MEC 654
Polytechnique-UPMC-Caltech
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Turbulence

teaching

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chapter 1

overture

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- **turbulent flows**

95% of industrial applications

- ✓ aeronautics
- ✓ automotive
- ✓ combustion
- ✓ nuclear power
- ✓ thermic
- ✓ meteorology
- ✓ magnetohydrodynamic
- ✓ process engineering
- ✓ …

overture

- **what is turbulence ?**

we dont know ! (no mathematical theory)

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- **synonyms of turbulence** in physics

✓ perturbation, agitation, exitment, instability, dissipation , disorder, dispersion incoherence, fragmentation , confusion, chaos , muddle, waste ...

✓ activity, animation, alert, mobility, vivacity, mixing , ardour, impetuosity, passion ...

- **antonyms**

✓ order composure, serenity, regularity , silence, peace ...

✓ resignation, stagnation , apathy, aphasia …

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• **turbulent systems**

✓ atmosphere
✓ flow interacting with an object
✓ stock exchange prices
✓ a crowd
✓ an individual
✓ …

• **tentative definitions of turbulence**

✓turbulence is a manifestation of the instability that characterizes the evolution of complex systems far from equilibrium

✓turbulence is a mechanism that regulates, on average, the behavior of systems out off equilibrium

overture

• **due to turbulence ...**

✓ energy consumption increases dramatically
✓ largest computers are far from being able to simulate the flow around
✓ an airplane
✓ …

• **without turbulence ...**

✓ no life on earth (huge winds, letal chimical elements stratified at ground level…)
✓ no combustion, no convection
✓ unsweetened coffee
✓ ...

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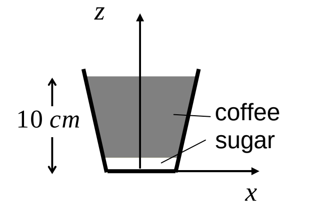
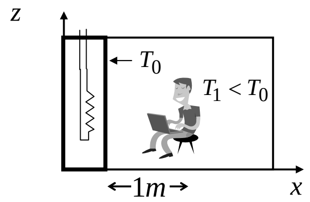
- **two molecular diffusion problems**



diffusion of sugar in a coffee cup



heat conduction in a room

- **simplification : semi-infinite geometries**

✓ invariance along x or z

✓ boundary conditions rejected at infinity in those directions

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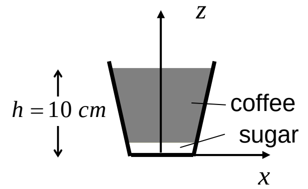
- **two molecular diffusion problems (…)**



diffusion of sugar in a coffee cup

✓ $n(z,t)$ = number of glucose molecules per volume

✓ D = molecular diffusivity of glucose in coffee , unit : $m^2 s^{-1}$

$$\boxed{\frac{\partial n}{\partial t} = D\,\frac{\partial^2 n}{\partial z^2}} \;\Rightarrow\; \boxed{n = f\left(z/\sqrt{Dt}\right)}$$

✓ molecular diffusivity : $D \approx 10^{-6}\ m^2 s^{-1}$

✓ time scale : $\tau_D\,(h) \sim h^2/D \approx 10^4 s \approx 3h$ (!) $\Rightarrow$ **turbulence may help**

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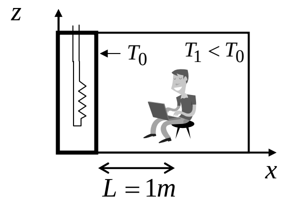
• **two molecular diffusion problems (…)**



heat conduction in a room

✓ $T(x,t)$ = temperature

✓ χ = thermal diffusivity, unit : $m^2 s^{-1}$

$$\frac{\partial T}{\partial t} = \chi \frac{\partial^2 T}{\partial x^2} \Rightarrow \frac{T - T_1}{T_0 - T_1} = g\left(x/\sqrt{\chi t}\right)$$

✓ thermal diffusivity (air at 0°) : $\chi \approx 10^{-7} m^2 s^{-1}$

✓ time scale : $\tau_\chi(L) \sim L^2/\chi \approx 10^7 s \approx 27\ jours\ (!)$ ⇨ **turbulence may help**

overture

• **turbulence**

✓ a challenge for **fondamental science**

✓ a daily problem for **engineers**

• **the course : objectives**

✓ understanding the **physical basis** of turbulence in fluids

✓ appraising **turbulence modelling** in engineering tools

• **the course : two aspects**

✓ in terms of trajectoiry : **irregularity**

✓ in term of statistics : **regularity**

chapter 2

continuum mechanics : a reminder

chapter 3

fluid mechanics : a reminder

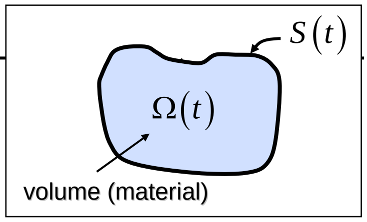
why ?

- ✓ turbulence is found in fluids, not in solids
- ✓ turbulence comes out from continuum mechanics equations, namely the Navier-Stokes equations if matter is a newtonian fluid
- ✓ observation shows that different fluids behaves differently in a turbulent regime
- ✓ low order descriptions are usually required to reduce the complexity of turbulence
- ✓ in practice, "modelling" turbulence states leads to describe turbulence as a "new state of matter", a sort of "new fluid" with specific constitutive laws

⇨ turbulence teaching must rely on the strong foundation of continuum mechanics

(*) my apologize to my former students : they have saw a large part of this stuff

2.1 conservation laws



• a volume of matter

✓ let Ω be any volume of fluid that we follow during its movement

✓ let b be any quantity (scalar, vector, …) which characterizes a fluid property

✓ denoting d/dt the material derivative, one may write :

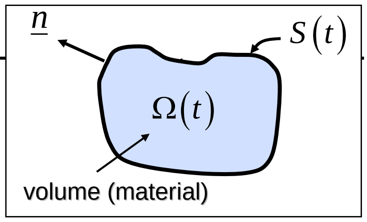
$$\frac{d}{dt}\iiint_{\Omega(t)} \rho b \, d\Omega = \text{source terms} = \iiint_{\Omega(t)} \mathcal{S}_b \, d\Omega$$

where $\begin{cases} b = \text{density per unit mass} \\ \rho b = \text{density per unit volume} \end{cases}$

✓ apply this to

$$\left.\begin{array}{l} \text{momentum } \; b = \underline{u} \\ \text{energy } \; b = E \\ \text{entropy } \; b = s \\ \text{mass } \; b = 1 \end{array}\right\} \Rightarrow \text{equations of mechanics}$$

2.1 conservation laws



• first term : transport theorem

$$\boxed{\frac{d}{dt}\iiint_{\Omega(t)} \rho b \, d\Omega} = \text{source terms} = \iiint_{\Omega(t)} \mathcal{S}_b \, d\Omega$$

✓ one can show that

$$\frac{d}{dt}\iiint_{\Omega(t)} \rho b \, d\Omega = \iiint_{\Omega(t)} \underbrace{\frac{\partial \rho b}{\partial t}}_{\text{time variation inside } \Omega(t)} d\Omega + \oiint_{S(t)} \underbrace{\rho b \underline{u} \, . \underline{n}}_{\text{flux across } S(t)} dS$$

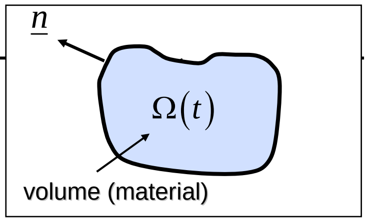
$\downarrow$ (*)

$$= \iiint_{\Omega(t)} \frac{\partial \rho b}{\partial t} d\Omega + \iiint_{\Omega(t)} \overbrace{div\left(\rho b \otimes \underline{u}\right)} d\Omega$$

$$\Rightarrow \boxed{\frac{d}{dt}\iiint_{\Omega(t)} \rho b \, d\Omega = \iiint_{\Omega(t)} \left[\frac{\partial \rho b}{\partial t} + div\left(\rho b \otimes \underline{u}\right) \right] d\Omega}$$

(*) $\iiint_{\Omega} div \; b \; dV = \iint_S b.\underline{n} \, dS$ - Green - Ostrogradski

2.1 conservation laws (…)



- **source term**

$$\frac{d}{dt}\iiint_{\Omega(t)} \rho b \, d\Omega = \text{source terms} = \boxed{\iiint_{\Omega(t)} \mathcal{S}_b \, d\Omega}$$

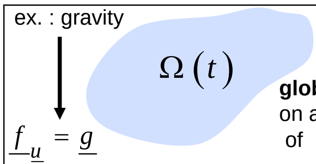
✓ two contributions

volumic source — surfacic source

$$\iiint_{\Omega(t)} \mathcal{S}_b \, d\Omega = \iiint_{\Omega(t)} \rho \, f_b \, d\Omega + \oiint_{S(t)} \varphi_b\left(\underline{n}\right) dS$$

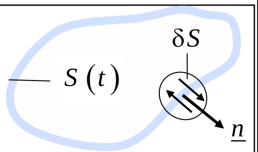
flux

✓ Cauchy's principle



global : acting on all fluid particules of $\Omega(t)$

local : restricted to the fluid particules that constitute the interface $S(t)$



- Cauchy's theorem : $\boxed{\varphi_b\left(\underline{n}\right) = \phi_b \cdot \underline{n}}$

(*) green - Ostrogradski

$$\Rightarrow \iiint_{\Omega(t)} \mathcal{S}_b \, d\Omega = \iiint_{\Omega(t)} \rho f_b \, d\Omega + \oiint_{S(t)} \phi_b \cdot \underline{n} \, dS = \iiint_{\Omega(t)} \left(\rho f_b + div\, \phi_b\right) d\Omega$$

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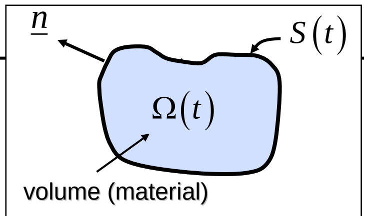
2.1 conservation laws (…)



- **total**

$$\frac{d}{dt}\iiint_{\Omega(t)} \rho b \, d\Omega = \iiint_{\Omega(t)} \frac{\partial \rho b}{\partial t} d\Omega + \oiint_{S(t)} \rho b \, \underline{u} \cdot \underline{n} \, dS$$

$$= \iiint_{\Omega(t)} \left[\frac{\partial \rho b}{\partial t} + div\left(\rho b \otimes \underline{u}\right) \right] d\Omega = \text{source terms} = \iiint_{\Omega(t)} \mathcal{S}_b \, d\Omega$$

$$= \iiint_{\Omega(t)} \rho f_b \, d\Omega + \oiint_{S(t)} \phi_b \cdot \underline{n} \, dS = \iiint_{\Omega(t)} \left(\rho f_b + div\, \phi_b\right) d\Omega$$

- **local form : volumic formulation**

✓ true whatever $\Omega(t)$ $\Rightarrow$ $\boxed{\frac{\partial \rho b}{\partial t} + div\left(\rho b \underline{u}\right) = \rho f_b + div\, \phi_b}$

local conservation laws of mechanics

- **apply this to :**
  - mass $b = 1$
  - momentum $b = \underline{u}$
  - energy $b = E$
  - entropy $b = s$

$\Rightarrow$ this provides the local equations of continuum mechanics, which include the equations of fluid mechanics

2.2 mass conservation : the continuity equation

$$\frac{\partial \rho b}{\partial t} + div\left(\rho b \underline{u}\right) = \rho f_b + div\, \phi_b$$

local conservation laws of mechanics

- **mass conservation :** $b = 1, f_b = \phi_b = 0$ (no source of mass) $\Rightarrow$ $\boxed{\frac{\partial \rho}{\partial t} + div\left(\rho \underline{u}\right) = 0}$

continuity equation
(conservative form)

- **decomposition of the divergence term :**

$$\underbrace{\frac{\partial \rho}{\partial t} + \underline{u}\,.\,\underline{grad}\,\rho}_{} + \rho\, div\, \underline{u} = 0 \quad \Rightarrow \quad \boxed{\frac{d\rho}{dt} + \rho\, div\, \underline{u} = 0}$$

material derivative : notation $\frac{d\rho}{dt}$

continuity equation
(non conservative form)

- **incompressible fluid flow = isovolumic flow (no volume variation along the trajectories)**

continuity equation $\frac{d\rho}{dt} + \rho\, div\, \underline{u} = 0$

incompressibility $\frac{d\rho}{dt} = 0$

$$\Rightarrow \boxed{div\, \underline{u} = \frac{1}{\rho}\frac{d\rho}{dt} = -\frac{1}{\Omega}\frac{d\Omega}{dt} = 0}$$

annexes -

material derivative

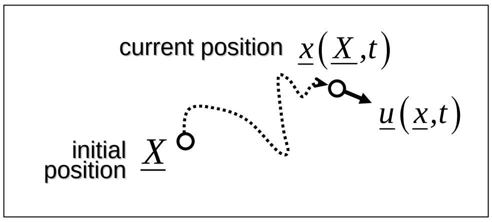
conservative and non conservative forms

annex - material derivative

material derivative

$$\underbrace{\frac{db}{dt}} = \frac{d}{dt} b\left[\, \underline{x}\left(\underline{X},t\right) \,\right]$$

$$= \frac{\partial b}{\partial t}\cancel{\frac{\partial t}{\partial t}} + \underbrace{\frac{\partial b}{\partial \underline{x}}}_{\nabla b} . \underbrace{\frac{\partial \underline{x}}{\partial t}}_{\underline{u}} = \frac{\partial b}{\partial t} + \nabla b . \underline{u}$$



- **vector** (e.g .$\underline{u}$) : $\nabla \underline{u}$ is a 2nd order tensor

$$\frac{d\underline{u}}{dt} = \frac{\partial \underline{u}}{\partial t} + \nabla \underline{u} . \underline{u} \quad \xrightarrow[\text{coordinates}]{\text{cartesian}} \quad \frac{du_i}{dt} = \frac{\partial u_i}{\partial t} + \frac{\partial u_i}{\partial x_j} u_j = \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}$$

$$= \frac{\partial u_i}{\partial t} + u_1 \frac{\partial u_i}{\partial x_1} + u_2 \frac{\partial u_i}{\partial x_2} + u_3 \frac{\partial u_i}{\partial x_3}$$

- **scalar** (e.g. ρ) : $\nabla \rho$ is a 1st order tensor, that is a vector. We not it $\underline{grad}\, \rho$

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \underline{u} . \underline{grad}\, \rho \quad \xrightarrow[\text{coordinates}]{\text{cartesian}} \quad \frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + u_j \frac{\partial \rho}{\partial x_j} = \ldots$$

annex - conservative and non conservative forms

$$\frac{\partial \rho b}{\partial t} + div\left(\rho b \otimes \underline{u}\right) = \rho f_b + div\, \phi_b \quad \text{conservative form}$$

✓ $$\frac{\partial \rho b}{\partial t} = b\frac{\partial \rho}{\partial t} + \rho \frac{\partial b}{\partial t}$$

✓ $$div\left(\rho b \otimes \underline{u}\right) = \rho b\, div\, \underline{u} + \nabla\left(\rho b\right).\underline{u}$$

$$= b\, div\left(\rho \underline{u}\right) - \left(\, b .\ \cancel{\underline{grad}\, \rho}\, \right) . \underline{u} + \left(\, \cancel{\underline{grad}\, \rho . b} + \rho \nabla b\, \right) . \underline{u}$$

$$\Longrightarrow \frac{\partial \rho b}{\partial t} + div\left(\rho b \otimes \underline{u}\right) = b\overbrace{\left[\frac{\partial \rho}{\partial t} + div\left(\rho \underline{u}\right)\right]}^{=0} + \rho \overbrace{\left[\frac{\partial b}{\partial t} + \nabla b .\ \underline{u}\right]}^{= db/dt} = \rho \frac{db}{dt}$$

mass conservation

$$\Longrightarrow \quad \rho \frac{db}{dt} = \rho f_b + div\, \phi_b \quad \text{non - conservative form}$$

annex - conservative and non conservative form (...)

- **example of a scalar** (e.g. ρ)

$$\frac{\partial \rho}{\partial t} + div(\rho \underline{u}) = 0 \qquad \text{conservative}$$

$$\frac{d\rho}{dt} + \rho\, div\underline{u} = 0 = \frac{\partial \rho}{\partial t} + \underline{u}\,.\,\underline{grad} + \rho\, div\underline{u} = 0 \qquad \text{non conservative}$$

- **discritising space :** 1st order forward scheme in one dimension between x and $x+\delta x$

✓ $$div(\rho \underline{u}) = \frac{\partial \rho u}{\partial x} = \frac{\rho(x+\delta x)u(x+\delta x) - \rho(x)u(x)}{\delta x} \qquad \text{conservative}$$

✓ $$\underline{u}\,.\,\underline{grad}\,\rho + \rho\, div\underline{u} = u(x)\frac{\rho(x+\delta x)-\rho(x)}{\delta x} + \rho(x)\frac{u(x+\delta x)-u(x)}{\delta x}$$
$$= \frac{\rho(x+\delta x)u(x+\delta x) - \rho(x)u(x)}{\delta x} - \frac{[\,u(x+\delta x)-u(x)\,][\,\rho(x+\delta x)-\rho(x)\,]}{\delta x}$$
$$= \frac{\partial \rho u}{\partial x} - \underbrace{\frac{[\,u(x+\delta x)-u(x)\,][\,\rho(x+\delta x)-\rho(x)\,]}{\delta x}}_{\text{residu}} \qquad \text{non conservative}$$

2.3 momentum : the law of dynamics

$$\boxed{\frac{\partial \rho b}{\partial t} + div(\rho b \underline{u}) = \rho f_b + div\, \phi_b}$$ **local conservation laws of mechanics**

- **momentum :** $b = \underline{u}$
  - ✓ volumic source : a force density $\underline{f_u} = \underline{f}$ e.g. : $\underline{f} = \underline{g}$ (gravity)
  - ✓ surfacic source : a constraint $\phi_{\underline{u}} = \underline{\underline{\sigma}}$ ⇦ a 2nd order tensor : the **Cauchy's strain tensor**

- **Cauchy's formulation : a principle and a theroem**

$$\oiint_{S(t)} \varphi_{\underline{u}}(\underline{n})\, dS = \oiint_{S(t)} \overbrace{\phi_{\underline{u}}\,.\,\underline{n}}^{\text{strain vector } \underline{\sigma}}\, dS = \oiint_{S(t)} \underline{\underline{\sigma}}\,.\,\underline{n}\, dS$$

Cauchy's principle — Cauchy's theorem — **Cauchy's strain tensor**

$$\Rightarrow \quad \boxed{\rho \frac{d\underline{u}}{dt} = \rho \underline{f} + div\, \underline{\underline{\sigma}}}$$ **the law of dynamics** (non conservative form)

2.3 momentum : the law of dynamics (…)

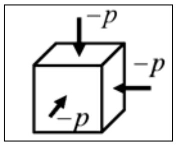
- **Cauchy's strain tensor : pressure and viscous stress**

✓ strain vector : $\underline{\sigma} = \underbrace{\underline{\underline{\sigma}}}_{\text{strain tensor}}.\underline{n}$

✓ static $(\underline{u} = 0)$: Pascal's law

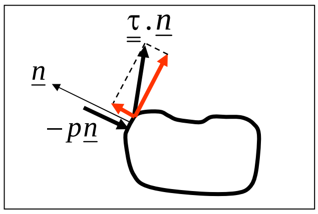
$$\underline{\sigma} = \underline{\underline{\sigma}}.\underline{n} = -p\,\underline{n} \;\Rightarrow\; \underline{\underline{\sigma}} = -p\,\underline{\underline{1}} \qquad \underline{\underline{1}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

pressure (↑ $-p\,\underline{n}$) — identity tensor ($\underline{\underline{1}}$)



✓ dynamics $(\underline{u} \neq 0)$: pressure and viscous stress

$$\underline{\sigma} = \underline{\underline{\sigma}}.\underline{n} = -p\underline{n} + \underbrace{\underline{\underline{\tau}}.\underline{n}}_{\text{friction}} \;\Rightarrow\; \underline{\underline{\sigma}} = -p\,\underline{\underline{1}} + \underbrace{\underline{\underline{\tau}}}_{\text{viscous stress tensor}}$$



- **constitutive law of matter :** $\underline{\underline{\tau}} = \underline{\underline{\tau}}(\nabla \underline{u})$

⇒ the constitutive law defines the ~~fluid~~ matter

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2.3 momentum : the law of dynamics (…)

- **the law of dynamics** $\boxed{\rho \dfrac{d\underline{u}}{dt} = \rho \underline{f} + div\, \underline{\underline{\sigma}}}$ (non conservative form)

- **pressure and viscous stress decomposition :**

$$\underline{\underline{\sigma}} = -p\,\underline{\underline{1}} + \underline{\underline{\tau}} \;\Rightarrow\; \boxed{\rho \frac{d\underline{u}}{dt} = \rho \underline{f} + div\left(-p\,\underline{\underline{1}} + \underline{\underline{\tau}} \right)}$$

- **other form :** use the identity $div\left(p\underline{\underline{1}}\right) = \underline{grad}\; p.\underline{\underline{1}} + p\; div\underline{\underline{1}} = \underline{grad}\; p$

$$\Rightarrow \boxed{\rho \frac{d\underline{u}}{dt} = \rho \underline{f} - \underline{grad}\; p + div\, \underline{\underline{\tau}}}$$

- **choice of the fluid = choice of the constitutive law** $\underline{\underline{\tau}} = \underline{\underline{\tau}}(\nabla \underline{u})$

⇒ this will be done latter

2.4 conservation of energy

why ?

✓ turbulence appears spontaneously in a continuum when the latter is strongly distorted, namely when mechanical energy is large

⇨ grasping turbulence requires understanding its role in the energy balance of continuum

2.4 conservation of energy

$$\boxed{\frac{\partial \rho b}{\partial t} + div\left(\rho b \underline{u}\right) = \rho f_b + div\, \phi_b}$$ **local conservation laws of mechanics**

- **energy** $b = E$ **has two origins : mechanical and thermal**

✓ volumic sources : $f_E = \underbrace{\underline{f} . \underline{u}}_{\text{mechanical power due to the volumic force } \underline{f}} + \overbrace{r/\rho}^{\text{volumic source of heat (e.g. radiation)}}$

✓ surfacic sources : $\phi_E = \underbrace{{}^t\underline{\underline{\sigma}} . \underline{u}}_{\text{mechanical power due to the contact force on } S(t)} - \overbrace{\underline{q}}^{\text{heat flux across } S(t)}$

⇨ $$\boxed{\rho \frac{dE}{dt} = \rho \underline{f} . \underline{u} + r + div\left({}^t\underline{\underline{\sigma}} . \underline{u} - \underline{q} \right)}$$ **energy conservation law** (non conservative form)

⇨ important physics hidden in this equation

2.4 conservation of energy (…)

$$\rho \frac{dE}{dt} = \rho \underline{f}.\underline{u} + r + div\left({}^t\underline{\underline{\sigma}}.\underline{u} - \underline{q} \right) \qquad \textbf{(1)}$$

- **energy has two components :** $E = e_k + e$ (kinetic energy, internal energy) $\qquad e_k = \frac{1}{2}\underline{u}^2 \;, e = c_v T$ (ideal gaz)

- **kinetic energy :** it is a mechanical energy, so use momentum equation

$$\underline{u}.\left(\rho \frac{d\underline{u}}{dt} = \rho \underline{f} + div\, \underline{\underline{\sigma}} \right) \Longrightarrow \boxed{\rho \frac{de_k}{dt} = \underbrace{\rho \underline{f}.\underline{u}}_{\text{work of the volumic force}} + \underbrace{\underline{u}.div\, \underline{\underline{\sigma}}}_{\text{work of the contact forces}}} \qquad \textbf{(2)}$$

- **thermal energy : (1) – (2)**

$$\boxed{\rho \frac{de}{dt} = r + \underbrace{div\left({}^t\underline{\underline{\sigma}}.\underline{u} \right) - \underline{u}.div\, \underline{\underline{\sigma}}}_{\text{internal work of the contact forces}} - div\, \underline{q}} \qquad \textbf{(3)}$$

2.4 the conservation of energy (…)

- **kinetic energy** $\boxed{\rho \frac{de_k}{dt} = \rho \underline{f}.\underline{u} + \underline{u}.div\, \underline{\underline{\sigma}}}$ $\Longleftarrow$ not a conservation law form (a $div\, \phi_{e_k}$ is lacking)

- **internal energy** $\boxed{\rho \frac{de}{dt} = r + div\left({}^t\underline{\underline{\sigma}}.\underline{u} \right) - \underline{u}.div\, \underline{\underline{\sigma}} - div\, \underline{q}}$

- **use the tensorial identity** $\boxed{div\left({}^t\underline{\underline{\sigma}}.\underline{u} \right) = \underline{u}.div\, \underline{\underline{\sigma}} + {}^t\underline{\underline{\sigma}} : \nabla \underline{u}}$

$$\Longrightarrow \left\{ \begin{array}{ll} \text{kinetic energy} & \rho \frac{de_k}{dt} = \overbrace{\rho \underline{f}.\underline{u} - {}^t\underline{\underline{\sigma}} : \nabla \underline{u}}^{\rho f_{e_k}} + div\left(\overbrace{{}^t\underline{\underline{\sigma}}.\underline{u}}^{\phi_{e_k}} \right) \\ \text{internal energy} & \rho \frac{de}{dt} = \overbrace{r + \underline{\underline{\sigma}} : \nabla \underline{u}}^{\rho f_e} - div\, \overbrace{\underline{q}}^{\phi_e} \end{array} \right\} \text{conservation law forms}$$

- **internal works :** ${}^t\underline{\underline{\sigma}} : \nabla \underline{u}$

$\Longrightarrow$ this volumic term is what sets the exchanges between kinetic and internal energy

2.4 the conservation of energy (...)

- **internal works** ${}^t\underline{\underline{\sigma}} : \nabla \underline{u}$: exchanges kinetic energy ↔ heat
- **pressure - viscous stress decomposition :** $\underline{\underline{\sigma}} = -p\,\underline{\underline{1}} + \underline{\underline{\tau}}$

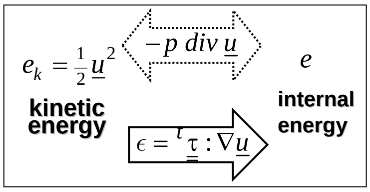
$$\Rightarrow \quad {}^t\underline{\underline{\sigma}} : \nabla \underline{u} = \left(-p\,\underline{\underline{1}} + {}^t\underline{\underline{\tau}} \right) : \nabla \underline{u} = \overbrace{-p\, div\, \underline{u}}^{\text{work of the pressure}} + \overbrace{{}^t\underline{\underline{\tau}} : \nabla \underline{u}}^{\text{work of viscosity}}$$

reversible (work of the pressure) — **irreversible** (work of viscosity)

- **introducing the dissipation function :** $\epsilon = {}^t\underline{\underline{\tau}} : \nabla \underline{u} > 0$

this is a key quantity for our topic : turbulence is something that increases considerably dissipation

$$\begin{cases} \rho \dfrac{de_k}{dt} = \overbrace{\rho \underline{f} . \underline{u} + \left(-p\, div\, \underline{u} + \epsilon \right)}^{\rho f_{e_c}} + div \overbrace{\left(-p\, \underline{u} + {}^t\underline{\underline{\tau}} . \underline{u} \right)}^{\phi_{e_c}} \\ \rho \dfrac{de}{dt} = \overbrace{r - \left(-p\, div\, \underline{u} + \epsilon \right)}^{\rho f_e} - div \overbrace{\underline{q}}^{\phi_e} \end{cases}$$

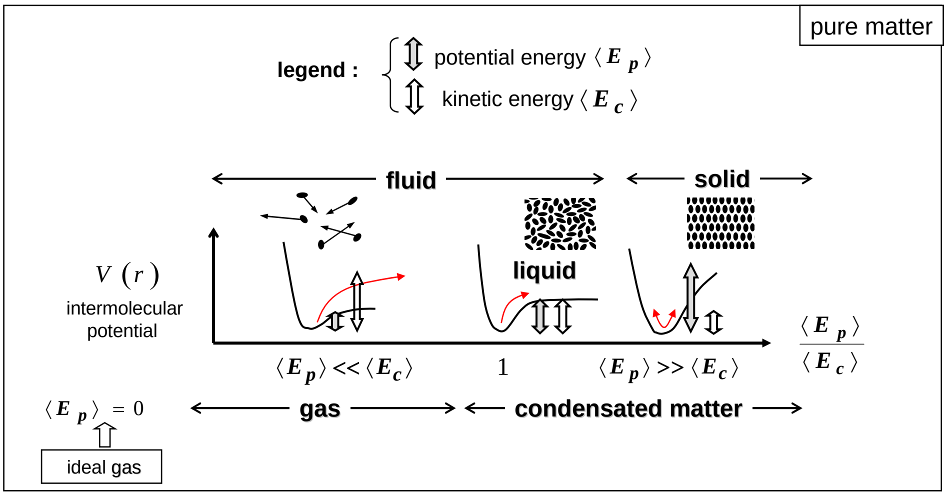


chapitre 3

fluid mechanics : a reminder

3.1 fluids

• **a fluid is an energetically disorganized state of matter**



Source : L. Jacquin, *Mécanique des fluides - Tome 1*, Ed. Polytechnique 2014

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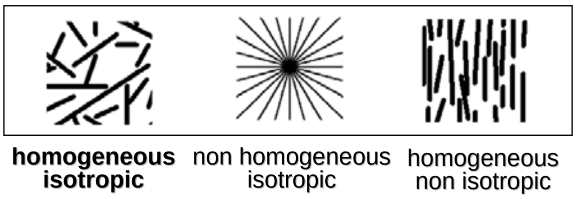
3.2 classical fluids

• **a classical (or simple) fluid has two properties**

✓ its is homogeneous and isotropic



✓ its is newtonian (it flows « newtonianly »)

• **what does it mean to be newtonian ?**

3.3 newtonian fluid

• **newton's law (1687)**

"*The resistance which arises from the lack of slipperiness of the parts of the liquid, other things being equal, is proportional to the velocity with which the parts of the liquid are separated from one another*"

• **in the modern language**

in a moving fluid, the stress is proportional to the velocity gradient

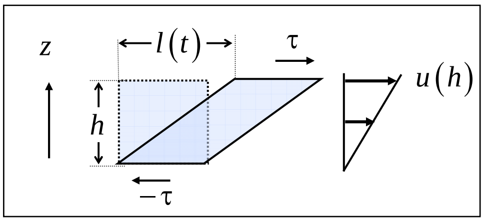
• **the newton's law for the case of a simple shear**

deformation :

$$e(t) = \frac{l(t)}{h}$$

deformation rate (s^{-1}) :

$$\dot{e} = \frac{\dot{l}}{h} = \frac{u(h)}{h} = \frac{du}{dz}$$



stress ($Pa = N.m^{-2}$)

$$\tau = \eta\,\dot{e} = \eta\,\frac{du}{dz}$$

Newton's law

shear viscosity η ($Pa.s$)

3.3 newtonian fluid (…)

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• **Newton's law for the case of a simple shear (…)**

$$\tau = \underbrace{\eta}_{\text{shear viscosity}} \frac{du}{dz}$$

← newtonian constitutive relation = linearity between $\underline{\underline{\tau}}$ and $\nabla \underline{u}$

• **general :** one can show that the most general linear form reads

$$\underline{\underline{\tau}} = \underbrace{\kappa}_{\text{bulk viscosity}}\, div\, \underline{u}\ \underline{\underline{1}} + 2\underbrace{\eta}_{\text{shear viscosity}} \left[\ \underline{\underline{d}} - \frac{1}{3} div(\underline{u})\underline{\underline{1}}\ \right]$$

where

$$\begin{cases} \underline{\underline{d}} = \frac{1}{2}\left(\nabla\underline{u} + {}^t\nabla\underline{u}\right) & \text{is the rate of strain tensor (symmetric part of } \nabla\underline{u}) \\ div\ \underline{u} = trace\{\underline{\underline{d}}\} \end{cases}$$

• **note :** the rotationnal part of $\nabla \underline{u}$, $\underline{\underline{\Omega}} = \frac{1}{2}\left(\nabla\underline{u} - {}^t\nabla\underline{u}\right)$, is absent because solid body rotation produces no stress

annex - properties of the newtonian constitutive law

$$\underline{\underline{\tau}} = \underbrace{\kappa}_{\text{bulk viscosity}} div\, \underline{u}\ \underline{\underline{1}} + 2 \underbrace{\eta}_{\text{shear viscosity}} \left[\ \underline{\underline{d}} - \frac{1}{3} div(\underline{u}) \underline{\underline{1}}\ \right]$$

- **velocity gradient tensor :** $\nabla \underline{u}$ has 9 components $(\nabla \underline{u})_{ij} = \frac{\partial u_i}{\partial x_j}$
- **rate of strain tensor :** $\underline{\underline{d}} = \frac{1}{2}(\ \nabla \underline{u} + {}^t\nabla \underline{u}\ )$ has 6 components $(\underline{\underline{d}})_{ij} = \frac{1}{2}(\ \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\ )$
- **identity tensor :** $\underline{\underline{1}}$ has 3 components $(\underline{\underline{1}})_{ij} = \delta_{ij}$
- **velocity divergence :** $div\, \underline{u} = trace\{\nabla \underline{u}\} = trace\{\underline{\underline{d}}\} = \frac{\partial u_i}{\partial x_i}$ is a scalar.

- **symmetry :** $\underline{\underline{d}}$ and $\underline{\underline{1}}$ are symmetric (no effect if *i* is changed by *j* and vice-versa) so that $\underline{\underline{\tau}}$ is symmetric. This results from the **homogeneous and isotropic** microscopic properties of a newtonian fluid

annexes - properties of the newtonian constitutive law (…)

bulk viscosity (active in violent changes of volume)

stokes hypothesis (plugged in every software product)

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annex - properties of the newtonian constitutive law (…)

$$\underline{\underline{\tau}} = \underbrace{\kappa}_{\text{bulk viscosity}} div\, \underline{u}\, \underline{\underline{1}} + 2\underbrace{\eta}_{\text{shear viscosity}} \left[\, \underline{\underline{d}} - \frac{1}{3} div(\underline{u}) \underline{\underline{1}}\, \right]$$

• **bulk viscosity :** evaluates departure from thermodynamic equilibrium

✓ total strain : $\underline{\underline{\sigma}} = \underbrace{\left(-\overbrace{p}^{\text{pressure}} + \kappa\, div\, \underline{u} \right)}_{\text{mechanical pressure } P} \underline{\underline{1}} + 2\,\eta \left[\, \underline{\underline{d}} - \frac{1}{3} div(\underline{u}) \underline{\underline{1}}\, \right]$

✓ the fluid is out of equilibrium when $P \neq p$

✓ a condition on time scales : compare $\tau_{compression} = \left(div\, \underline{u} \right)^{-1}$ with $\tau_{microscopic}$

$$\begin{cases} \tau_{compression} >> \tau_{microscopic} & \text{equilibrium} \\ \tau_{compression} \approx \tau_{microscopic} & \text{out of equilibrium} \end{cases}$$

✓ $\tau_{microscopic}$ is usually very small ($\sim 10^{-9} s$ for a gaz)

⇨ bulk viscosity may be considered as neglectable, except in very rapid changes of volume (hypersonic flows, combustion, detonations…)

annex - properties of the newtonian constitutive law (…)

• **approximation : Stokes hypothesis**

✓ Stokes' (equilibrium) hypothesis : $\kappa = 0$ ⇨ $\boxed{\underline{\underline{\tau}} = -\frac{2\,\eta}{3} div(\underline{u}) \underline{\underline{1}} + 2\,\eta\, \underline{\underline{d}}}$

3.4 flow of a newtonian incompressible and homogeneous fluid with constant viscosity an no external force : momentum

$$\underline{\underline{\tau}} = \underbrace{\kappa}_{\text{bulk viscosity}} \, div\, \underline{u} \; \underline{\underline{1}} + 2 \underbrace{\eta}_{\text{shear viscosity}} \left[\, \underline{\underline{d}} - \frac{1}{3} div\left(\underline{u}\right) \underline{\underline{1}} \, \right]$$

bulk viscosity shear viscosity

• approximation : incompressibility

✓ continuity $\dfrac{d\rho}{dt} + \rho \, div\, \underline{u} = 0$

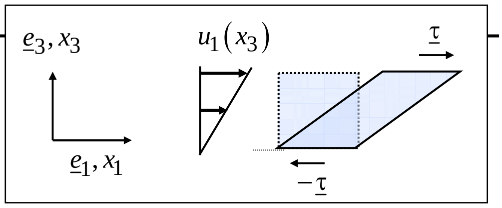
✓ incompressibility $\dfrac{d\rho}{dt} = 0$

$$\Rightarrow \quad div\, \underline{u} = 0 \quad \Rightarrow \quad \underline{\underline{\tau}} = 2\,\eta\, \underline{\underline{d}}$$

$$\Rightarrow \quad \underline{\underline{\tau}} = 2\,\eta\, \underline{\underline{d}} = \eta\left(\nabla \underline{u} + {}^t\nabla \underline{u} \right)$$

$$\tau_{ij} = \eta \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

3.4 flow of a newtonian incompressible and homogeneous fluid with constant viscosity an no external force : momentum



$$\underline{\underline{\tau}} = 2\,\eta\, \underline{\underline{d}} = \eta\left(\nabla \underline{u} + {}^t\nabla \underline{u} \right)$$

• training : apply this to a constant shear

✓ velocity $u_i = u_1(x_3)\,\delta_{i1}$

✓ velocity gradient

$$\begin{cases} \nabla \underline{u} = \dfrac{du_1}{dx_3} \underline{e}_1 \otimes \underline{e}_3 \\ {}^t\nabla \underline{u} = \dfrac{du_1}{dx_3} \underline{e}_3 \otimes \underline{e}_1 \end{cases}$$

meaning

$$\nabla \underline{u} = \begin{pmatrix} 0 & 0 & du_1/dx_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad {}^t\nabla \underline{u} = \begin{pmatrix} 0 & 0 & 0 \\ du_1/dx_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

✓ rate of deformation

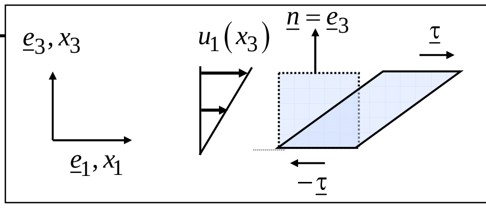
$$\underline{\underline{d}} = \frac{1}{2}\left(\nabla \underline{u} + {}^t\nabla \underline{u} \right) = \frac{1}{2} \frac{du_1}{dx_3} \left(\underline{e}_1 \otimes \underline{e}_3 + \underline{e}_3 \otimes \underline{e}_1 \right)$$

meaning

$$\underline{\underline{d}} = \frac{1}{2} \begin{pmatrix} 0 & 0 & du_1/dx_3 \\ du_1/dx_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

✓ viscous stress tensor $\underline{\underline{\tau}} = 2\,\eta\, \underline{\underline{d}} = \eta \dfrac{du_1}{dx_3} \left(\underline{e}_1 \otimes \underline{e}_3 + \underline{e}_3 \otimes \underline{e}_1 \right)$

3.4 flow of a newtonian incompressible and homogeneous fluid with constant viscosity an no external force : momentum



$$\underline{\underline{\tau}} = 2\,\eta\,\underline{\underline{d}} = \eta\,\frac{du_1}{dx_3}\left(\underline{e}_1 \otimes \underline{e}_3 + \underline{e}_3 \otimes \underline{e}_1\right)$$

✓ viscous stress vector :

$$\underline{\tau} = \underline{\underline{\tau}}\,.\,\underline{n} = \frac{du_1}{dx_3}\left(\underline{e}_1 \otimes \underline{e}_3 + \underline{e}_3 \otimes \underline{e}_1\right).\,\underline{e}_3$$

✓ apply the rule : $(\,\underline{e}_i \otimes \underbrace{\underline{e}_j\,).\,\underline{e}_k}_{\delta_{jk}} = \underline{e}_i\,\delta_{jk}$

✓ this gives : $\underline{\tau} = \frac{du_1}{dx_3}\left[\,(\,\underline{e}_1 \otimes \underbrace{\underline{e}_3\,).\,\underline{e}_3}_{1} + (\,\underline{e}_3 \otimes \underbrace{\underline{e}_1\,).\,\underline{e}_3}_{0}\,\right] = \frac{du_1}{dx_3}\underline{e}_1 \;\Rightarrow\; \boxed{\underline{\tau} = \frac{du_1}{dx_3}\underline{e}_1}$

3.4 flow of a newtonian incompressible and homogeneous fluid with constant viscosity an no external force : momentum

• **momentum** $\qquad \boxed{\rho\frac{d\underline{u}}{dt} = div\left(\,-p\,\underline{\underline{1}} + \underline{\underline{\tau}}\,\right)}$

• **incompressibility :** $div\,\underline{u} = 0$

• **newtonian :** $\underline{\underline{\tau}} = 2\eta\,\underline{\underline{d}} \;\Rightarrow\; \boxed{\frac{d\underline{u}}{dt} = -\frac{1}{\rho}\underline{grad}\;p + \frac{1}{\rho}div\left(\,2\eta\,\underline{\underline{d}}\,\right)}$

• **constant viscosity :** $\eta = const. \;\Rightarrow\; div\,\underline{\underline{\tau}} = 2\eta\,div\,\underline{\underline{d}} = \eta\,div\left(\nabla\underline{u} + {}^t\nabla\underline{u}\,\right)$

tensorial Identities $\left.\begin{aligned} div\,\nabla\underline{u} &= \Delta\underline{u} \\ div\,{}^t\nabla\underline{u} &= \underline{grad}\left(\,div\,\underline{u}\,\right)\end{aligned}\right\} \;\Rightarrow\; = \eta\,\Delta\underline{u} + \eta\,\underline{grad}\;\cancel{div\,\underline{u}} = \eta\,\Delta\underline{u}$

$$\Rightarrow\; \boxed{\frac{d\underline{u}}{dt} = -\frac{1}{\rho}\underline{grad}\;p + \nu\,\Delta\underline{u}} \qquad \nu = \frac{\eta}{\rho}\ \text{kinematic viscosity}$$

• **homogeneous fluid :** $\rho = const. \;\Rightarrow\; \boxed{\frac{d\underline{u}}{dt} = -\underline{grad}\;\frac{p}{\rho} + \nu\,\Delta\underline{u}}$

3.4 flow of a newtonian incompressible and homogeneous fluid with constant viscosity an no external force : energy

- **kinetic energy** $e_k = \frac{1}{2}\underline{u}^2$ **:** $\rho \frac{de_k}{dt} = p \, div \underline{u} + div\left(-p\,\underline{u} + {}^t\underline{\underline{\tau}}.\underline{u} \right) - \epsilon$
- **newtonian with constant viscosity :** ${}^t\underline{\underline{\tau}} = \underline{\underline{\tau}} = 2\eta \underline{\underline{d}}$, $\eta = const.$
- **dissipation function :** $\epsilon = {}^t\underline{\underline{\tau}} : \nabla\underline{u} = 2\eta\, \underline{\underline{d}} : \nabla\underline{u}$

✓ other form : use decomposition $\nabla\underline{u} = \frac{1}{2}\left(\nabla\underline{u} + {}^t\nabla\underline{u}\right) + \frac{1}{2}\left(\nabla\underline{u} - {}^t\nabla\underline{u}\right) = \underline{\underline{d}} + \underline{\underline{\Omega}}$

antisymmetric — symmetric

$$\Rightarrow \quad \epsilon = 2\eta\, \underline{\underline{d}} : \nabla\underline{u} = 2\eta\, \underline{\underline{d}} : \underline{\underline{d}} + 2\eta\, \underline{\underline{\Omega}} : \underline{\underline{d}} = 2\eta\, \underline{\underline{d}} : \underline{\underline{d}}$$

see annex

$$\Rightarrow \quad \begin{cases} \rho \frac{de_k}{dt} = div\left(-p\,\underline{u} + 2\eta\, \underline{\underline{d}}.\underline{u} \right) - \epsilon \\ \epsilon = 2\eta\, \underline{\underline{d}} : \underline{\underline{d}} \end{cases}$$

- dissipation rate per unit mass

annex – demonstration

- $\underline{\underline{\Omega}} : \underline{\underline{d}} = 0$

✓ $\underline{\underline{\Omega}} : \underline{\underline{d}} = \frac{1}{4}\left(\nabla\underline{u} - {}^t\nabla\underline{u}\right) : \left(\nabla\underline{u} + {}^t\nabla\underline{u}\right) = \frac{1}{4}\left(\nabla\underline{u} : \nabla\underline{u} - {}^t\nabla\underline{u} : {}^t\nabla\underline{u} + \nabla\underline{u} : {}^t\nabla\underline{u} - {}^t\nabla\underline{u} : \nabla\underline{u} \right)$

✓ $\nabla\underline{u} : \nabla\underline{u} = \left(\frac{\partial u_i}{\partial x_j} \underline{e}_i \otimes \underline{e}_j \right) : \left(\frac{\partial u_k}{\partial x_l} \underline{e}_k \otimes \underline{e}_l \right) = \frac{\partial u_i}{\partial x_j}\frac{\partial u_k}{\partial x_l} \overbrace{\left(\underline{e}_i \otimes \underbrace{\underline{e}_j \right) : \left(\underline{e}_k}_{\delta_{jk}} \otimes \underline{e}_l \right)}^{\delta_{il}} = \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial xi}$

||

✓ ${}^t\nabla\underline{u} : {}^t\nabla\underline{u} = \left(\frac{\partial u_i}{\partial x_j} \underline{e}_j \otimes \underline{e}_i \right) : \left(\frac{\partial u_k}{\partial x_l} \underline{e}_l \otimes \underline{e}_k \right) = \frac{\partial u_i}{\partial x_j}\frac{\partial u_k}{\partial x_l} \overbrace{\left(\underline{e}_j \otimes \underbrace{\underline{e}_i \right) : \left(\underline{e}_l}_{\delta_{il}} \otimes \underline{e}_k \right)}^{\delta_{jk}} = \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}$

✓ we also find that : $\nabla\underline{u} : {}^t\nabla\underline{u} = \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j} = {}^t\nabla\underline{u} : \nabla\underline{u}$

3.4 flow of a newtonian incompressible and homogeneous fluid with constant viscosity an no external force : energy (…)

- **dissipation function : alternative forms**

$$\epsilon = 2\eta \underline{\underline{d}} : \underline{\underline{d}} = \eta \left| \underbrace{\nabla \underline{u}}_{\text{velocity gradients}} \right|^2 + \eta \, div \overbrace{\left(\nabla \underline{u} . \underline{u} \right)}^{\text{flux}} \quad (1)$$

$$\epsilon = 2\eta \underline{\underline{d}} : \underline{\underline{d}} = \eta \underbrace{\underline{\omega}}_{\text{vorticity}}^2 + \eta \, div \underbrace{\left(2 \nabla \underline{u} . \underline{u} \right)}_{\text{flux}} \quad (2)$$

see annex

- **kinetic energy**

(1) ⇒ see annex

$$\begin{cases} \dfrac{de_k}{dt} = div\left(-\dfrac{p}{\rho}\underline{u} + \nu \, \underline{grad} \, e_k \right) - \varepsilon_1 \\ \varepsilon_1 = \nu \left| \nabla \underline{u} \right|^2 \text{ - pseudo dissipation rate per unit mass} \end{cases}$$

we will use this latter on

(2) ⇒ see annex

$$\begin{cases} \dfrac{de_k}{dt} = div\left(-\dfrac{p}{\rho}\underline{u} - \nu \, \underline{\omega} \wedge \underline{u} \right) - \varepsilon_2 \\ \varepsilon_2 = \nu \, \underline{\omega}^2 \text{ - pseudo dissipation rate per unit mass} \end{cases}$$

annex - demonstration

- **dissipation function** $\epsilon(\underline{x},t) = 2\eta \, \underline{\underline{d}} : \underline{\underline{d}}$ **- formulation (1)**

✓ $$2\,\underline{\underline{d}} : \underline{\underline{d}} = 2\left(d_{ij} \underline{e}_i \otimes \underline{e}_j \right) : \left(d_{kl} \underline{e}_k \otimes \underline{e}_l \right) = 2 d_{ij} d_{kl} \overbrace{\left(\underline{e}_i \otimes \underbrace{\underline{e}_j \right) : \left(\underline{e}_k}_{\delta_{jk}} \otimes \underline{e}_l \right)}^{\delta_{il}} = 2 d_{ij} d_{kl} \delta_{il} \delta_{jk} = 2 d_{ij} d_{ji}$$

✓ $$d_{ij} = \tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

✓ $$2 d_{ij} d_{ji} = \tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right) = \left(\frac{\partial u_i}{\partial x_j} \right)^2 + \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}$$

$$= \underbrace{\left(\frac{\partial u_i}{\partial x_j} \right)^2}_{|\nabla \underline{u}|^2} + \underbrace{\frac{\partial}{\partial x_i}\left(u_j \frac{\partial u_i}{\partial x_j} \right)}_{div(\nabla \underline{u}.\underline{u})} - u_j \frac{\partial}{\partial x_j} \underbrace{\left(\frac{\partial u_i}{\partial x_i} \right)}_{div\, \underline{u} = 0}$$

$$\Rightarrow \boxed{\epsilon(\underline{x},t) = 2\eta \, \underline{\underline{d}} : \underline{\underline{d}} = \eta \left| \nabla \underline{u} \right|^2 + \eta \, div\left(\nabla \underline{u} . \underline{u} \right)}$$

annex - demonstration (...)

• **kinetic energy**

✓ $$\frac{de_k}{dt} = div\left[-\frac{p}{\rho}\underline{u} + \nu\overbrace{\left(\nabla\underline{u} + {}^t\nabla\underline{u} \right)}^{2\eta\underline{\underline{d}}}.\underline{u} \right] - \nu\left|\nabla\underline{u}\right|^2 - \nu\, div\left(\nabla\underline{u}.\underline{u}\right)$$

⇨ $$\frac{de_k}{dt} = div\left(-\frac{p}{\rho}\underline{u} + \nu\, {}^t\nabla\underline{u}\,.\,\underline{u} \right) - \nu\left|\nabla\underline{u}\right|^2$$

✓ $${}^t\nabla\underline{u}.\underline{u} = \left[\left({}^t\nabla\underline{u} \right)_{ij} \underline{e}_i \otimes \underline{e}_j \right].u_k\, \underline{e}_k = u_k \left({}^t\nabla\underline{u} \right)_{ij} \underline{e}_i \delta_{jk}$$

$$= u_j \left({}^t\nabla\underline{u} \right)_{ij} \underline{e}_i = u_j \frac{\partial u_j}{\partial x_i}\underline{e}_i = \frac{\partial}{\partial x_i}\left(\frac{1}{2}\underline{u}^2 \right)\underline{e}_i = \underline{grad}\left(\frac{1}{2}\underline{u}^2 \right) = \underline{grad}\; e_k$$

⇨ $$\boxed{\frac{de_k}{dt} = div\left(-\frac{p}{\rho}\underline{u} + \nu\,\underline{grad}\; e_k \right) - \nu\left|\nabla\underline{u}\right|^2}$$

annex - demonstration (...)

• **dissipation function** $\epsilon\left(\underline{x},t\right) = 2\eta\, \underline{\underline{d}} : \underline{\underline{d}}$ **- formulation 2**

✓ $\underline{\omega}_i = \varepsilon_{ijk}\dfrac{\partial u_k}{\partial x_j}$ - vorticity

✓ $$\left|\underline{\omega}\right|^2 = \left(\varepsilon_{ijk}\frac{\partial u_k}{\partial x_j} \right)\left(\varepsilon_{ipq}\frac{\partial u_q}{\partial x_p} \right) = \overbrace{\varepsilon_{ijk}\varepsilon_{ipq}}^{(*)}\frac{\partial u_k}{\partial x_j}\frac{\partial u_q}{\partial x_p}$$

$$= \left(\delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp}\right)\frac{\partial u_k}{\partial x_j}\frac{\partial u_q}{\partial x_p} = \frac{\partial u_k}{\partial x_j}\frac{\partial u_k}{\partial x_j} - \frac{\partial u_k}{\partial x_j}\frac{\partial u_j}{\partial x_k}$$

$$= \left(\frac{\partial u_i}{\partial x_j} \right)^2 - \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i} = \left(\frac{\partial u_i}{\partial x_j} \right)^2 - \frac{\partial}{\partial x_j}\left(u_i \frac{\partial u_j}{\partial x_i} \right) + u_i \frac{\partial}{\partial x_i}\left(\cancel{\frac{\partial u_j}{\partial x_j}} \right)$$

$$= \left|\nabla\underline{u}\right|^2 - div\left(\nabla\underline{u}.\underline{u}\right)$$

⇨ $$\boxed{\left|\underline{\omega}\right|^2 = \left|\nabla\underline{u}\right|^2 - div\left(\nabla\underline{u}.\underline{u}\right)}$$

(*) *identity* : $\varepsilon_{ijk}\varepsilon_{ipq} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}$

annex - demonstration (…)

- **kinetic energy**

✓ $$\begin{cases} \dfrac{de_k}{dt} = div\left(-\dfrac{p}{\rho}\underline{u} + \nu\, {}^t\nabla\underline{u}\,.\,\underline{u} \right) - \nu\left|\nabla\underline{u}\right|^2 \\ \left|\nabla\underline{u}\right|^2 = div\left(\nabla\underline{u}.\underline{u}\right) + \left|\,\underline{\omega}\,\right|^2 \end{cases}$$

⇨ $$\frac{de_k}{dt} = div\left[-\frac{p}{\rho}\underline{u} + \nu\left({}^t\nabla\underline{u} - \nabla\underline{u} \right).\underline{u} \right] - \nu\left|\,\underline{\omega}\,\right|^2$$

⇨ $$\frac{de_k}{dt} = div\left(-\frac{p}{\rho}\underline{u} - 2\eta\,\underline{\underline{\Omega}}.\underline{u} \right) - \nu\left|\,\underline{\omega}\,\right|^2$$

✓ identity $\underline{\underline{\Omega}}.\underline{A} = \frac{1}{2}\underline{\omega}\wedge\underline{A}$ see annex

⇨ $$\boxed{\frac{de_k}{dt} = div\left(-\frac{p}{\rho}\underline{u} - \eta\,\underline{\omega}\wedge\underline{u} \right) - \nu\left|\,\underline{\omega}\,\right|^2}$$

annex - demonstration (…)

✓ identity $\underline{\underline{\Omega}}.\underline{A} = \frac{1}{2}\underline{\omega}\wedge\underline{A}$

✓ $$\underline{\underline{\Omega}}\,.\,\underline{A} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right]\left(\underline{e}_i \otimes \underline{e}_j \right).\left(A_l\,\underline{e}_l \right)$$

$$= \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right] A_l\,\underline{e}_i\delta_{jl} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right] A_j\,\underline{e}_i$$

✓ $$\frac{1}{2}\underline{\omega}\wedge\underline{A} = \frac{1}{2}\omega_k A_j\left(\underline{e}_k \wedge \underline{e}_j\right) = \frac{1}{2}\omega_k \underbrace{\left[\underline{e}_i.\left(\underline{e}_k \wedge \underline{e}_j\right) \right]}_{\varepsilon_{ikj}} A_j\,\underline{e}_i = \frac{1}{2}\varepsilon_{ikj}\omega_k A_j\,\underline{e}_i$$

✓ check that $\varepsilon_{ikj}\omega_k A_j = \left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j} \right) A_j$ with $\varepsilon_{123} = \varepsilon_{312} = \varepsilon_{231} = 1$

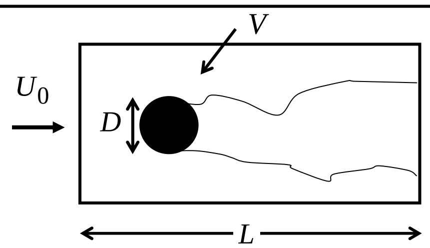
$\varepsilon_{132} = \varepsilon_{213} = \varepsilon_{321} = -1$ and $\underline{\omega} = \underline{rot}\ \underline{u} = \begin{pmatrix} \partial/\partial x_1 \\ \partial/\partial x_2 \\ \partial/\partial x_3 \end{pmatrix} \wedge \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} \partial u_3/\partial x_2 - \partial u_2/\partial x_3 \\ \partial u_1/\partial x_3 - \partial u_3/\partial x_1 \\ \partial u_2/\partial x_1 - \partial u_1/\partial x_2 \end{pmatrix}$

chapitre 4

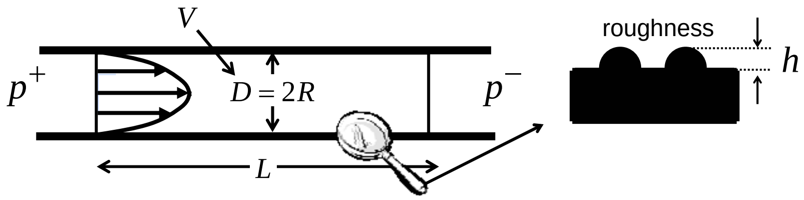
turbulence : a paradox

4.1 dissipation

• a sphere



• a rough pipe



• question : how much energy dissipated in a given fluid volume V ?

⇨ need a convenient definition for this quantity

4.1 dissipation (…)

$\nu = \eta/\rho$ - kinematic viscosity

- **dissipation rate per unit mass** $\varepsilon = \dfrac{\epsilon(\underline{x},t)}{\rho} = 2\nu\, \underline{\underline{d}} : \underline{\underline{d}} > 0$
  - newtonian
  - incompressible

- **mean value**

  ✓ spatial average in V : $\langle(\,.\,)\rangle_V \equiv \dfrac{1}{V}\iiint_V (\,.\,)\, dV$

  ✓ time average : $\overline{(\,.\,)}^T \equiv \lim_{T\to\infty} \dfrac{1}{T}\int (\,.\,)\, dt$

  $\Rightarrow$ $\varepsilon_V \equiv \overline{\langle \varepsilon \rangle_V}^T$ mean dissipation rate per unit mass

- **note**

  ✓ one observes that turbulent flows are « statistically steady » :

$$\overline{(\,.\,)(\underline{x},t)}^T = \lim_{T\to\infty} \frac{1}{T}\int (\,.\,)(\underline{x},t)\, dt = \overline{(\,.\,)}^T(\underline{x})$$

  ✓ commutation :

$$\overline{\langle(\,.\,)\rangle_V}^T = \langle \overline{(\,.\,)}^T \rangle_V$$

4.1 dissipation (…)

$$\varepsilon_V \equiv \overline{<\varepsilon>_V}^T = 2\nu < \overline{\underline{\underline{d}} : \underline{\underline{d}}}^T >_V$$

mean dissipation rate per unit mass

- **can we measure this quantity ?**

- **this means :**

  ✓ measuring $\nabla \underline{u}(\underline{x},t)$, namely 9 gradients $\left(\partial u_i/\partial x_j\ , i,j = 1,2,3\right)$

  ✓ time integration

  ✓ everywhere in volume V

  $\Rightarrow$ this has never been done

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4.1 dissipation (…)

- **the case of a flow of a newtonian incompressible and homogeneous fluid with constant viscosity without external force**

- **kinetic energy** $$\frac{de_k}{dt} = div\left(-\frac{p}{\rho}\underline{u} + 2\nu\,\underline{\underline{d}}.\underline{u} \right) - \varepsilon$$

- **identity** $$\frac{de_k}{dt} = \frac{\partial e_k}{\partial t} + div\left(e_k\,\underline{u}\right)$$

$$\Rightarrow \frac{\partial e_k}{\partial t} = div\underbrace{\left[-\left(\frac{p}{\rho} + e_k\right)\underline{u} + 2\nu\,\underline{\underline{d}}.\underline{u} \right]}_{\phi_{e_k}} - \varepsilon$$

- **suppose the flow is statistically steady :** time and volumic averaging give

$$\Rightarrow \begin{cases} 0 = \dfrac{1}{V}\iint_S \overline{\phi_{e_k}}^T .\,\underline{n}\,dS - \varepsilon_V \\ \phi_{e_k} = -\left(\dfrac{p}{\rho} + e_k\right)\underline{u} + 2\nu\,\underline{\underline{d}}.\underline{u} \quad \text{- power of surfacic efforts} \\ \varepsilon_V \quad \text{- mean dissipation rate per unit mass} \end{cases}$$

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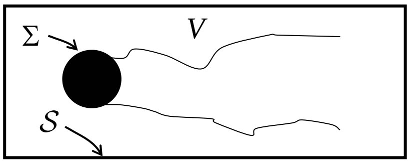
4.2 dissipation : the case of a sphere

$$\begin{cases} 0 = \dfrac{1}{V}\iint_S \overline{\phi_{e_k}}^T .\,\underline{n}\,dS - \varepsilon_V \\ \phi_{e_k} = -\left(\dfrac{p}{\rho} + e_k\right)\underline{u} + 2\nu\,\underline{\underline{d}}.\underline{u} \quad \text{- power of surfacic efforts} \\ \varepsilon_V \quad \text{- mean dissipation rate per unit mass} \end{cases}$$

- **sphere** $S = \mathcal{S} \cup \Sigma$



$$\Rightarrow \varepsilon_V = \cancel{\frac{1}{V}\oiint_\Sigma \overline{\phi_{e_k}}^T .\,\underline{n}\,dS} + \frac{1}{V}\oiint_{\mathcal{S}} \overline{\phi_{e_k}}^T .\,\underline{n}\,dS$$

no slip

✓ supposing $\mathcal{S}$ wide enough for having $p \approx const.$ and $\left|2\nu\,\underline{\underline{d}}\right| << e_k$ (ideal fluid) on $\mathcal{S}$

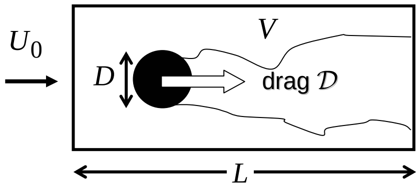
$$\Rightarrow \varepsilon_V = -\frac{1}{V}\oiint_{\mathcal{S}} \overline{e_k\,\underline{u}}^T .\,\underline{n}\,dS$$

flux of kinetic energy balance on the control surface $\mathcal{S}$

not accurate when $\mathcal{S}$ is large

4.2 dissipation : the case of a sphere (…)

• a « trick » : use the drag



$$\varepsilon_V \equiv \overline{<\varepsilon>_V}^T = 2\nu < \overline{\underline{\underline{d}} : \underline{\underline{d}}}^T >_V$$

✓ dissipation rate = power per unit mass : $\varepsilon_V = \dfrac{\text{force} \times \text{velocity}}{\text{mass}}$

✓ for large enough V, the force is the mean drag : $\lim_{V\to\infty}\left(\rho V \varepsilon_V\right) \approx \overline{\mathcal{D}}^T U_0$

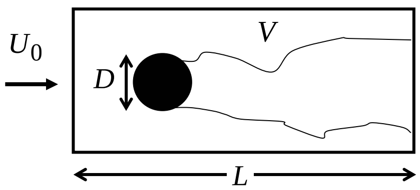
✓ introducing the drag coefficient : $\overline{C_{\mathcal{D}}}^T = \overline{\mathcal{D}}^T \Big/ \frac{1}{2}\rho U_0^2 D^2$

$$\Longrightarrow \quad \boxed{\lim_{V\to\infty}\left(\rho V \varepsilon_V\right) \approx \overline{C_{\mathcal{D}}}^T \times \frac{1}{2}\rho U_0^3 D^2}$$

✓ values of $\overline{C_{\mathcal{D}}}^T$ are available

4.2 dissipation : the case of a sphere (…)

• mean dissipation rate per unit volume of a sphere



$$\lim_{V\to\infty}\left(\rho V \varepsilon_V\right) = \overline{C_{\mathcal{D}}}^T \times \frac{1}{2}\rho U_0^3 D^2$$

• dimensional analysis $\overline{\mathcal{D}}^T = \mathcal{F}\left(U_0, D, \rho, \eta\right) \Longrightarrow \boxed{\dfrac{\overline{\mathcal{D}}^T}{\rho U_0^2 D^2} = \overline{C_{\mathcal{D}}}^T \left(\mathrm{Re} = \dfrac{\rho U_0 D}{\eta} \right)}$

drag coefficient Reynolds

• conclusion $\boxed{\lim_{V\to\infty}\left(\rho V \varepsilon_V\right) = \underbrace{\overline{C_{\mathcal{D}}}^T\left(\mathrm{Re}\right)}_{\text{viscosity}} \times \underbrace{\rho U_0^3 D^2}_{\text{injected power}}}$

• experiments provide $\overline{C_{\mathcal{D}}}^T\left(\mathrm{Re}\right)$

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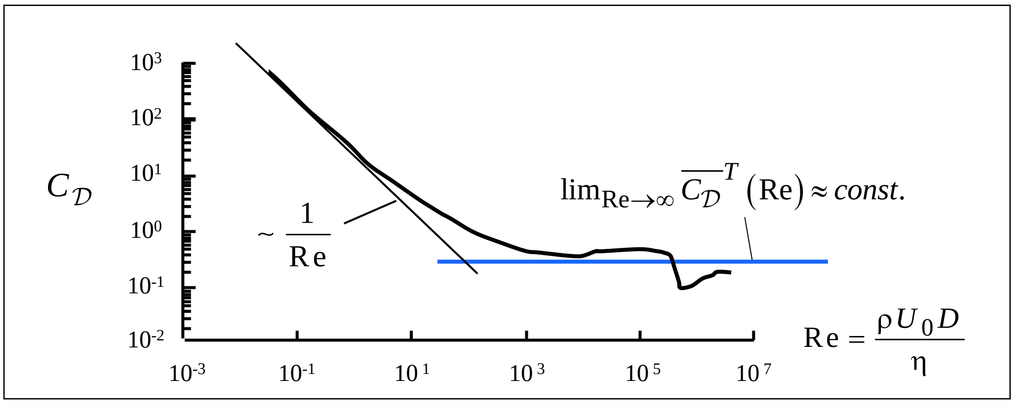
4.2 dissipation : the case of a sphere (…)

- **experiments on spheres**



- **so we observe that :** $\lim_{\mathrm{Re}>>1} \lim_{V\to\infty} (\rho V \varepsilon_V) = const. \times \underbrace{\rho U_0^3 D^2}_{\text{injected power}}$

- **which means :** at large Reynolds numbers, power absorbed by viscous friction does not longer depend on viscosity !!

4.3 a paradox

- **so we make the paradoxal following observation :**

at large Reynolds numbers, power absorbed by viscous friction no longer depends on viscosity !!

- **lets have a look on such flows**

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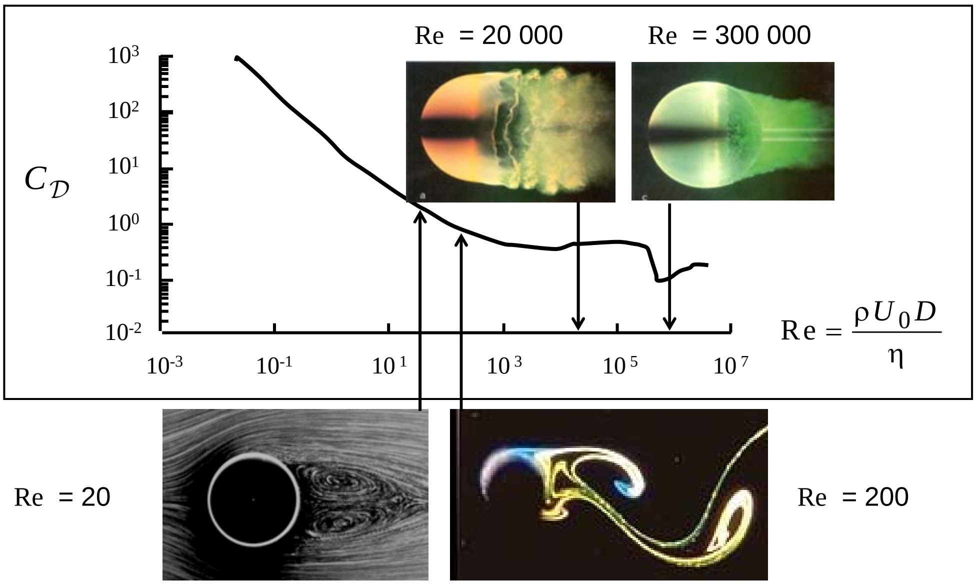
4.3 a paradox (…)



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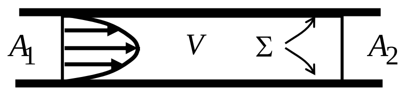
4.4 dissipation : the case of a pipe

we skip this case today

4.4 dissipation : the case of a pipe

$$\begin{cases} 0 = \frac{1}{V} \iint_S \overline{\phi_{e_k}}^T . \underline{n}\, dS - \varepsilon_V \\ \phi_{e_k} = -\left(\frac{p}{\rho} + e_k \right) \underline{u} + 2\nu \underline{\underline{d}} . \underline{u} \quad \text{- power of surfacic efforts} \\ \varepsilon_V \quad \text{- mean dissipation rate per unit mass} \end{cases}$$

- **pipe** $S = \Sigma \cup A_1 \cup A_2$



$$\Rightarrow \varepsilon_V = \underbrace{\cancel{\frac{1}{V} \iint_\Sigma \overline{\phi_{e_k}}^T . \underline{n}\, ds}}_{\text{no slip}} + \frac{1}{V} \iint_{A_1 \cup A_2} \overline{\phi_{e_k}}^T . \underline{n}\, ds$$

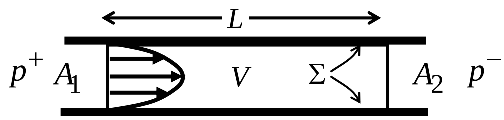
✓ supposing the **velocity field** statistically homogenous in the direction parallel to the pipe axis

$$\Rightarrow \varepsilon_V = -\frac{1}{V} \iint_{A_1 \cup A_2} \frac{1}{\rho} \overline{p\,\underline{u}}^T . \underline{n} dS \underbrace{\cancel{- \frac{1}{V} \iint_{A_1 \cup A_2} \overline{e_k\,\underline{u}}^T . \underline{n}\, dS + \frac{1}{V} \iint_{A_1 \cup A_2} 2\nu \overline{(\underline{\underline{d}} . \underline{u})}^T . \underline{n}\, dS}}_{\text{statistical homogeneity along the pipe axis}}$$

$$\Rightarrow \boxed{\varepsilon_V = -\frac{1}{V} \iint_{A_1 \cup A_2} \frac{1}{\rho} \overline{p\,\underline{u}}^T . \underline{n}\, dS}$$

energy of the pressure force is transformed into heat

4.4 dissipation : the case of a pipe (...)



$$\boxed{\varepsilon_V = -\frac{1}{V} \iint_{A_1 \cup A_2} \frac{1}{\rho} \overline{p\,\underline{u}}^T . \underline{n}\, dS}$$

- **approximation : introducing**

✓ the mean surfacic pressures $p^+ = \frac{1}{A_1} \iint_{A_1} \overline{p}^T dS \;,\; p^- = \frac{1}{A_2} \iint_{A_2} \overline{p}^T dS$

✓ the mean bulk velocity $\overline{U_b}^T = \frac{1}{A_1} \iint_{A_1} \overline{\underline{u}}^T . \underline{n}\, dS = \frac{1}{A_2} \iint_{A_2} \overline{\underline{u}}^T . \underline{n}\, dS = const.$

$$\left. \begin{array}{r} A_1 = A_2 = A \\ V = A \times L \end{array} \right\} \Rightarrow \boxed{\varepsilon_V \approx \underbrace{\frac{p^+ - p^-}{L}}_{\text{pressure gradient}} \underbrace{\overline{U_b}^T}_{\text{bulk velocity}}}$$

mean pressure power per unit mass

$$\Rightarrow \boxed{\rho V \varepsilon_V \approx \left(p^+ - p^- \right) \times \overline{U_b}^T \times A}$$

mean pressure power

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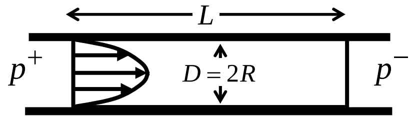
4.4 dissipation : the case of a pipe (...)



$$\rho V \varepsilon_V \approx (\, p^+ - p^- \,) \times \overline{U_b}^T \times A$$

- **dimensional analysis** $(\, p^+ - p^- \,) = \mathcal{G}\left(\overline{U_b}^T, D, L, h, \rho, \eta\right)$

roughness

$$\Rightarrow \quad \boxed{\frac{p^+ - p^-}{\rho(\, \overline{U_b}^T \,)^2} = \overline{\lambda}^T \left(\, \mathrm{Re}, \frac{h}{D} \,\right)} \qquad \mathrm{Re} = \frac{\rho\, \overline{U_b}^T D}{\eta} \qquad \frac{h}{D}$$

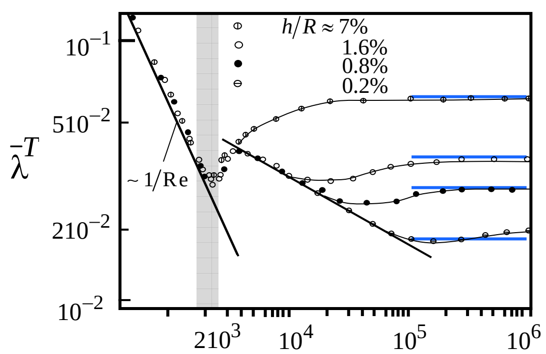
pipe friction coefficient — Reynolds — roughness

- **conclusion** $\rho V \varepsilon_V \approx \underbrace{\overline{\lambda}^T \left(\, \mathrm{Re}, \frac{h}{D} \,\right)}_{\text{viscosity, roughness}} \times \underbrace{\rho(\, \overline{U_b}^T \,)^3 D^2}_{\text{injected power}}$

- **experiments provide** $\overline{\lambda}^T \left(\, \mathrm{Re}, \frac{h}{D} \,\right)$

4.4 dissipation : the case of a pipe (...)

- **experiments on pipes**



$$\lim_{\mathrm{Re}\to\infty} \overline{\lambda}^T (\mathrm{Re}, h/R) = const.(h/R)$$

$$\mathrm{Re} = \frac{\rho U_0 D}{\eta}$$

Nykuradse (1932)

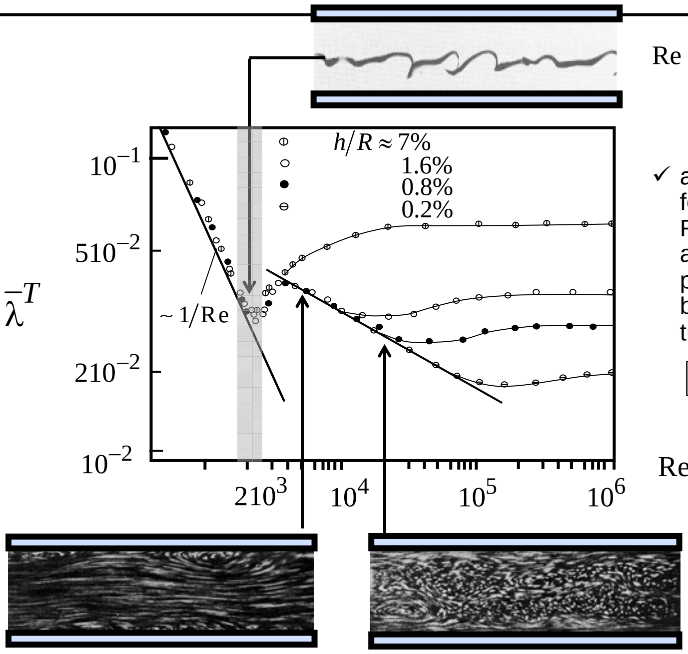
- **so we observe that :**

$$\lim_{\mathrm{Re}>>1} (\rho V \varepsilon_V) \approx const.(\, h/R \,) \times \underbrace{\rho(\, \overline{U_b}^T \,)^3 D^2}_{\text{injected power}}$$

- **which shows again that :**

at large Reynolds numbers, the power absorbed by viscous friction becomes independ on viscosity !!

4.4 dissipation : the case of a pipe (...)



✓ a concensus for the critical Reynolds above which a pipe flow becomes turbulent is :

$$\boxed{Re \approx 2000}$$

$$Re = \frac{\rho U_0 D}{\eta}$$

4.4 dissipation : the case of a pipe (...)

✓ a concensus for the critical Reynolds above which a pipe flow becomes turbulent is :

$$\boxed{Re \approx 2000}$$

➢ **exemple**

✓ domestic hydraulic network pipe

diameter $D \approx 10^{-2}\ m$

✓ turbulent regime :

$$Re = \frac{U\ D}{\nu} > 2000 \quad \Rightarrow \quad U > \frac{2\,10^3 \times 10^{-6}}{10^{-2}} = 20\ cm.s^{-1}$$

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4.5 Richardson-Kolmogorov cascade

• **so we make the paradoxical following observation :**

at large Reynolds numbers, power absorbed by viscous friction no longer depends on viscosity

• **we also observe that**

for sufficiantly large Reynolds numbers, flows are « turbulent »

• **we conclude that :**

in a turbulent flow, viscous dissipation results from a non-viscous process !!

• **a clever phenomenological scheme is needed to overcome this paradox**

the model of energy cascade by Richardson-Kolmogorov

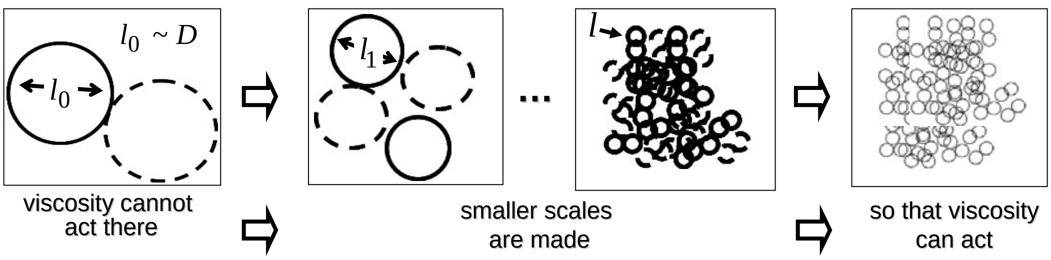
4.5 Richardson-Kolmogorov cascade

• **the basic principle : a cascade of energy**

in a turbulent regime flow breaks down in movments scales that allow the flow to adjust its dissipation to the mean power involved

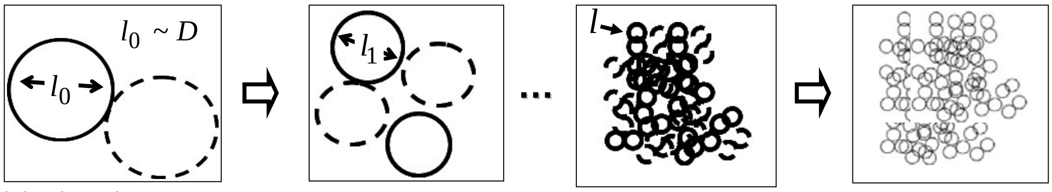
• **a four point reasoning**

1 - turbulence is reduced to a set of nested “flow stuctures“ whose characteristic size l and characteristic velocity u_l result from a dynamic process leading to a successive rupture of the “flow structures" into “flow structures” of decreasing scales



4.5 Richardson-Kolmogorov cascade (…)

2 - kinetic energy is injected to smaller « flow structures » from the largests ones (l_0, u_0) where external forces act. The rate of this initial energy transfer is $\varepsilon_0 \sim u_0^2/\tau_0$ where $\tau_0 \sim l_0/u_0$
τ_0 denotes the « life time » of the largest fluide structures. Putting $\tau_0 \sim l_0/u_0$, one gets $\varepsilon_0 \sim u_0^3/l_0$

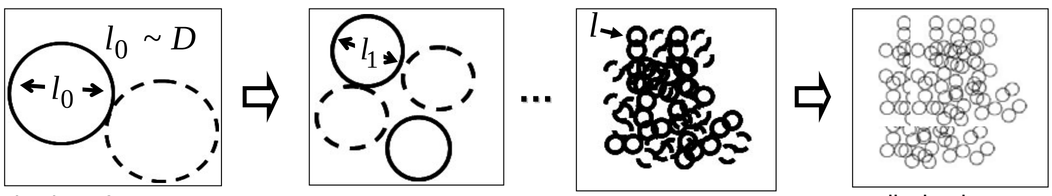


injection of energy
rate $\varepsilon_0 \sim u_0^3/l_0$

4.5 Richardson-Kolmogorov cascade (…)

3 - kinetic energy is dissipated at the same rate by « flow structures » of sufficiently small scales for viscous friction to be effective. By definition such scale, denoted (l_ν, u_ν) must fulfill $u_\nu l_\nu/\nu \approx 1$ (Reynolds number of order 1). This is the so called viscous regime where kinetic energy is transformed into heat.

viscous regime
$\mathrm{Re} = l_\nu u_\nu/\nu \sim 1$

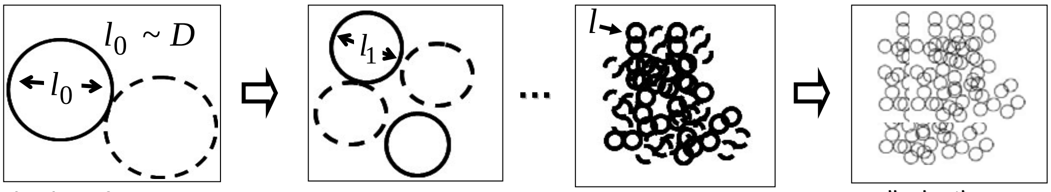


injection of energy
rate $\varepsilon_0 \sim u_0^3/l_0$

dissipation
rate $\varepsilon_0 \sim u_0^3/l_0$

4.5 Richardson-Kolmogorov cascade (…)

4 - for $l_\nu << l << l_0$, which defines the so-called inertial regime, energy transfer is self-similar, meaning identical for all scalles l , and it is **local**, meaning that energy of a given structure is entirely transfered to the smaller structures it makes during its life time. This is a purely inertial process where viscosité plays no role. Energy transfer rate at scales l is $\varepsilon_l \sim u_l^2 / \tau_l \sim u_l^3 / l$. It is constant and equal to ε_0, whatever l. Turbulence dynamics in the inertial regime only depends on l and ε_0 .



injection of energy rate $\varepsilon_0 \sim u_0^3 / l_0$

energy transfer rate $\varepsilon_l \sim u_l^3 / l \sim \varepsilon_0$

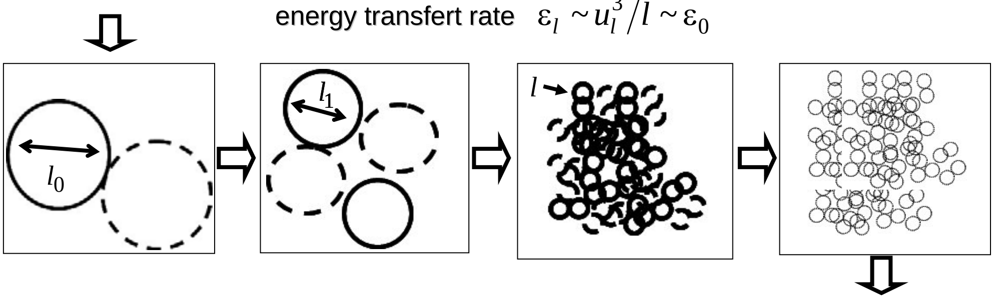
dissipation rate $\varepsilon_0 \sim u_0^3 / l_0$

4.5 Richardson-Kolmogorov cascade (…)

• the cascade process self adapts

ε_0

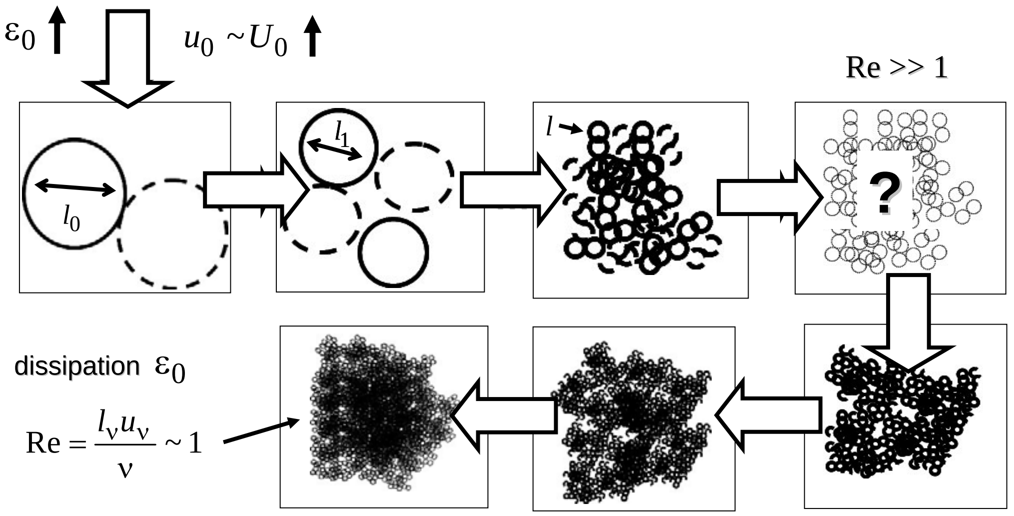
energy transfert rate $\varepsilon_l \sim u_l^3 / l \sim \varepsilon_0$



dissipation rate ε_0

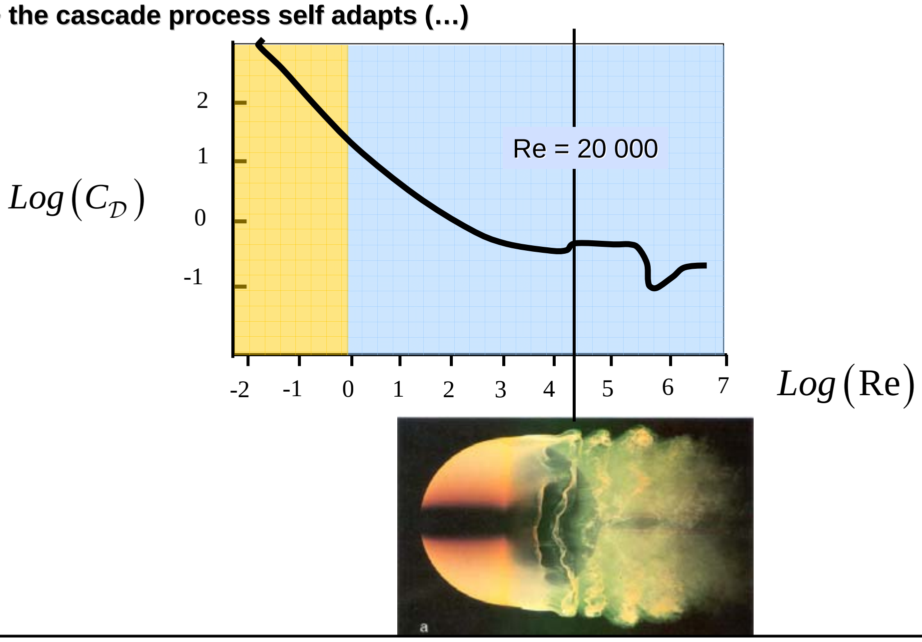
4.5 Richardson-Kolmogorov cascade (…)

- **the cascade process self adapts (…)**



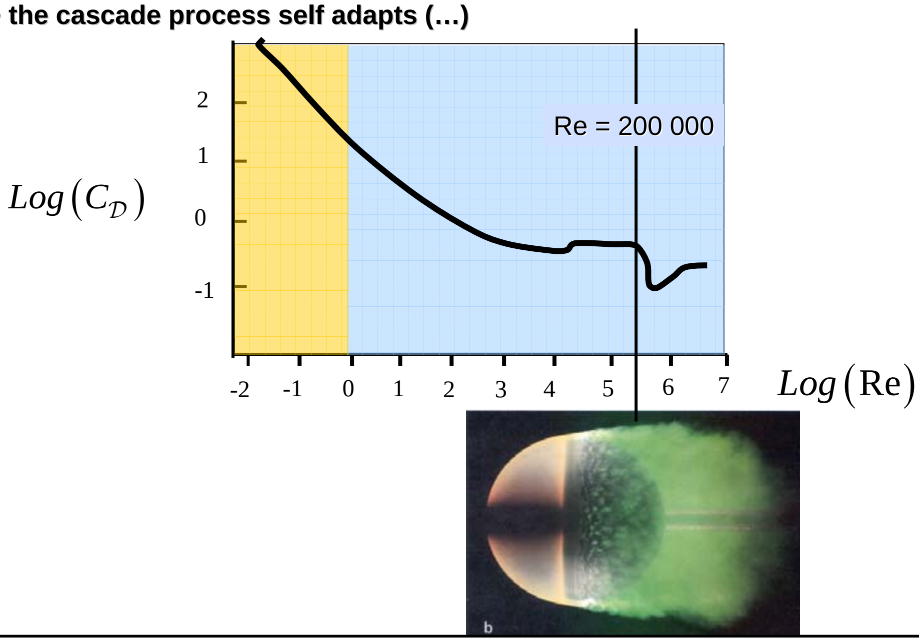
4.5 Richardson-Kolmogorov cascade (…)

- **the cascade process self adapts (…)**



4.5 Richardson-Kolmogorov cascade (...)

• the cascade process self adapts (...)



4.5 Richardson-Kolmogorov cascade (...)

• the cascade process self adapts (...)