gamma(t) \tag{19}$$

in order to obtain consistency between both definitions.

The identification of the averaged Green function $G_{\alpha\beta\gamma}$ with the product of average Green functions $G_\alpha G_\beta G_\gamma$ is incorrect: Kraichnan (1971) and Sulem (1975) have shown that (19) induces an over-estimation of relaxation by non-local interactions. This deficiency however can be cured if one uses a modified estimate derived from the Test Field Model, a spherical version of which is given in the appendix. We shall incorporate this correction at the end of Section 5.

The markovianisation technique applied to Eq. (14) turns out to be only a slight modification if the relaxation time of the triple correlations proves to be negligible compared with the time of evolution of the energy spectrum. This assumption can be tested in the case of homogeneous and isotropic turbulence on an infinite plane. In the enstrophy-cascading inertial range, the energy spectrum is $E(k) = C\varepsilon^{-2/3}k^{-3}(\ln k/k_I)^{-1/3}$ (Kraichnan, 1971), where ε is the enstrophy dissipation rate and C a constant of order unity. The characteristic time of spectral transfers in the vicinity of k is about $\varepsilon^{-1/3}(\ln k/k_I)^{-1/3}$, whereas the evolution of the energy spectrum is governed by the eddy-turnover time at the lower end of the cascade, approximately $\varepsilon^{-1/3}$. These two time-scales differ by the ratio $(\ln k/k_I)^{1/3}$, which justifies markovianisation. The assumption seems to be less justified in the $k^{-5/3}$ inverse energy cascade, where the local turnover time of eddies is equal to the time required for energy to cascade from k_I to k. However, we know experimentally that the EDQNM approximation is able to develop both inertial ranges in high Reynolds numbers simulations (Pouquet *et al.*, 1975).

New difficulties due to the existence of waves appear in the present model. In the absence of waves ($\omega_\alpha \equiv 0$), and if the damping rate $\mu_{\alpha\beta\gamma}(t)$ varies but slightly over the interval $[s, t]$, the characteristic time is given by

$$\theta_{\alpha\beta\gamma}(t) = \{1 - \exp[-v^*_{\alpha\beta\gamma}(t)]\}/v^*_{\alpha\beta\gamma}(t), \tag{20}$$

where

$$v^*_{\alpha\beta\gamma}(t) = v_{\alpha\beta\gamma} + \mu_{\alpha\beta\gamma}(t).$$

Relation (20) gives $\theta_{\alpha\beta\gamma} \simeq t$ for short t [an estimate exact to $O(t^2)$ for a gaussian initial state], and $\theta_{\alpha\beta\gamma} \simeq 1/v^*_{\alpha\beta\gamma}$ for large t. In all cases, $\theta_{\alpha\beta\gamma}$ is positive as long as $\mu_{\alpha\beta\gamma} > 0$, a condition which holds because $b_{\alpha\beta\gamma}$ is itself typically positive (André, 1974). In the presence of waves, (20) is replaced by

$$\theta_{\alpha\beta\gamma}(t) = \frac{1}{v^{*2}_{\alpha\beta\gamma} + \omega^2_{\alpha\beta\gamma}}[v^*_{\alpha\beta\gamma} + \exp\{-v^*_{\alpha\beta\gamma}t\}(\omega_{\alpha\beta\gamma}\sin\omega_{\alpha\beta\gamma}t - v^*_{\alpha\beta\gamma}\cos\omega_{\alpha\beta\gamma}t)]. \tag{21}$$

Near $t = 0$, we have again $\theta_{\alpha\beta\gamma} \simeq t$ and the model is still exact at first order. This first stage is followed by a transient stage during which the system keeps some memory of the initial state. $\theta_{\alpha\beta\gamma}$ is then affected by oscillations which can yield negative values if $\omega_{\alpha\beta\gamma}/v^*_{\alpha\beta\gamma} \gtrsim 16$: we may consider this an evident failure of the model. Starting with a gaussian initial state, we see that it takes some time for the nonlinear transfers to become established; in fact they increase as $O(t)$, while propagative effects act instantaneously. Indeed, this initial prevalence of propagative effects in the dynamics of transfer renders somewhat doubtful the very idea of markovianisation, at least during the transient stage.

Far from initial times, when turbulence is fully developed, expression (21) regains its full sense, at least from the formal standpoint, and reaches the stationary form already given by Holloway (1977)

$$\theta_{\alpha\beta\gamma}(t)=\nu^*_{\alpha\beta\gamma}(t)/[\nu^{*2}_{\alpha\beta\gamma}(t)+\omega^2_{\alpha\beta\gamma}]. \tag{22}$$

This expression is always positive and as required by realisability. When introduced in (16), (22) causes the damping rate to decrease in comparison with the case without waves, a behaviour which expresses the increase of the memory time of correlations. Simultaneously, nonlinear transfer is inhibited in (15). The particular set of triads which satisfy the resonance condition $\omega_{\alpha\beta\gamma}=0$ plays a very special part in the transfer: for such triads, the inhibition effect does not exist and one can expect a great portion of the transfer to be due to them. Such interactions only are retained in the Resonant Interaction model (RI) of Longuet-Higgins and Gill (1967) in the limiting case of weak modes on an infinite β-plane. However the RI fails to explain the growth of zonal modes which are left unchanged by resonant interactions; it is thus necessary to involve higher order interactions where quartet resonances may occur (Loesch, 1978).

Another problem still, pointed out by Holloway (1979), arises when the RI is applied to the discrete problem (periodic β-plane or sphere): the only resonances retained are then the intersections of the lattice of possible triads with the hypersurface of ξ^3 which is the locus of resonant interactions. One can show easily that, save for a few pathological cases which disappear after taking in account the free surface condition, the only resonant interactions are isosceles interactions, the apex of which are zonal modes: in spherical geometry, they read $(l_0, 0; l, m; l, -m)$. Such interactions produce only exchanges between (l, m) and $(l, -m)$, namely a simple phase rotation of the whole associated complex mode; the zonal mode remains unchanged. Thus, the RI applied to the discrete problem yields a completely linear system; spherical harmonics (or Fourier modes in the β plane) are still eigenfunctions of the whole system, but the frequencies depend now not only on the zonal solid body rotation but on the whole zonal energy spectrum. This limit for weak modes or large β is obtained in numerical results reported in Section 6. Nevertheless, according to (22) resonance broadening occurs for finite amplitude modes: among all triads with at least one mode in the wave domain, the essential part of the transfer is due to the almost resonant ones which satisfy $\omega_{\alpha\beta\gamma} \simeq \nu^*_{\alpha\beta\gamma}$. This condition may be fulfilled in local triads where the three frequencies are of the same order. For nonlocal triads, the frequency of the mode of lowest degree, if non-zonal, dominates, and thus we are far from the resonance conditions; if the mode of lowest degree is zonal, while the other two lie in the turbulent domain, we are close to resonance and the interaction is not inhibited.

This suggests that dispersive effects yield more local transfer than expected in ordinary two dimensional turbulence; they systematically inhibit non-local transfer, save those driven by zonal modes.

5. COMPLEX DAMPING

We have so far regarded the damping coefficient as real and positive. In the absence of propagation effects, this property arises in a natural way from the vorticity equation which we write here symbolically† as

$$(d/dt+\nu)\zeta=i\lambda\zeta\zeta, \tag{23}$$

in the spherical case; λ stands for a real coefficient of interaction. If we suppose that in the initial state, all even order moments are real and all odd order moments purely imaginary, it is clear from (23) that this remains true during the ensuing evolution. In order to conserve this property in the EDQNM model, the relaxation rate of triple correlations must be real.

Now, in the presence of waves, the vorticity equation reads (again in symbolic form)

$$(d/dt+\nu+i\omega)\zeta=i\lambda\zeta\zeta, \tag{24}$$

so that propagative terms do exist at all orders of the hierarchy. Because of the phase rotation they induce, all non-quadratic moments become complex during the evolution of the system, whatever the initial conditions, and the *a priori* obligation to obtain real relaxation coefficients disappears; it will be shown in what follows that one must indeed associate an imaginary part to the damping. In other words, the action of $n+1$st order moments upon nth order moments is not only characterised by a relaxation but also by a phase displacement. Everything which has been said about our previous formulation of EDQNM extends at once to the complex damping case. In fact, only the real part of the Green function $G_{\alpha\beta\gamma}$ has been taken into account in (15).

The method we shall follow relies on a formulation propounded by André (1974) leading to a systematic calculation of damping rates in the case of isotropic and homogeneous two-dimensional turbulence. In order

†In planar geometry, one writes $(d/dt+\nu)\zeta=\lambda\zeta\zeta$, so that the argument is valid if he supposes all moments initially real.

to clarify the ideas involved, we only give a symbolic exposition of the method, referring the reader interested in more technical details to the aforementioned paper.

The statistical distribution of flow realisations may be described using a probability law $P\{\zeta\}$, defined over the set ζ of all allowable values for harmonic coefficients. We are concerned with the moments $T^{(j)}_{\alpha_1\ldots\alpha_j} = \langle\zeta_{\alpha_1}\ldots\zeta_{\alpha_j}\rangle$ which can be defined as derivatives of the first characteristic function,

$$\mathscr{T}(A)=\int e^{A\zeta}P(\zeta)\,d\zeta,$$

in the form

$$\partial^j\mathscr{T}/\partial A_{\alpha_1}\ldots\partial A_{\alpha_j}=T^{(j)}_{\alpha_1\ldots\alpha_j}.$$

The cumulants $C^{(j)}_{\alpha_1\ldots\alpha_j}$ are defined as the derivatives of the second characteristic function $\mathscr{L}(A)=\log\mathscr{T}(A)$ in the form

$$\partial^j\mathscr{L}/\partial A_{\alpha_1}\ldots\partial A_{\alpha_j}=C^{(j)}_{\alpha_1\ldots\alpha_j}.$$

It is easy to show that whenever the first order averages $\langle\zeta_\alpha\rangle$ vanish cumulants are identical with moments up to the third order; at fourth order, the homogeneity conditions yield

$$\begin{aligned}C^{(4)}_{\alpha_1\alpha_2\alpha_3\alpha_4}=T^{(4)}_{\alpha_1\alpha_2\alpha_3\alpha_4}&-\delta(m_{\alpha_1}+m_{\alpha_2})\delta(m_{\alpha_3}+m_{\alpha_4})T^{(2)}_{\alpha_1\alpha_2}T^{(2)}_{\alpha_3\alpha_4}\\&-\delta(m_{\alpha_1}+m_{\alpha_3})\delta(m_{\alpha_2}+m_{\alpha_4})T^{(2)}_{\alpha_1\alpha_3}T^{(2)}_{\alpha_2\alpha_4}\\&-\delta(m_{\alpha_1}+m_{\alpha_4})\delta(m_{\alpha_2}+m_{\alpha_3})T^{(2)}_{\alpha_1\alpha_4}T^{(2)}_{\alpha_2\alpha_3}.\end{aligned}$$

The major interest of using cumulants rather than moments in the formulation of the statistical theory of turbulence comes from the fact that cumulants all vanish beyond the third order for a gaussian distribution. They allow simpler formulations of high order closures when the flow is supposed to be almost gaussian. Denoting that jth order cumulant by the abridged form C_j, the complete heirarchy of cumulant equations reads symbolically

$$(d/dt+\nu_2)C_2=\sum_{\{2\}\to\{3\}}i\lambda C_3, \tag{25.2}$$

$$(d/dt+\nu_3+i\omega_3)C_3=i\sum_{\{3\}\to\{4\}}\lambda C_2C_2+i\sum_{\{3\}\to\{4\}}\lambda C_4, \tag{25.3}$$

$$(d/dt+\nu_4+i\omega_4)C_4=i\sum_{\{4\}\to\{5\}}\lambda C_2C_3+i\sum_{\{4\}\to\{5\}}\lambda C_5, \tag{25.4}$$

$$(d/dt+\nu_5+i\omega_5)C_5=i\sum_{\{5\}\to\{6\}}\lambda C_2C_4+i\sum_{\{5\}\to\{6\}}\lambda C_3C_3+i\sum_{\{5\}\to\{6\}}\lambda C_6, \tag{25.5}$$

$$(d/dt+\nu_j+i\omega_j)C_j=i\sum_{\{j\}\to\{j+1\}}\sum_{l=2}^{j-1}\lambda C_lC_{j+1-l}+i\sum_{\{j\}\to\{j+1\}}\lambda C_{j+1}. \tag{25.j}$$

On the right hand side of Eq. (25.j) for the jth order cumulant, one finds a sum of $(j+1)$st-order cumulants and a series of products of lower order cumulants. The symbol $\{j\}\to\{j+1\}$ means building from the j-tuplet $\{j\}$ all the $(j+1)$-tuplets $\{j+1\}$ obtained by replacing one mode of $\{j\}$ by two other ones coupled with it in a triad. In addition, homogeneity implies that a number of cancellations occur in the first summation of the right hand side of (25.j).

In André's approximation, which can be considered as a weak form of the quasi-normal approximation, one neglects all products of cumulants of order greater than 3 in the right-hand side of (25.j). A simplified hierarchy obtains, whose general equation reads

$$(d/dt+\nu_j+i\omega_j)C_j=i\sum_{\{j\}\to\{j+1\}}\lambda C_2C_{j-1}+i\sum_{\{j\}\to\{j+1\}}\lambda C_{j+1}. \tag{26.j}$$

We suppose that energy (namely cumulants C_2) varies on a time scale larger than those of all other cumulants C_j, $j\geqq 3$; this assumption is *a priori* all the more true as j becomes greater. Thus, if one keeps the energy C_2 fixed, the simplified hierarchy is linear for all orders greater than 3. Suppose now an Nth-order closure of eddy damping type, denoted by

$$i\sum\lambda C_{N+1}=-{}^N\mu C_N.$$

This assumption is found consistent because it implies for all orders $2\leqq j<N$

$$i\sum\lambda C_{j+1}=-{}^j\mu C_j. \tag{27}$$

where ${}^j\mu$ depends only on j and the energy spectrum. Without rotation, ${}^j\mu$ must be real; here, we generalize it by allowing the existence of an imaginary part. The previous expression (27) may be replaced in (26.j) and gives

$$(d/dt+\nu_j+i\omega_j+{}^j\mu)C_j=i\sum_{\{j\}\to\{j+1\}}\lambda C_2C_{j-1}, \tag{28.j}$$

which is the jth order closure of the hierarchy. C_j is then eliminated in Eq. (26.$(j-1)$) using (28.j) and the markovianisation method in the same way as in Section 4 for triple correlations. One obtains

$$(d/dt+\nu_{j-1}+i\omega_{j-1})C_{j-1}=i\sum_{(j-1)\to(j)}\lambda C_2 C_{j-2} - \sum_{(j-1)\to(j)}\lambda\lambda[C_2/(i\omega_j+\nu_j+{}^j\mu)]C_{j-1},$$

which is consistent with closure (28.$(j-1)$) if we assume the following recurrence relation

$$^{j-1}\mu=\sum_{(j-1)\to(j)}\lambda\lambda C_2/(i\omega_j+\nu_j+{}^j\mu), \tag{29}$$

or, more precisely, dropping the symbolic formulation,

$$^{j-1}\mu_{\alpha_1\ldots\alpha_{j-1}}=\left\{\sum_{\beta,\gamma}\frac{b_{\bar\alpha_1\beta\gamma}Z_\beta}{{}^j\mu_{\beta\gamma\alpha_2\ldots\alpha_{j-1}}+\nu_{\beta\gamma\alpha_2\ldots\alpha_{j-1}}i\omega_{\beta\gamma\alpha_2\ldots\alpha_{j-1}}}\right\} +\{\alpha_1\leftrightarrow\alpha_2\}+\ldots+\{\alpha_1\leftrightarrow\alpha_j\}, \tag{30}$$

where the symbol $\{\alpha_1\leftrightarrow\alpha_i\}$ represents the term obtained by applying the prescribed permutation in the first bracket of the right-hand side. At this point, relation (30) defines a complex damping rate suitable for the whole truncated hierarchy if a boundary condition for μ is provided at the truncation order $j=N$. Both effects, damping and dephasing, are present in (30) as expected. However, expression (30) is too complicated to be practically tractable: we thus need further simplifications.

Suppose now that the damping of the cumulant $C_{\alpha_1\ldots\alpha_j}$ results from the addition of independent dampings for each mode. This implies

$$^j\mu_{\alpha_1\ldots\alpha_j}={}^j\mu_{\alpha_1}+{}^j\mu_{\alpha_2}+\ldots+{}^j\mu_{\alpha_j},$$

where ${}^j\mu_\alpha={}^j\bar\mu_\alpha$. This assumption is analogous to relation (19), assumed for the EDQNM; it leads to similar difficulties, which we examine below. Consistency with relation (30) holds if one can identify ${}^j\mu_{\alpha_1}$ with the first bracket of the right hand side, ${}^j\mu_{\alpha_2}$ with the second one and etc.... We thus need to replace in the first bracket all terms which depend on $\alpha_2,\ldots,\alpha_{j-1}$ by mean quantities; we assume

$$^j\mu_{\alpha_2}+{}^j\mu_{\alpha_3}+\ldots+{}^j\mu_{\alpha_{j-1}}\simeq(j-2)\langle{}^j\mu\rangle_{\alpha_1},$$

$$\nu_{\alpha_2}+\nu_{\alpha_3}+\ldots+\nu_{\alpha_{j-1}}\simeq(j-2)\langle\nu\rangle_{\alpha_1},$$

$$\omega_{\alpha_2}+\omega_{\alpha_3}+\ldots+\omega_{\alpha_{j-1}}\simeq(j-2)\langle\omega\rangle_{\alpha_1},$$

where $\langle {}^j\mu\rangle_{\alpha_1}$, $\langle \nu\rangle_\alpha$ and $\langle\omega\rangle_\alpha$ are averages to be precised. One *a priori* expects $Re\langle {}^j\mu_\alpha\rangle$ to increase and $Im({}^j\mu_\alpha)$ to decrease on the average with the degree. The identification of $Re({}^{j-1}\mu_{\alpha_1})$ to the real part of the first bracket of (30) is thus meaningful only if degrees l_{α_i} greatly superior to l_{α_1} are not present among $l_{\alpha_2},\ldots,l_{\alpha_{j-1}}$; otherwise, $Re({}^{j-1}\mu_{\alpha_1})$ would be dominated by $Re({}^{j-1}\mu_{\alpha_i})$ in the sum $Re({}^{j-1}\mu_{\alpha_1}\ldots\alpha_{j-1})$, and the identification would lead to an error of the same order as $Re({}^{j-1}\mu_{\alpha_1})$. On the other hand, identifying $Im({}^{j-1}\mu_{\alpha_1})$ with the imaginary part of the first bracket is only meaningful if degrees much smaller than l_{α_1} are not present among $l_{\alpha_2},\ldots,l_{\alpha_{j-1}}$. This suggests defining $\langle\nu\rangle_{\alpha_1}$ and $\langle {}^jRe(\mu)\rangle_{\alpha_1}$ as the average of ν and $Re(\mu)$ over all modes of degree l_α smaller than l_{α_1}. For the imaginary part of the denominator, we assume the incoherence hypothesis

$$\langle\omega\rangle_{\alpha_1}=0,\quad \langle Im\,{}^j\mu\rangle_{\alpha_1}=0,$$

which yields the following recursion relation

$${}^j\mu_\alpha=\sum_{\beta,\gamma}\frac{b_{\bar\alpha\beta\gamma}Z_\beta}{{}^{j+1}\mu_\beta+{}^{j+1}\mu_\gamma+(j-1)\langle {}^{j+1}\mu\rangle_\alpha+\nu_\beta+\nu_\gamma+(j-1)\langle\nu\rangle_\alpha+i\omega_\beta+i\omega_\gamma}.\tag{31}$$

Like the usual EDQNM damping (19), relation (31) overestimates the effects of non-local interactions. The most natural way of introducing the TFM correction in this new model is to replace the coefficients $b_{\alpha\beta\gamma}$ in (31) by $\hat b_{\alpha\beta\gamma}$, derived in the appendix:

$${}^j\mu_\alpha=g^2\sum_{\beta,\gamma}\frac{\hat b_{\bar\alpha\beta\gamma}Z_\beta}{{}^{j+1}\mu_\beta+{}^{j+1}\mu_\gamma+(j-1)\langle {}^{j+1}\mu\rangle_\alpha+\nu_\beta+\nu_\gamma+(j-1)\langle\nu\rangle_\alpha+i\omega_\beta+i\omega_\gamma}.\tag{32}$$

In order to define the complex damping completely, one needs to initiate the recursion process. For sufficiently large values of j, the denominator of (32) is dominated by $(j-1)[\langle {}^{j+1}\mu\rangle_\alpha+\langle\nu\rangle_\alpha]$; moreover, $Re\,{}^j\mu_\alpha$ is $O(1/j)$ so that the term $\langle {}^{j+1}\mu\rangle_\alpha$ is also negligible compared with $\langle\nu\rangle_\alpha$, and $Im\,{}^j\mu_\alpha$ is $O(1/j^2)$. At higher orders, stochastic damping is dominated by viscosity and (32) can be completed by ${}^{j_0}\mu_\alpha\equiv 0$ for a sufficiently large j_0.

The damping ${}^3\mu_\alpha$ can be used in a modified version of EDQNM: in the equation for triple correlations, the dissipative term is renormalised by the real part of the damping to give

$$\nu^*_{\alpha\beta\gamma}=\nu_{\alpha\beta\gamma}+Re({}^3\mu_\alpha+{}^3\mu_\beta+{}^3\mu_\gamma),$$

whereas the imaginary part renormalises the dispersive term which in turn yields

$$\omega^*_{\alpha\beta\gamma}=\omega_{\alpha\beta\gamma}+Im({}^3\mu_\alpha+{}^3\mu_\beta+{}^3\mu_\gamma).$$

The characteristic time $\theta_{\alpha\beta\gamma}$ becomes

$$\theta_{\alpha\beta\gamma}=v^*_{\alpha\beta\gamma}/(v^{*2}_{\alpha\beta\gamma}+\omega^*_{\alpha\beta\gamma}). \tag{33}$$

From the standpoint of resonance interactions theory, expression (33) is understandable as a broadening associated with a shift of resonance; in Holloway's expression (22), only broadening is present. The existence of a frequency shift is not surprising: indeed, one knows that deterministic systems with a few non-linear waves do show frequency shifts; as recent examples, one can refer to Loesch (1978) for the β-plane problem and to Chakraborty and Chandra (1978) for a wave problem in a cold plasma. The existence of an imaginary part of the stochastic damping is nothing but a statistical extension of these results.

We are now interested in the behaviour of stochastic damping for high degree modes. If on the average, the real part of damping increases with degree, there must exist a degree l_0 beyond which $Re\mu_\alpha$ dominates all Rossby frequencies of the system, as well as frequency shifts; this defines Rhines' turbulent domain, within which damping by non-local processes dominate, like in two-dimensional ordinary turbulence.† In non-local interactions $(\bar{\alpha},\beta,\gamma)$, $l_\beta \ll l_\alpha \sim l_\gamma$, $Re({}^{j+1}\mu_\beta)$ is negligible compared to $Re({}^{j+1}\mu_\alpha)$ which itself is very close to $Re({}^{j+1}\mu_\gamma)$. Similar inequalities hold for the viscosity terms; moreover $\langle {}^{j+1}\mu\rangle_\alpha$ and $\langle v\rangle_\alpha$ may be replaced by $Re({}^{j+1}\mu_\alpha)$ and v_α respectively, after noting that modes of degrees close to l_α dominate within the averages. A simpler form of expression (32) thus reads:

$$Re^j\mu_\alpha \simeq g^2 \sum_{\substack{\beta,\gamma\\ NL}} \hat{b}_{\bar{\alpha}\beta\gamma} Z_\beta/[j(Re^{j+1}\mu_\alpha+v_\alpha)], \tag{34}$$

where summation is over nonlocal interactions. The contribution of non-local interactions to the imaginary part of the stochastic damping is

$$Im({}^j\mu_\alpha)_{NL}=-g^2\sum_{\beta,\gamma}\hat{b}_{\bar{\alpha}\beta\gamma}Z_\beta[\omega_\beta+\omega_\gamma+Im({}^{j+1}\mu_\beta) + Im({}^{j+1}\mu_\gamma)]/j^2[Re({}^{j+1}\mu_\alpha)+v_\alpha]^2. \tag{35}$$

†Non localness is strictly valid for k^{-3} or steeper energy spectrum. In a $k^{-5/3}$ spectrum, local transfers are not negligible.

Apparently, the terms with index β dominate on the right-hand side since ω_β is large compared to ω_γ. But in fact, to a non-local interaction $(\bar{\alpha}, \beta, \gamma)$ with $l_\beta \ll l_\alpha \sim l_\gamma$, γ close to α, one can generally associate another interaction $(\bar{\alpha}, \bar{\beta}, \gamma')$ with γ' again close to α; because of the regularity of 3-l symbols involved in (A.6), the corresponding coefficients $\hat{b}_{\alpha\beta\gamma}$ and $\hat{b}_{\alpha\bar{\beta}\gamma'}$ may be considered equal, so that when both interactions (α, β, γ) and $(\alpha, \bar{\beta}, \gamma')$ are coupled in (35) the $[\omega_\beta + Im({}^{j+1}\mu_\beta)]$ terms cancel. One therefore gets

$$Im({}^j\mu_{\alpha_{NL}}) \simeq -g^2 \sum_{\substack{\beta,\gamma \\ NL}} \hat{b}_{\bar{\alpha}\beta\gamma} Z_\beta [\omega_\alpha + Im({}^{j+1}\mu_\alpha)]/j^2[Re({}^{j+1}\mu_\alpha) + \nu_\alpha]^2. \quad (36)$$

This contribution is large compared with the one coming from local interactions where all quantities of the same kind indexed by either α or β have the same order of magnitude: therefore, the contributions from local $(\alpha', \alpha, \alpha'')$ and nonlocal $(\alpha', \beta, \alpha'')$ interactions typically differ by a ratio $3Z_\alpha/2Z_\beta$, which is small because $l_\beta \ll l_\alpha$. Thus

$$Im({}^j\mu_\alpha) \simeq Im({}^j\mu_{\alpha_{NL}}).$$

Relations (34) and (36) suggest a self-similarity hypothesis for stochastic damping: we suppose that both parts of the damping depend on order j only by a scaling factor

$$Re({}^{j+1}\mu_\alpha) = e_j Re({}^j\mu_\alpha), \quad Im({}^{j+1}\mu_\alpha) = f_j Im({}^j\mu_\alpha).$$

From relation (34), one thus obtains

$$[je_j - (j+1)e_{j+1}e_j^2] Re({}^j\mu_\alpha) = [(j+1)e_j - j]\nu_\alpha.$$

For large j, ν_α dominates $Re({}^j\mu_\alpha)$ and we have $e_j = j/(j+1) \simeq 1$; for small j and modes outside the dissipation range, $Re({}^j\mu_\alpha)$ dominates dissipation, so that

$$Re({}^j\mu_\alpha) = g[(je_j)^{-1} \sum_{\beta,\gamma} \hat{b}_{\alpha\beta\gamma} Z_\beta]^{1/2}, \quad (37)$$

together with the recursion relation $e_j e_{j+1} = j/(j+1)$, which, iterated from $e_\infty = 1$ gives $e_3 = 0.85$. Thus

$$Re^3\mu_\alpha \simeq \lambda \left[\sum_{\substack{\beta \\ l_\beta < l_\alpha}} Z_\beta \right]^{1/2}, \quad (38)$$

with $\lambda = 0.63g$ since $\hat{b}_{\alpha\beta\gamma}$ is always close to unity for nonlocal triads. The coefficient λ is given equal to 0.53 by Basdevant *et al.* (1977), which yields

$g=0.84$. Expression (37) is now used to eliminate $Re(^j\mu_\alpha)$ in (36). If viscosity can be neglected, one obtains

$$Im(^j\mu_\alpha) = -\omega_\alpha/(je_j+f_j), \tag{39}$$

with f_j satisfying the recurrence relation $f_j=je_j[(j/e_j)+f_{j+1}-1]^{-1}$, which iterated from $f_\infty=1$ gives $f_3=0.76$ and $Im(^3\mu_\alpha)=-0.30\omega_\alpha$. In the turbulent domain, the ratio of frequency shift to Rossby frequency appears to be a constant over all modes. This constant, dependant neither on g nor on the spectrum (as long as non-localness applies), is equal to -0.3 for $j=3$. Notice that, since e_j and f_j are always close to 1, a good approximation for all j is $Im(^j\mu_\alpha) \simeq -\omega_\alpha/(j+1)$.

6. NUMERICAL RESULTS

To illustrate the results obtained in the last section, we have computed the stochastic damping as defined by (28) for several energy spectra and several values of the dispersion parameter.

Since interactions between waves and turbulence occur essentially in the largest scales, it is unnecessary to require high resolution. We used a spherical model with triangular truncation at $L_{max}=21$; 483 modes are thus retained, coupled by 100,701 triadic interactions in the TFM. Indeed, we can hardly reach much higher resolutions if we retain all interactions since their number grows as L^5_{max}.

To obtain $^3\mu_\alpha$, the damping which acts on triple correlations, we have iterated (28) from a purely viscous damping at order j_0 ($^{j_0}\mu_\alpha \equiv 0$): j_0 has been chosen large enough to ensure convergence of the series. In every case studied, the number of iterations has been no more than a few tens. The g factor has been taken equal to 1. We characterise dispersion by the Rossby number $R_0=\sqrt{Z}/2\Omega$ where Z is the relative enstrophy, and dissipation by the Reynolds number $R_L=R^2\sqrt{Z}/\nu$. The spherical counterpart for the transition wavenumber k_β of Rhines is $l_\Omega=R_0^{-1}$. In the atmosphere, R_0 is close to 0.1; transition between waves and turbulence domains is thus expected to occur for $l_\Omega \simeq 10$.

Figure 1 shows the real damping $Re(^3\mu_\alpha)$ for j^{-3} inertial range with $R_0=0.1$ and $R_L=100{,}000$. Sections of constant order m with respect to degree are drawn with a solid line. The homogeneous damping obtained for $R_0=\infty$ pure turbulence, is drawn with a discontinuous line. Clearly, the presence of waves induces a decrease of the damping more and more pronounced as one goes away from zonality ($m=0$). Damping is observed to drop near truncation due to the cutoff of interactions involving modes

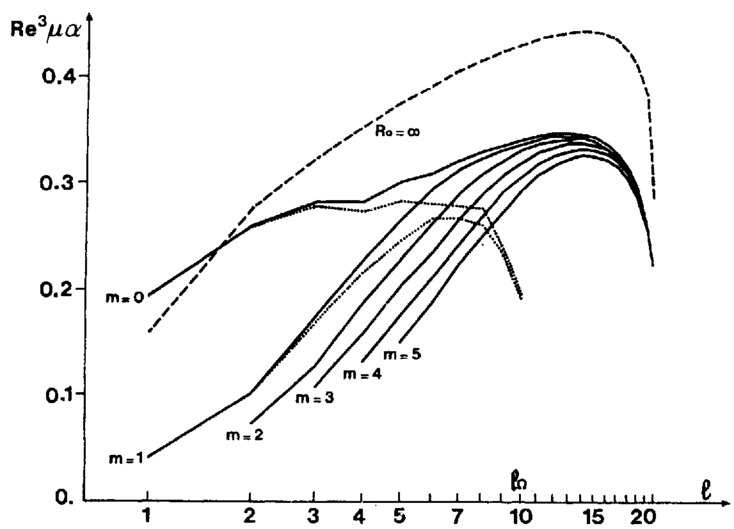

FIGURE 1 Real damping $Re^3\mu_\alpha$ for an homogeneous isotropic j^{-3} inertial range with $R_0=0.1$ and $R_L=100{,}000$. In solid lines: the sections of constant order, m, drawn with respect to degree. In dotted lines: curves obtained by setting $l_{max}=10$. In discontinuous line: the homogeneous damping obtained with $R_0=\infty$.

beyond L_{max}. The dotted line curves were obtained by setting $L_{max}=10$, everything else remaining as before; below $l=6$ where the drop now occurs, the curves agree with the previous ones, obtained with $L_{max}=21$. The value chosen for R_L is not critical. Indeed, the damping remains almost unchanged for any value $R_L>10{,}000$; thus, the dissipation degree is $l_d>100$ for a j^{-3} energy spectrum, largely beyond L_{max}. Viscosity acts only as a germ from which the damping builds up by stochastic processes.

Figure 2 reports the variation of damping with respect to R_0 for $m=0-1-2$, using a j^{-3} spectrum again. For zonal modes, it remains of the same order as R_0 decreases; one even observes an increase of damping at lower degrees as expected for the TFM (cf. Appendix). For non-zonal modes (here $m=1,2$), the decrease of damping as R_0 decreases is more pronounced at lower degrees and higher orders. These curves are to be compared to those reported in Figure 3 for transfers η_α.† One can clearly see the inhibition of zonal transfers: for $R_0=0.1$, the ratio to the transfer in the absence of waves is 0.1; for $R_0=0.001$ the ratio is only 10^{-5}.

†Let us state that η_α is the linear part of the equation for Z_α, viz. (16). The total transfer involves also terms like $Z_\beta Z_\gamma$ which can be considered as sources since they do not vanish with Z_α.

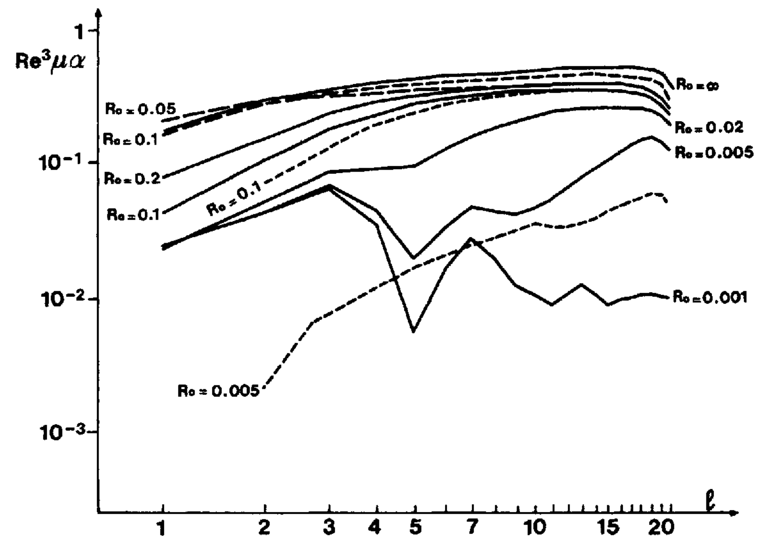

FIGURE 2 Real damping $Re^3\mu_\alpha$ for several values of R_0 with a j^{-3} inertial range. In solid line: $Re^3\mu_l^1$. In short broken-line: $Re^3\mu_l^2$. In long broken line: ${}^3\mu_l^0$.

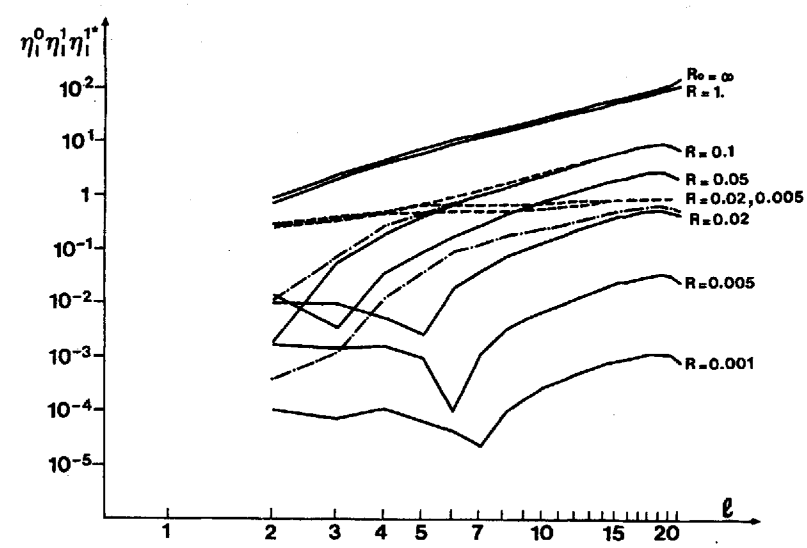

FIGURE 3 Transfers η_α for a j^{-3} inertial range. In solid line: η_l^0. In broken line: η_l^1. In mixed line: η_l^{1*}.

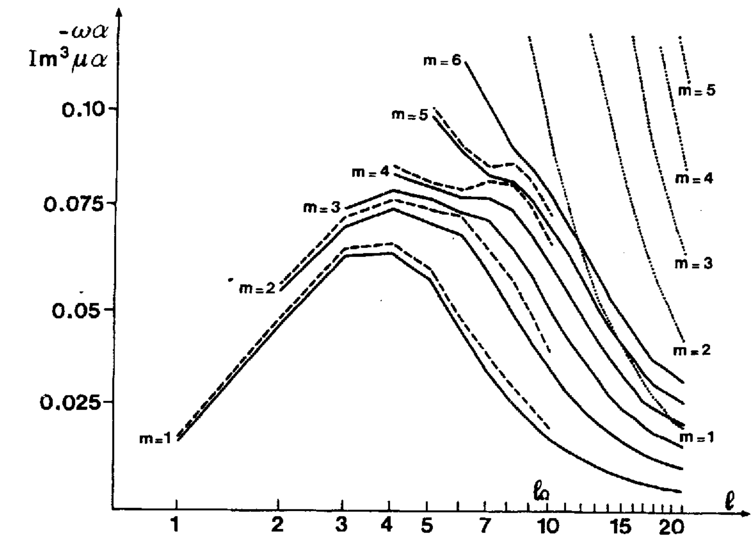

FIGURE 4 Frequency shift $Im^3\mu_\alpha$ for a j^{-3} inertial range. Solid line: the section of constant order m drawn with respect to degree. In broken line: curves obtained by setting $l_{max}=10$. Dotted line: negative of Rossby frequencies.

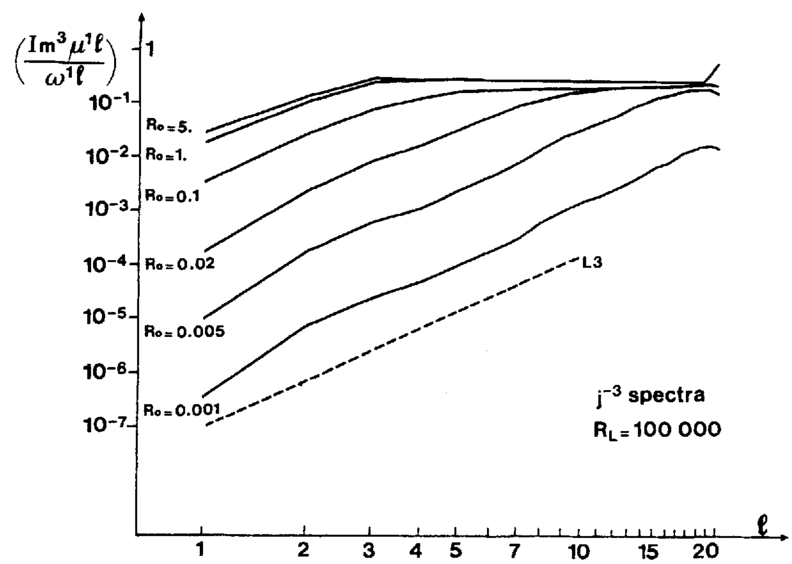

FIGURE 5 Relative shift $Im^3\mu_l^1/\omega_l^1$ for a j^{-3} spectrum. The slope l^3 is plotted with a broken line.

Further, the inhibition is found to have autosimilarity for high degrees. On the other hand, the non-zonal transfer η_l^1 tends to a limit due to the existence of resonances which transfer energy between the two components of each complex mode (cf. Section 3). With such interactions dropped, the remaining transfer η_l^{1*} is inhibited and does not tend to a limit; nevertheless, inhibition is smaller than for zonal transfer, due to the effect of transfer by almost resonant triads.

Figure 4 shows the frequency shift $Im(^3\mu_\alpha)$ for the same values of the parameters as in Figure 1. On the same figure are plotted in dotted lines the negative of the Rossby frequencies for $R_0=0.1$; the shifts appear to be opposite in sign to the Rossby frequencies. On low order curves, the shift is maximum for $l_M=4$ and decreases beyond l_M as expected, since Rossby frequencies themselves decrease. At lower degrees, the shifts are negligible compared to the Rossby frequencies themselves, but it appears in Figure 5 that their relative importance increases with degree. Figure 5 shows the relative shift $Im(^3\mu_l^1)/\omega_l^1$ for several values of R_0. In the limit $R_0 \rightarrow 0$, nonlinear effects become negligible (cf. Section 3), the problem tends to be linear and thus no shift at all can occur. It may be observed that the increase of the relative shift with degree follows approximately a j^3 law. In the high degree limit, the relative correction tends towards an asymptotic value equal to 0.28 which is very close to the value found in Section 5. The limit is reached very rapidly when R_0 is large enough, practically at $l=3$ in the case $R_0=5$. Figure 6 reports the relative shifts obtained with several energy spectra and with $R=0.1$. The curves associated with the j^{-3} spectrum and energy equipartition respectively bracket the other ones; these extreme cases correspond to maximum and minimum concentration of enstrophy in the lowest degrees. The most interesting feature is that all curves tend towards a nearly identical limit; this confirms the independence of the limit ratio with regard to the spectrum even if interactions are strongly local as happens in the case of an energy equipartition spectrum. This behaviour seems to be more general than simply suggested by the results of Section 5.

7. CONCLUDING REMARKS

We have shown the existence of a statistical dephasing effect, induced by turbulent processes on Rossby waves. An estimate of the shift, based on a simple model, shows that its relative value tends, as l increases, towards a limit which does not depend on the energy spectrum. This effect is

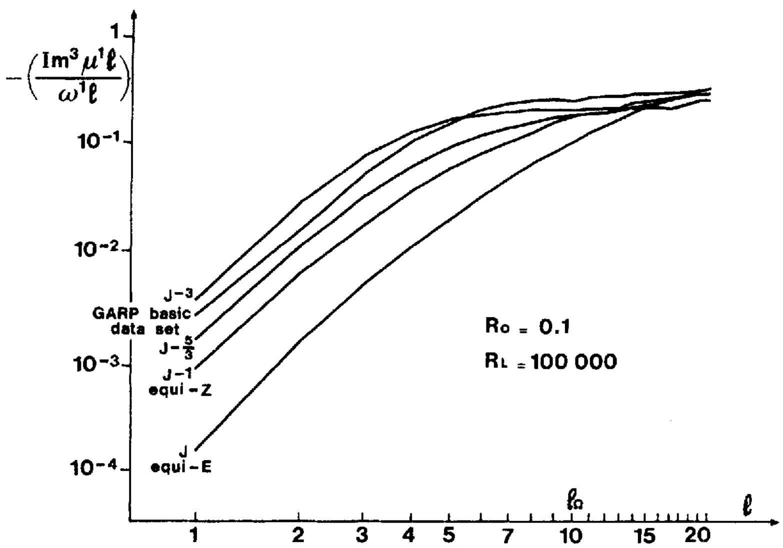

FIGURE 6 Relative shift $Im^3\mu_l^1/\omega_l^1$ for several energy spectra with $R_0=0.1$.

certainly not negligible, especially in the transition zone between wave and turbulence domains.

Clearly, the easiest criticism to level at the present method arises from the existence of high order resonances which are poorly taken in account by the simplifying processes which yield (31). In fact, high order resonances are less sharp than low order ones, since real damping is more important there. A refinement of (31) would consist in a more accurate estimate of the last iterations, near $j=3$.

One may hope that the use of renormalisation techniques, recently introduced in turbulence, will bring further clarification of this problem, at least from the phenomenological standpoint.

Acknowledgements

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Appendix

SPHERICAL TEST FIELD MODEL

We do consider here the overestimation of damping in the EDQNM approximation. This problem can be considered independently from waves: it was originally pointed out by Kraichnan (1964) within the frame of the DIA equations. Non-linear interactions ($l_\alpha \ll l_\beta \sim l_\gamma$) are largely dominated by simple advection of ζ_β and ζ_γ by ζ_α: small scale oscillations observed from a eulerian standpoint are produced by this sweeping process rather than by intrinsic deformation of structures. By using the coefficients $b_{\alpha\beta\gamma}$ in building the damping mechanism, we do not separate the sweeping effect from the deformation effect and interpret it as a relaxation process although it is an extrinsic bias related to the observer's viewpoint. Thus we overestimate the relaxation of correlations more and more as non-localness increases.

The Test Field Model (Kraichnan, 1971; Sulem *et al.*, 1975) remedies this problem in a semi-heuristic way. In the TFM, one notices first that the distortion of structures in non-local interactions is caused by the pressure term in the Navier Stokes equation. Since the same pressure term is also responsible for the incompressibility of the flow, one can imagine evaluating its effects by measuring the rate of growth of the compressible part of the flow when the assumption of incompressibility is relaxed. This is modelled by considering a vector test field **U** simply advected by the velocity field of the flow.

We give below a brief sketch of the TFM formulation in spherical geometry, using scalar function notation. A description of the planar form of the TFM using the same notations can be found in Holloway (1977); we shall therefore reproduce here the principal results of this paper without demonstration. The incompressible advecting field is defined by a streamfunction Ψ, so that $\mathbf{v} = \mathbf{k} \times \nabla\Psi$. The advected field possesses a solenoidal part and a divergent part: it is therefore defined by a streamfunction ψ and a velocity potential χ: $\mathbf{u} = \mathbf{k} \times \nabla\psi + \nabla\chi$.

The advection equations in the inviscid case read

$$\begin{cases} (\partial/\partial t)\nabla^2\psi + K(\Psi, \chi) + H(\Psi, \psi) = 0, & \text{(A.1a)} \\ (\partial/\partial t)\nabla^2\chi - K(\Psi, \psi) + H(\Psi, \chi) = 0, & \text{(A.1b)} \end{cases}$$

where H is an auto-advective operator which reduces to the jacobian if $\mathbf{u} = \mathbf{v}$ and K a symmetric operator expressed as

$$K(\Psi, \chi) = \Psi_{xx}\chi_{yy} + \Psi_{yy}\chi_{xx} - 2\Psi_{xy}\chi_{xy}$$

on the plane, or

$$K(\Psi,\chi)=\sec^2\phi[\Psi_\lambda\chi_{\phi\phi}+\Psi_{\phi\phi}\chi_\lambda]_\lambda-\sec\phi[\Psi_\phi\chi_\phi\sin\phi]_\phi-\sec\phi\left[\frac{1}{\cos\phi}\psi_\lambda\chi_\lambda\right]_{\phi\phi}. \tag{A.2}$$

on the sphere (the subscripts here indicate derivatives). In (A.1) we are especially interested in the K-terms which give the distortion rate by coupling each part of the test field to the other: removing the self-advective terms expressed by the jacobians, we obtain

$$\begin{cases}(\partial/\partial t)\nabla^2\psi+K(\Psi,\chi)=0, & \text{(A.3a)}\\ (\partial/\partial t)\nabla^2\chi-K(\Psi,\psi)=0. & \text{(A.3b)}\end{cases}$$

It can be shown, by integrating by parts, that Eqs. (A.3) are conservative for the total energy of advected field; in planar geometry we have

$$\int d\sigma\,\psi K(\Psi,\chi)=\int \alpha\sigma[\psi_x\Psi_x\chi_{yy}+\psi_{yy}\Psi_x\chi_x+\psi_x\Psi_{yy}\chi_x]$$

and in spherical geometry

$$\int d\sigma\psi K(\Psi,\chi)=\int d\sigma[-\sec^2\phi(\psi_\lambda\Psi_\lambda\chi_{\phi\phi} +\psi_{\phi\phi}\Psi_\lambda\chi_\lambda+\psi_\lambda\Psi_{\phi\phi}\chi_\lambda)+\psi_\phi\psi_\phi\chi_\phi\tan\phi]. \tag{A.4}$$

Conservation of energy is thus a simple consequence of the three-fold symmetry in (A.4). One must remark however that enstrophy is not conserved.

In order to write the spectral equations in a consistent way with (4), we define the vorticities of **U** and **V** and the divergence of **V** by

$$\zeta=\nabla^2\Psi,\quad \xi=\nabla^2\psi,\quad \delta=\nabla^2\chi.$$

The spectral equations then read

$$\begin{cases}(\partial/\partial t)\bar{\xi}_\alpha(t)=-\sum_{\beta,\gamma}K_{\alpha\beta\gamma}\zeta_\beta(t)\delta_\gamma(t), & \text{(A.5a)}\\ (\partial/\partial t)\bar{\delta}_\alpha(t)=\sum_{\beta,\gamma}K_{\alpha\beta\gamma}\zeta_\beta(t)\xi_\gamma(t), & \text{(A.5b)}\end{cases}$$

with the following expression for the real coefficient K in terms of 3-l symbols:

$$K_{\alpha\beta\gamma}=(4j_\beta^2 j_\gamma^2)^{-1}[(2l_\alpha+1)(2l_\beta+1)(2l_\gamma+1)]^{1/2}$$
$$\times[2j_\alpha^2 j_\beta^2+2j_\beta^2 j_\gamma^2+2j_\gamma^2 j_\alpha^2-j_\alpha^4-j_\beta^4-j_\gamma^4]$$
$$\times\begin{pmatrix} l_\alpha & l_\beta & l_\gamma \\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} l_\alpha & l_\beta & l_\gamma \\ m_\alpha & m_\beta & m_\gamma\end{pmatrix}. \qquad \text{(A.6)}$$

Expression (A.6) can be obtained using the Wigner–Eckart theorem and relations between 3-l symbols given by Edmonds (1974). The corresponding coefficient in plane geometry is more simply

$$K_{\mathbf{kpq}}=|\mathbf{p}\times\mathbf{q}|^2/p^2q^2.$$

In planar geometry, the selection rules for non-linear interactions in the TFM and in the vorticity equation are almost identical, namely $\mathbf{k}+\mathbf{p}+\mathbf{q}=0$; the only difference is that isosceles triads are allowed in the TFM. In spherical geometry, the coefficient $K_{\alpha\beta\gamma}$ is non zero if the triad (α,β,γ) satisfies

$$\begin{cases} m_\alpha+m_\beta+m_\gamma=0, & \text{(A.7a)}\\ l_\alpha+l_\beta+l_\gamma \quad \text{even}, & \text{(A.7b)}\\ l_\alpha+l_\beta\geqq l_\gamma;\quad l_\beta+l_\gamma\geqq l_\alpha;\quad l_\gamma+l_\alpha\geqq l_\beta. & \text{(A.7c)}\end{cases}$$

Selection rules thus differ from those of the vorticity equation: the parity rule for the sum of degrees is inverted and the triangular inequalities are no longer strict. We shall see that this causes some ambiguity when waves are introduced in the TFM.

The damping rate is calculated from (A.5) in the same way as in Section 4. After a few manipulations, one obtains

$$\mu_\alpha(t)=g^2\sum_{\beta,\gamma}\hat{b}_{\alpha\beta\gamma}\hat{\theta}_{\alpha\beta\gamma}(t)Z_\beta(t), \qquad \text{(A.8a)}$$

where

$$\hat{b}_{\alpha\beta\gamma}=2K_{\alpha\beta\gamma}K_{\gamma\beta\alpha}. \qquad \text{(A.8b)}$$

A consistent expression for $\hat{\theta}_{\alpha\beta\gamma}$ is once more

$$\hat{\theta}_{\alpha\beta\gamma}=[\mu_\alpha+\mu_\beta+\mu_\gamma]^{-1}. \qquad \text{(A.8c)}$$

It is indeed the positivity of $\hat{b}_{\alpha\beta\gamma}$ which ensures the realizability of the model. g is a scaling factor, the value of which is adjusted by comparison of TFM results with other data. In two dimensional turbulence, one agrees to take g in the interval 0.7–1.0.

At present, the TFM appears to be the most efficient operational closure for developed turbulence problems (Pouquet *et al.*, 1975; Herring and Kraichnan, 1974). However, one may notice one important arbitrariness in the TFM formulation: for sufficiently local interactions, advection distorts the structures as much as pressure does, so that TFM phenomenology is hardly relevant. The TFM applies strictly as long as non-local processes dominate in the non-linear viscosity effect; this is true for homogeneous isotropic turbulence in two dimensions. Its validity, however, extends beyond this particular case: for example, the TFM has been successfully applied to homogeneous isotropic turbulence in three dimensions. This indicates that the TFM is able to give correct scalings even if the interactions are local. It is not obvious, nevertheless, that this remains true when finer effects like those due to anisotropy are studied.

Keeping these considerations in mind, the most natural way of extending the TFM to problems with viscosity and waves is to introduce the natural dissipative and dispersive terms in Eq. (A.5). This modification replaces thus (A.8c) by

$$\theta_{\alpha\beta\gamma} = |\mu_\alpha + \mu_\beta + \mu_\gamma + \nu_{\alpha\beta\gamma} + i\omega_{\alpha\beta\gamma}|^{-1},$$

where

$$\nu_{\alpha\beta\gamma} = \nu_\alpha + \nu_\beta + \nu_\gamma, \quad \omega_{\alpha\beta\gamma} = \omega_\alpha + \omega_\beta + \omega_\gamma.$$

In planar geometry, the definition of $\nu_{\alpha\beta\gamma}$ and $\omega_{\alpha\beta\gamma}$ is straightforward because TFM interactions and natural interactions act on the same triads; this property is no longer true in spherical geometry where natural and TFM interactions are "interleaved" with alternating parity rules. This peculiar feature poses no real problem as far as the dissipative term $\nu_{\alpha\beta\gamma}$ is concerned, because it is homogeneous and smoothly varying over the range of wavevectors. On the contrary, the dispersive term $\omega_{\alpha\beta\gamma}$ oscillates rapidly over wavevector space and requires a more precise definition. By computing it on TFM triads, one is likely to introduce displacements of resonance. For instance, triads composed of three zonal modes do transfer energy in the TFM, and further, will be automatically resonant with such a definition; one can thus expect an overestimation of relaxation for zonal modes.

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